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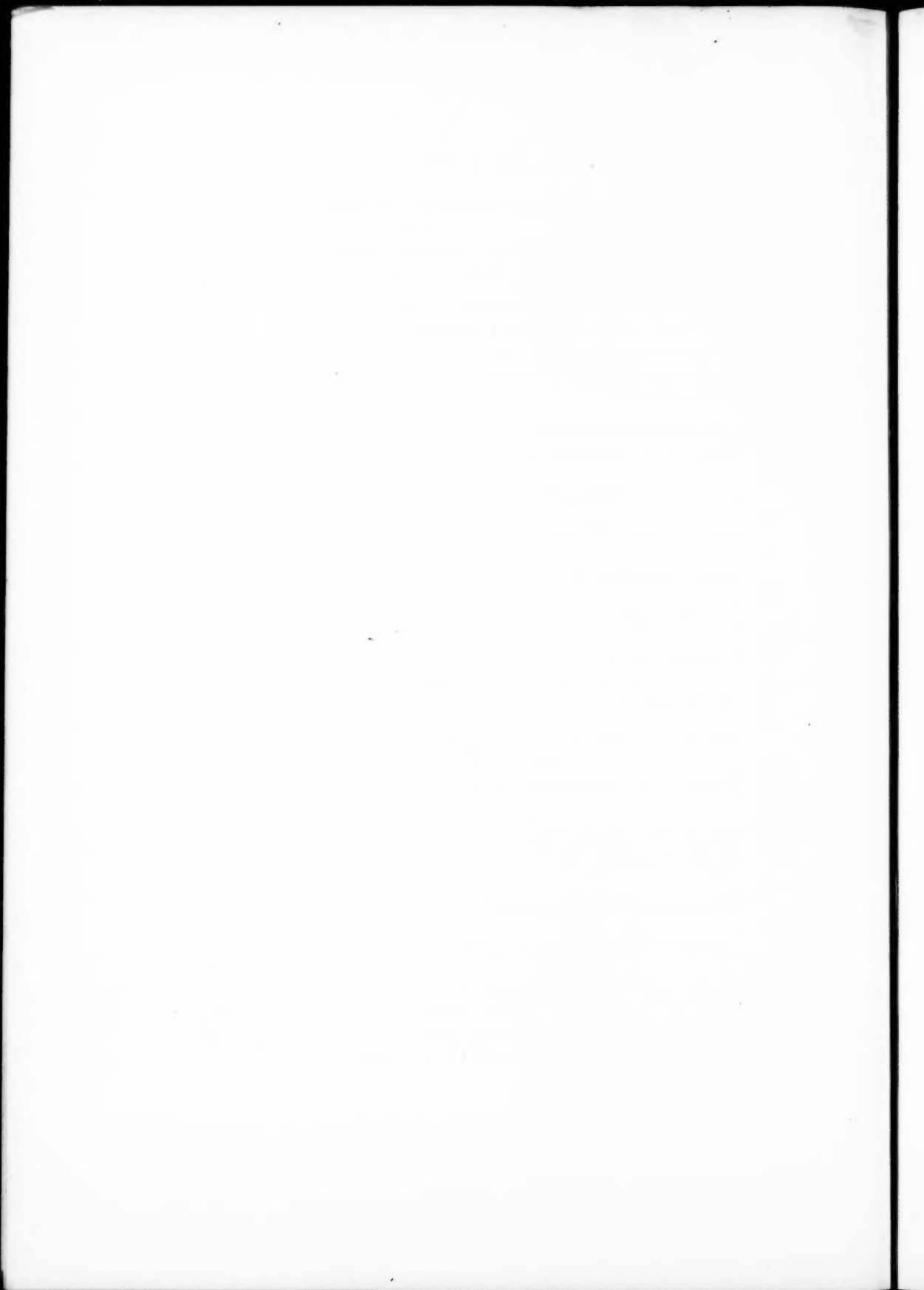
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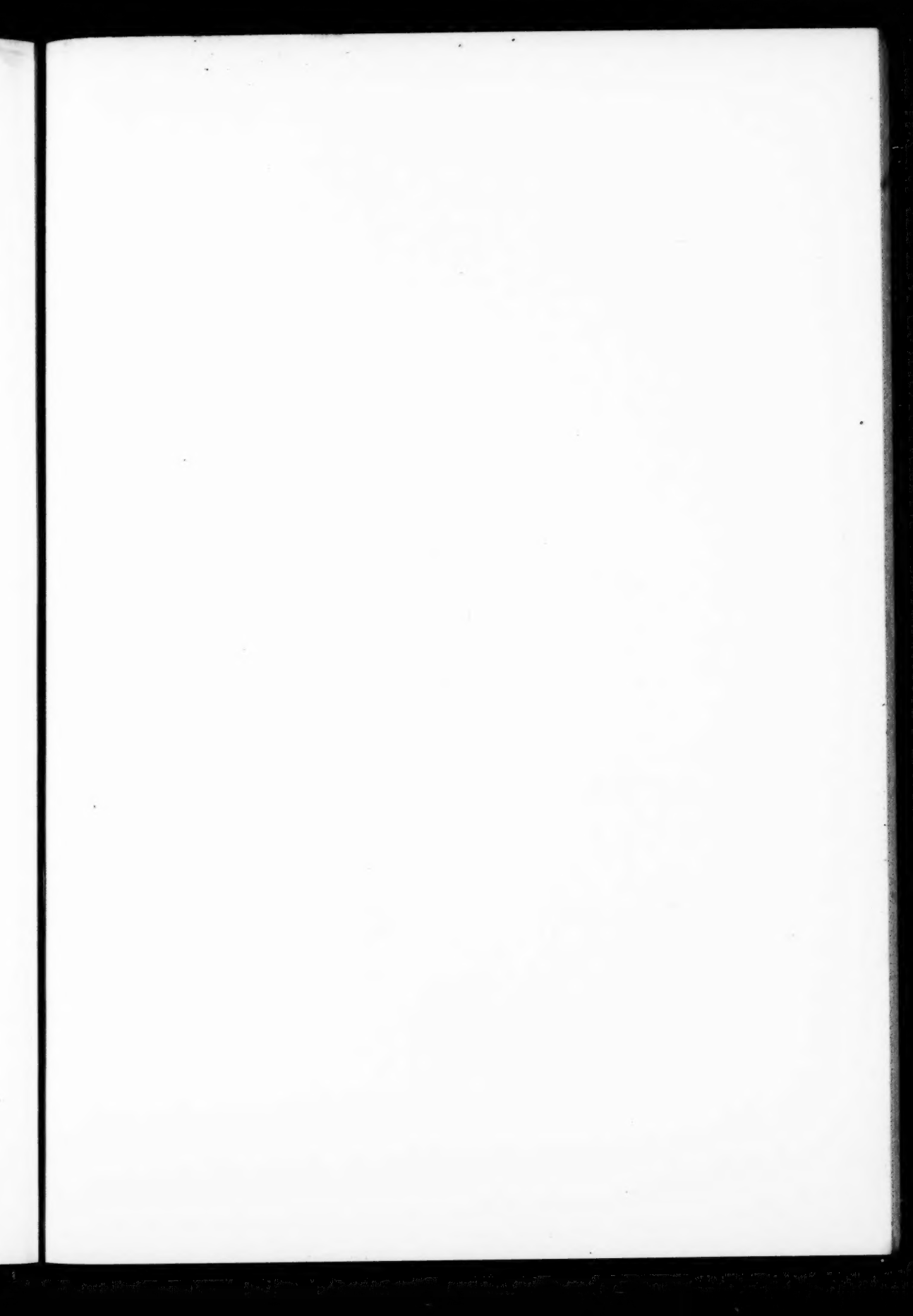
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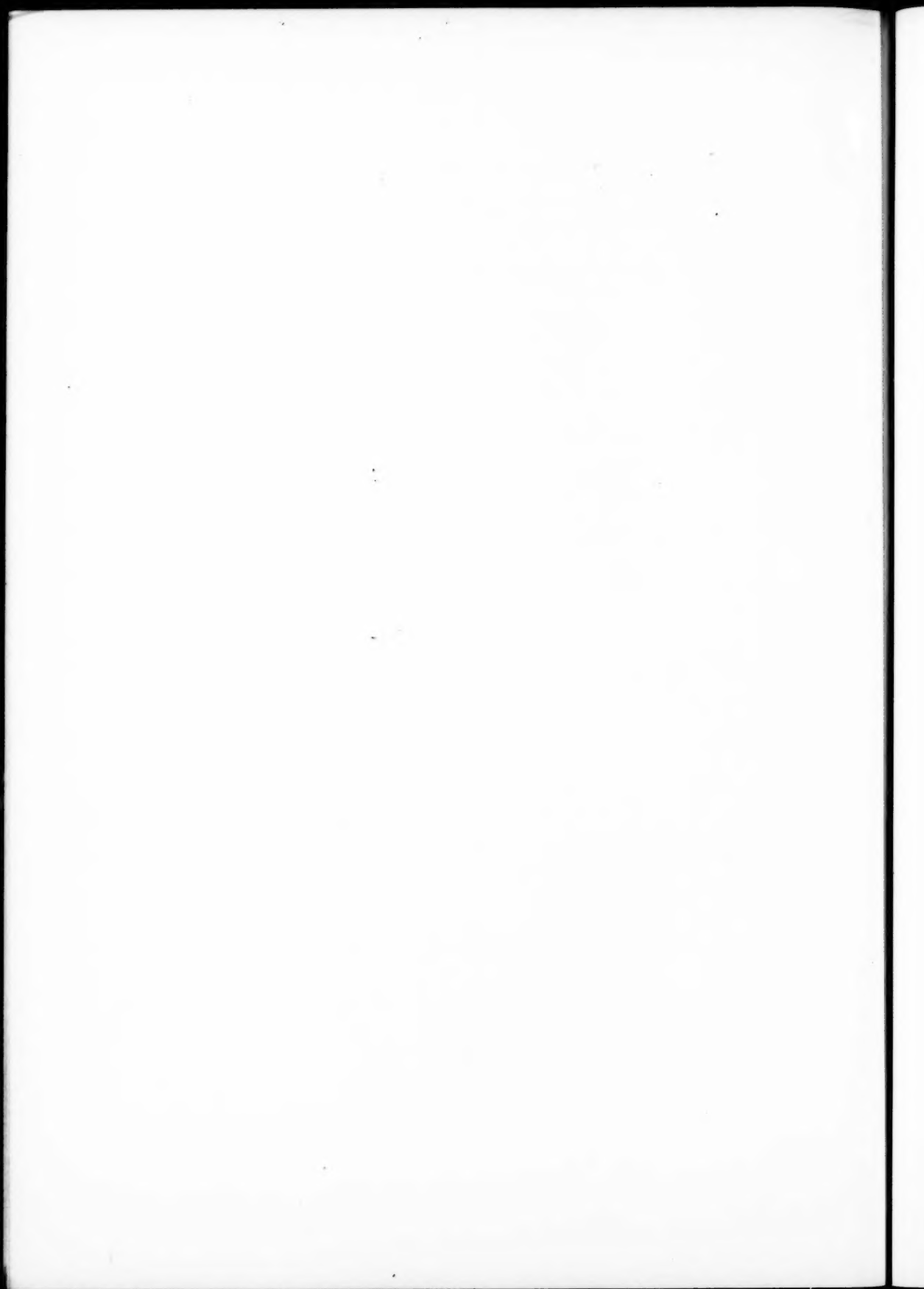
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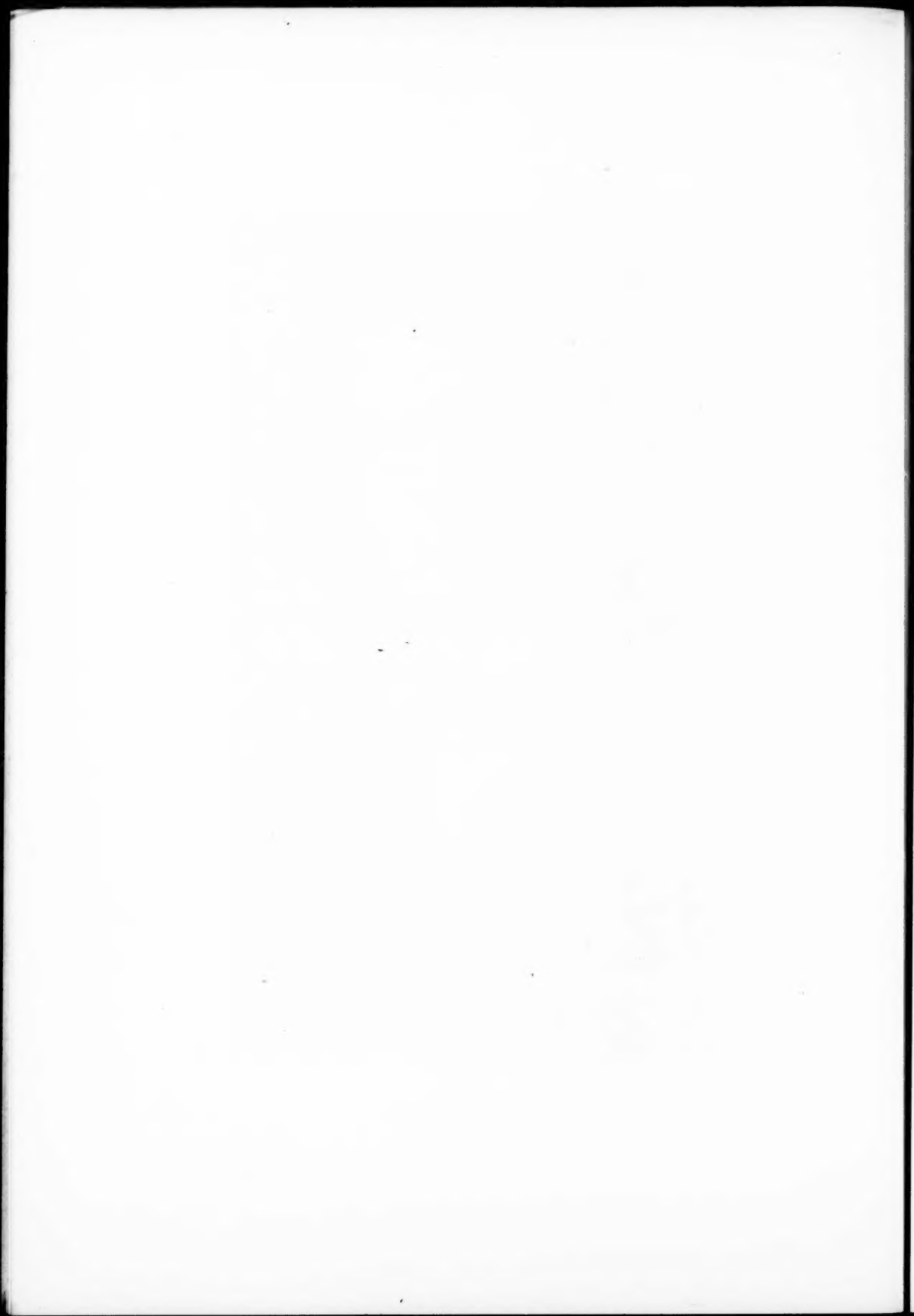




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WARING'S PROBLEM FOR CUBIC FUNCTIONS*

BY
L. E. DICKSON

1. INTRODUCTION

In 1921 it was proved by Kamke that if $f(x)$ is a polynomial with rational coefficients whose value is an integer ≥ 0 for every integer $x \geq 0$, then every integer ≥ 0 is a sum of a limited number u of 1's and a limited number v of values of $f(x)$ for integers $x \geq 0$. This existence theorem was later proved by the method of Hardy and Littlewood by Winogradow and Landau.

For the case of any quadratic function, the writer (and later Dr. Pall) evaluated the limits u and v .

We shall here treat cubic functions (1). The case in which a term x^2 occurs is under investigation by my students. The main result is Theorem 2. For special cubic functions, Theorems 4 and 5 give universal Waring theorems.

2. DETERMINATION OF ALL FUNCTIONS (1) WITH CERTAIN PROPERTIES

We restrict attention to cubic functions of the form

$$(1) \quad f(x) = \frac{\alpha x^3 + \beta x}{d} \quad (\alpha \neq 0, d > 0),$$

$$(2) \quad \alpha, \beta, d \text{ integers without a common factor } > 1.$$

We assume that $f(x)$ is an integer for every integer $x \geq 0$. By the values 1 and 2 of x , we see that $\alpha + \beta$ and $8\alpha + 2\beta$ are divisible by d , whence

$$(3) \quad 6\alpha \text{ and } 6\beta \text{ are divisible by } d.$$

If d has a prime factor $p > 3$, then α and β are divisible by p , contrary to (2). Hence 2 and 3 are the only possible prime factors of d .

To discuss only a pure Waring problem, we assume that $f(x) \geq 0$ if x is any integer $x \geq 0$, and that 1 is a value of $f(\xi)$ for a certain integer $\xi \geq 0$ (otherwise, sums of values of f would never give the number 1).

I. *Case d a multiple of 6.* Write $d = 6t$. By (3), α and β are divisible by t , whence $t = 1$, by (2), and $d = 6$. By $f(\xi) = 1$,

$$(4) \quad \alpha\xi^3 + \beta\xi = 6,$$

whence ξ is a positive divisor of 6.

* Presented to the Society, June 19, 1933; received by the editors June 23, 1933.

I₁. Let $\xi = 3w$, $w = 1$ or 2 . By (4), $9\alpha w^3 + \beta w = 2$. Elimination of β gives

$$f(x) = \frac{1}{6} \left[\alpha x^3 + \frac{2 - 9\alpha w^3}{w} x \right],$$

$$f(w) = \frac{1}{6}(1 - 4\alpha w^3) \geq 0, \alpha \leq 0; \quad f(6w) = 2 + 27\alpha w^3 \geq 0, \alpha \geq 0,$$

whence $\alpha = 0$, contrary to hypothesis. Thus Case I₁ is excluded.

I₂. Let $\xi = 2$. By (4), $4\alpha + \beta = 3$. Then

$$f(1) = \frac{1}{6}(1 - \alpha) \geq 0, \alpha \leq 1; \quad f(3) = \frac{1}{6}(15\alpha + 9) \geq 0, \alpha \geq 0,$$

whence $\alpha = 1$, and $f(x)$ is the pyramidal number

$$(5) \quad P(x) = \frac{1}{6}(x^3 - x).$$

This function satisfies all of our preceding assumptions.

I₃. There remains only the case $\xi = 1$. Then $\alpha + \beta = 6$,

$$(6) \quad f(x) = x + \frac{\alpha}{6}(x^3 - x) \equiv x + \alpha P(x), \alpha > 0,$$

which is an integer ≥ 0 for every integer $x \geq 0$, since the same is true of $P(x)$.

II. Case d a multiple of 3, but not of 6. Then $d = 3t$, where t is a positive odd integer. By (3), α and β are divisible by t , whence $t = 1$ by (2). Thus

$$f(x) = \frac{1}{6}(\alpha x^3 + \beta x), \quad \alpha\xi^3 + \beta\xi = 3,$$

whence ξ is a positive divisor of 3.

II₁. Let $\xi = 3$. Then $\beta = 1 - 9\alpha$,

$$f(1) = \frac{1}{6}(1 - 8\alpha) \geq 0, \alpha \leq 0;$$

$$f(4) = \frac{1}{6}(4 + 28\alpha) \geq 0, \alpha \geq 0.$$

Since $\alpha \neq 0$, Case II₁ is excluded.

II₂. Hence $\xi = 1$, $\beta = 3 - \alpha$,

$$(7) \quad f(x) = x + \frac{\alpha}{3}(x^3 - x) \equiv x + 2\alpha P(x), \alpha > 0.$$

III. Case d a multiple of 2, but not of 3. Similarly as in II, we find that $d = 2$, $\xi \neq 2$, $\xi = 1$, $\alpha + \beta = 2$,

$$(8) \quad f(x) = x + \frac{\alpha}{2}(x^3 - x) \equiv x + 3\alpha P(x), \alpha > 0.$$

IV. $d = 1$. Then $\xi = 1$, $\alpha + \beta = 1$, $f(x) = x + 6\alpha P(x)$.

THEOREM 1. If $f(x) = (\alpha x^3 + \beta x)/d$ ($\alpha \neq 0$) is an integer ≥ 0 for every integer $x \geq 0$, and if $f(x) = 1$ for some integer $x > 0$, then $f(x)$ is either a pyramidal number $P(x) = \frac{1}{6}(x^3 - x)$, or is $x + \epsilon P(x)$, where ϵ is a positive integer. Conversely, each of the resulting functions is an integer ≥ 0 for every integer $x \geq 0$, while $f(x) = 1$ has a positive integral solution.

3. THE MAIN THEOREM AND THREE LEMMAS

THEOREM 2. To each positive integer ϵ prime to 3 there correspond positive integers C and $\nu \geq 8$ such that every integer $\geq C \cdot 3^\nu$ is a sum of nine values of

$$(9) \quad f(x) = x + \frac{1}{6}\epsilon(x^3 - x)$$

for integral values ≥ 0 of x .

We shall first give the parts of the proof which hold both for $\epsilon = 1$ and $\epsilon > 1$, and then establish the few simple additional facts required in the case $\epsilon = 1$, and hence prove Theorem 2 for $\epsilon = 1$ with $C = 168$, $\nu = 8$. Then we shall present the more elaborate theory for $\epsilon > 1$ (which does not hold for $\epsilon = 1$). That theory gives a reconstructed proof which provides an explicit program actually to express any sufficiently large integer as a sum of nine values of $f(x)$.

LEMMA 1. There exists an integer m' such that any given integer is congruent to $f(3m')$ modulo 3^n .

The difference $f(z+3r) - f(z)$ has the value

$$(10) \quad \Delta = \frac{1}{2}\epsilon(3rz^2 + 9r^2z + 9r^3) + 3r - \frac{1}{2}\epsilon r.$$

Since 3 is a factor of all terms except the last, while ϵ is prime to 3, $\Delta \not\equiv 0$ if $r \not\equiv 0 \pmod{3^n}$. Take $r = m' - k$, $z = 3k$, $0 < r < 3^n$. Then

$$f(3m') - f(3k) = f[3k + 3(m' - k)] - f(3k) \not\equiv 0 \pmod{3^n}.$$

Hence for $j = 0, 1, \dots, 3^n - 1$, the 3^n integers $f(3j)$ are incongruent modulo 3^n , so that any integer is congruent to one of them.

LEMMA 2. If η is an odd constant integer, $v(\eta - v)$ is even and can be made congruent to any assigned even integer modulo 2^k by choice of an integer v .

Let $V(\eta - V) \equiv v(\eta - v) \pmod{2^k}$. Then the product of $V - v$ by $V + v - \eta$ is divisible by 2^k , while the factors are of unlike parity. Hence one factor is odd and the other is divisible by 2^k . Thus $V \equiv v$ or $\eta - v \pmod{2^k}$. Hence when v ranges over the 2^k values $0, 1, \dots, 2^k - 1$, we obtain at most (and hence exactly) $\frac{1}{2} \cdot 2^k$ values of $v(\eta - v)$ incongruent modulo 2^k , and the latter values are all even. This proves Lemma 2.

LEMMA 3. If $n > 1$ and $m < \epsilon \cdot 3^n$, then $f(3m) < \gamma \cdot 3^{3n}$, where

$$(11) \quad \gamma = \frac{1}{2}(9\epsilon^4 + 1).$$

Since $9m^3 - m$ increases with m ,

$$f(3m) = 3m + \frac{1}{2}\epsilon(9m^3 - m) < 3\epsilon 3^n + \frac{1}{2}\epsilon(9\epsilon^3 3^{3n} - \epsilon 3^n),$$

which will be $< \gamma 3^{3n}$ if $3\epsilon - \frac{1}{2}\epsilon^2 < \frac{1}{2}3^{2n}$. The latter holds for every n if $\epsilon \geq 6$ and holds for $n > 1$ if $\epsilon \leq 5$, since the maximum of $6\epsilon - \epsilon^2$ is its value 9 for $\epsilon = 3$.

4. PLAN OF PROOF WITH THE NECESSARY FORMULAS

If s and C are given positive numbers, we can evidently choose a positive integer n so that

$$C \cdot 27^n \leq s < C \cdot 27^{n+1}.$$

In Theorem 2, $s \geq C \cdot 27^n$. Hence we may take $n \geq \nu \geq 8$.

Any such s is one of the integers s_i falling in the following three sub-intervals:

$$3^{i-1}C3^{3n} \leq s_i < 3^iC3^{3n} \quad (i = 1, 2, 3).$$

By Lemma 1 we can choose an integer m_i so that

$$s_i = f(3m_i) + 3^n M_i, \quad 0 \leq m_i < 3^n,$$

where M_i is an integer. Since $f(m_i) \geq 0$, $3^n M_i \leq s_i < 3^i C 3^{3n}$. Using also Lemma 3, we get

$$(3^{i-1}C - \gamma)3^{3n} < M_i < 3^i C 3^{3n}.$$

Write $M_i = \epsilon 3^{3n} + N_i$. Then

$$(12) \quad (3^{i-1}C - \gamma - \epsilon)3^{3n} < N_i < (3^i C - \epsilon)3^{3n} \quad (i = 1, 2, 3).$$

Take $l = 3^n$ in the identity

$$f(l - x) + f(l + x) \equiv 2l + \frac{1}{3}\epsilon(l^3 - l + 3lx^2)$$

and sum for three values x_i of x . Thus

$$\sum_{i=1}^3 [f(3^n - x_i) + f(3^n + x_i)] = T,$$

$$T = \epsilon 3^{3n} + 3^n(\epsilon Q - \epsilon + 6), \quad Q = x_1^2 + x_2^2 + x_3^2.$$

Write ϕ_i for $f(v_i) + f(w_i)$ and Q_i for Q . Then will

$$s_i = f(3m_i) + 3^n(\epsilon 3^{3n} + N_i) = f(3m_i) + \phi_i + T$$

(and hence s_i will be a sum of nine values of f) if

$$(13) \quad 3^n(N_i + \epsilon - 6) = \phi_i + \epsilon 3^n Q_i \quad (i = 1, 2, 3).$$

We impose on the unknowns v_i, w_i the restrictions

$$(14) \quad v_i + w_i = 3b_i 3^n \quad (i = 1, 2, 3; b_i \text{ a positive odd integer}).$$

The identity

$$\phi_i = (v_i + w_i) \left[1 + \frac{\epsilon}{6} \{ (v_i + w_i)^2 - 3v_i w_i - 1 \} \right]$$

gives $\phi_i = 3^n B_i$, where

$$(15) \quad B_i = 3b_i \left[1 + \frac{\epsilon}{6} \{ 9b_i^2 3^{2n} - 1 - 3v_i(3b_i 3^n - v_i) \} \right].$$

Inserting $\phi_i = 3^n B_i$ into (13) and cancelling 3^n , we get

$$(16) \quad \epsilon Q_i = N_i + \epsilon - 6 - B_i.$$

We shall later choose the v_i so that

$$(17) \quad 0 \leq v_i \leq 3b_i 3^n, \quad 0 \leq N_i + \epsilon - 6 - B_i \leq \epsilon 3^{2n}.$$

These and (14) and (16) imply

$$(18) \quad 0 \leq w_i, \quad 0 \leq Q_i \leq 3^{2n}.$$

We shall later prove that we may take Q_i to be an integer which is a sum of three squares of integers $x_j \geq 0$. Thus $x_j \leq 3^n$ by (18). It will then follow that s_i is the sum of the values of $f(x)$ for the nine values $3m_i, v_i, w_i, 3^n - x_j, 3^n + x_j$ of x , each an integer ≥ 0 .

Employ the abbreviation

$$(19) \quad V_i = v_i - \frac{3}{2} b_i 3^n.$$

In the right member of (16), we insert the value of B_i and get

$$S_i = N_i + \epsilon - 6 - 3b_i \left[1 + \frac{\epsilon}{6} \left(3V_i^2 + \frac{9}{4} b_i^2 3^{2n} - 1 \right) \right].$$

The final condition (17) is $0 \leq S_i \leq \epsilon 3^{2n}$. Now $S_i \geq 0$ if $\frac{1}{3} A_i \geq V_i^2$, where

$$(20) \quad A_i = \frac{6}{\epsilon} \left[\frac{N_i + \epsilon - 6}{3b_i} - 1 \right] - \frac{9}{4} b_i^2 3^{2n} + 1.$$

Hence $S_i \geq 0$ if

$$(21) \quad A_i \geq 0, \quad V_i \geq 0, \quad (\frac{1}{3} A_i)^{1/2} \geq V_i.$$

Next, $S_i \leq \epsilon 3^{2n}$ if $\frac{1}{3}G_i \leq V_i^2$, where

$$(22) \quad G_i = A_i - 2 \cdot 3^{2n}/b_i,$$

and hence if

$$(23) \quad G_i \geq 0, \quad V_i \geq 0, \quad (\frac{1}{3}G_i)^{1/2} \leq V_i.$$

If we assume that $G_i \geq 0$ (whence $A_i \geq 0$), as well as

$$(24) \quad \frac{3}{2} b_i 3^n + (\frac{1}{3}G_i)^{1/2} \leq v_i \leq \frac{3}{2} b_i 3^n + (\frac{1}{3}A_i)^{1/2}, \quad (\frac{1}{3}A_i)^{1/2} \leq \frac{3}{2} b_i 3^n,$$

we see that (21) and (23) follow and that $v_i \leq 3b_i 3^n$, whence (17) hold.

By using the values (20) and (22) of A_i and G_i , we see that condition $G_i \geq 0$ and the final inequality (24) are equivalent to

$$l_i \leq N_i \leq L_i, \quad l_i = \epsilon 3^{2n} + \frac{9}{8} \epsilon b_i^3 3^{2n} + \beta_i,$$

where

$$(25) \quad L_i = \frac{9}{2} \epsilon b_i^3 3^{2n} + \beta_i, \quad \beta_i = b_i \left(3 - \frac{\epsilon}{2} \right) + 6 - \epsilon = (1 + \frac{1}{2}b_i)(6 - \epsilon).$$

This inequality will evidently follow if l_i is \leq the lower limit in (12) and if L_i is \geq the upper limit in (12), and hence if

$$(26) \quad 2\epsilon + \gamma + \frac{9}{8} \epsilon b_i^3 + \frac{\beta_i}{3^{2n}} \leq 3^{i-1}C \leq \frac{3}{2} \epsilon b_i^3 + \frac{\epsilon}{3} + \frac{\beta_i}{3^{2n+1}} \quad (i = 1, 2, 3).$$

When $\epsilon = 1$, $n \geq 8$, inequalities* (26) all hold if

$$(27) \quad b_1 = 5, \quad b_2 = 7, \quad b_3 = 11, \quad C = 168.$$

Since we shall need to assign to v_i a prescribed residue modulo 8, we desire that at least 8 consecutive integral values of v_i satisfy the first inequality (24) for every $i = 1, 2, 3$. The difference between its limits is

$$D_i = (\frac{1}{3}A_i)^{1/2} - (\frac{1}{3}G_i)^{1/2}.$$

Write $\mu_i = 2 \cdot 3^{2n}/(b_i A_i)$. By (22) and (23), $0 < \mu_i \leq 1$. Thus D_i is the product of $(\frac{1}{3}A_i)^{1/2}$ by

$$1 - (1 - \mu_i)^{1/2} = \frac{\mu_i}{1 + (1 - \mu_i)^{1/2}} > \frac{\mu_i}{2}.$$

Hence

* The limits for C are approximately 147, 188 if $i=1$; 130.8, 172.6 if $i=2$; 167.1, 221.9 if $i=3$. Since we desire a minimum C independent of i , we take $C=168$.

$$(28) \quad D_i > \frac{3^{2n}}{b_i(3A_i)^{1/2}}.$$

By (12) and (20),

$$(29) \quad 3A_i \leq \frac{18}{\epsilon} \left[\frac{(3^i C - \epsilon)3^{2n} + \epsilon - 6}{3b_i} - 1 \right] - \frac{27}{4} b_i^2 3^{2n} + 3.$$

5. PROOF OF THEOREM 2 WHEN $\epsilon = 1$

When $\epsilon = 1$, $n \geq 8$, (27)–(29) give

$$3A_1 < 435 \cdot 3^{2n} < 21^2 \cdot 3^{2n},$$

$$D_1 > 3^n/105 > 62,$$

and $D_2 > 29$, $D_3 > 14$, whence each $D_i > 8$. By (15),

$$(30) \quad 2B_i - 6b_i = b_i \epsilon F,$$

where F denotes the quantity in $\{ \}$ in (15). By Lemma 2, we can choose $v_i \pmod{8}$ so that F is congruent modulo 8 to any assigned even integer. Thus in (16) we can choose $v_i \pmod{8}$ so that $2\epsilon Q_i \equiv 2z \pmod{8}$, where z is an arbitrary integer. Take $z = \epsilon = 1$. Then $Q_i \equiv 1 \pmod{4}$. But $Q_i > 0$. Hence Q_i is a sum of three integral squares. This proves Theorem 2 when $\epsilon = 1$, with $C = 168$, $\nu = 8$.

6. PROOF OF THEOREM 2 WHEN ϵ IS PRIME TO 6 AND $\epsilon > 1$

We shall first determine b_1 , b_2 , b_3 , C , ν so that all three inequalities (26) hold when $n \geq \nu$, viz.,

$$(31) \quad I_i \leq 3^{i-1}C \leq S_i \quad (i = 1, 2, 3).$$

Minimum values of C and b_i may be found by the following scheme. Take b_1 to be the least positive odd integer for which $I_1 \leq S_1$. Take C to be the least integer $\geq I_1$. Take b_2 and b_3 to be the least positive odd integers for which $S_2 \geq 3C$, $S_3 \geq 9C$. We find that

| ϵ | b_1 | b_2 | b_3 | C |
|------------|-------|-------|-------|--------|
| 5 | 13 | 19 | 27 | 15182 |
| 7 | 17 | 25 | 35 | 49510 |
| 11 | 27 | 39 | 57 | 309485 |
| 13 | 31 | 45 | 65 | 564244 |

For these values we find that $I_2 \leq 3C$, $I_3 \leq 9C$, whence (31) hold.

For a general ϵ , we shall choose b_i to be a linear function of ϵ which has the value in the tablette when $\epsilon=5, \dots, 13$, and is such that (31) hold as regards the coefficients of the highest power of ϵ .

A. For $\epsilon=6e+1$, we take* $b_1=14e+3$ and get

$$(32) \quad C = 24354e^4 + 18882e^3 + 5508e^2 + 728e + 38.$$

Then $I_1 < C < S_1$.

A₁. If e is odd, take $b_2=21e+4$. We find that I_2 is termwise (as to coefficients of powers of e) less than $3C$, and that S_2 is termwise $>3C$. Taking $b_3=29e+6$, we find that $I_3 < 9C < S_3$.

A₂. If e is even, $e > 0$, take $b_2=21e+3$, $b_3=29e+7$. Since b_3 exceeds b_3 in Case A₁, evidently $S_3 > 9C$. Similarly, $I_2 < 3C$. Computation gives $S_2 > 3C$, $I_3 < 9C$, since $e \geq 2$.

B. For $\epsilon=6e-1$, take $b_1=14e-1$. Then

$$(33) \quad C = 24354e^4 - 10944e^3 + 1917e^2 - 150e + 5,$$

and $I_1 < C < S_1$.

B₁. If e is odd, take $b_2=21e-2$, $b_3=29e-2$. For every $e \geq 1$ computation gives $I_2 \leq 3C$, $S_3 \geq 9C$. See B₁₂.

B₂. If e is even, we accent the letters b, I, S . Take $b'_2=21e-3$, $b'_3=29e-1$. Computation gives $S'_2 \geq 3C$ if $e \geq 2$, $I'_3 \leq 9C$ for $e \geq 1$. See B₁₂.

B₁₂. Since $b'_2 < b_2$, $b_3 < b'_3$, we have $I'_2 < I_2$, $S'_2 < S_2$, $I_3 < I'_3$, $S_3 < S'_3$. The results proved in Cases B₁ and B₂ therefore imply $I'_2 < 3C$, $I_3 < 9C$, $S'_3 > 9C$, and also $S_2 > 3C$ if $e \geq 2$ (while $S_2 \geq 3C$ by the remark above our tablette). These inequalities together with those in B₁ and B₂ give all the inequalities (31).

By (29) and the square of (28) we see that $D_i > 8$ if 3^{2n} exceeds a certain function of e and i , and hence if n is sufficiently large.

If s_i is any given integer, Lemma 1 shows the existence of an integer m'_i such that

$$s_i = f(3m'_i) + 3^n M'_i, \quad 0 \leq m'_i < 3^n,$$

where M'_i is an integer. In (10) take $z=3m'_i$, $r=3^n y_i$. Since $\Delta \equiv 3r \pmod{\epsilon}$ and since Δ has the factor r and hence 3^n , we have $\Delta = 3^n E$. Since ϵ is prime to 3^n , we get $E \equiv 3y_i \pmod{\epsilon}$. Write

$$m_i = m'_i + 3^n y_i, \quad M_i = M'_i - E.$$

Then

$$f(3m_i) - f(3m'_i) = \Delta = 3^n E, \quad s_i = f(3m_i) + 3^n M_i.$$

* Actually the least odd b_1 when $e=1, 2, 3$, or 4 .

Since ϵ is prime to 3, we can choose integers y_i so that

$$M'_i - 3y_i - 6 - 3b_i \equiv 0 \pmod{\epsilon}, \quad 0 \leq y_i < \epsilon.$$

The last inequality shows that the maximum m_i is $3^n - 1 + 3^n(\epsilon - 1) = \epsilon \cdot 3^n - 1$. Hence $0 \leq m_i < \epsilon \cdot 3^n$, as desired in Lemma 3.

As before, we write $M_i = \epsilon 3^{2n} + N_i$. Thus

$$\begin{aligned} N_i &\equiv M'_i - 3y_i, \\ N_i - 6 - 3b_i &\equiv 0 \pmod{\epsilon}. \end{aligned}$$

It has been noted that the quantity in $\{ \}$ in (15) is even, whence $B_i \equiv 3b_i \pmod{\epsilon}$. Hence

$$N_i + \epsilon - 6 - B_i \equiv 0 \pmod{\epsilon}.$$

Hence (16) yields an integral value of Q_i .

We proved that there exist more than 8 consecutive integral values of v_i which satisfy inequalities (24). By (15),

$$2B_i \equiv 6b_i - 3b_i v_i (\eta - v_i) \pmod{8}, \quad \eta = 3b_i 3^n.$$

By Lemma 2, we can choose $v_i \pmod{8}$ so that $v_i(\eta - v_i) \equiv 2\zeta_i \pmod{8}$, where ζ_i is any assigned integer. Then

$$N_i + \epsilon - 6 - B_i \equiv N_i + \epsilon - 6 - 3b_i + 3b_i \zeta_i \pmod{4}.$$

We choose ζ_i so that the second member is $\equiv \epsilon \pmod{4}$. Then (16) gives $Q_i \equiv 1 \pmod{4}$.

Since inequalities (26) were satisfied at the outset, we know that $G_i \geq 0$ and that the final inequality (24) holds. Also (24₁) was shown to hold. Hence (17) hold. By (17₂), $0 \leq \epsilon Q_i \leq \epsilon 3^{2n}$. Since Q_i is an integer ≥ 0 such that $Q_i \equiv 1 \pmod{4}$, it is well known that $Q_i = \sum_{j=1}^3 x_j^2$, where the x_j are integers ≥ 0 . But $Q_i \leq 3^{2n}$, whence each $x_j \leq 3^n$.

In view of (17₁), the integer w_i defined by (14) is ≥ 0 . Then $f(v_i) + f(w_i) = \phi_i$ has the value $3^n B_i$. Hence (16) yields (13), which is the condition that s_i be the sum of the values of $f(x)$ for the nine values $3m_i, v_i, w_i, 3^n - x_j, 3^n + x_j$ of x , each an integer ≥ 0 .

7. PROOF OF THEOREM 2 WHEN $\epsilon = 2\delta$, δ PRIME TO 3

We shall determine the b_i and C to satisfy (26), viz., (31). For the following four values of δ , our later results do not apply:

| δ | b_1 | b_2 | b_3 | C |
|----------|-------|-------|-------|---------|
| 1 | 7 | 11 | 15 | 1025 |
| 2 | 11 | 17 | 25 | 7943 |
| 5 | 23 | 31 | 45 | 145900 |
| 8 | 37 | 55 | 77 | 1206699 |

In the preceding table, the b_i and C have their minimum values and all inequalities (31) hold.

J. When $\delta = 3d + 1$, $d \geq 1$, we take $b_1 = 14d + 5$ and get

$$(34) \quad C = 24354d^4 + 33795d^3 + 17591d^2 + 4082d + 358,$$

and find that $I_1 < C < S_1$.

J₁. If d is odd, take $b_2 = 21d + 8$, $b_3 = 29d + 12$. Then $S_2 \geq 3C$, $I_3 \leq 9C$.

J₂. If d is even, take $b_2 = 21d + 7$, $b_3 = 29d + 11$. Then $I_2 < 3C$, $S_3 > 9C$.

J₁₂. The remaining inequalities (31) follow from J₁ and J₂ as in B₁₂.

K. When $\delta = 3d - 1$, $d \geq 4$, take $b_1 = 14d - 5$. Then

$$(35) \quad C = 24354d^4 - 33795d^3 + 17591d^2 - 4058d + 350.$$

K₁. If d is even, take $b_2 = 21d - 9$, $b_3 = 29d - 9$. Then $I_2 \leq 3C$, $S_3 \geq 9C$ for $d \geq 1$.

K₂. If d is odd, take $b_2 = 21d - 10$, $b_3 = 29d - 8$. Then $S_2 \geq 3C$, $I_3 \leq 9C$ for $d \geq 4$.

K₁₂. Exactly as in B₁₂, the remaining inequalities (31) follow from K₁, K₂.

Let $\epsilon = 2^*E$, where E is odd and $e \geq 1$. By Lemma 2, we can choose $v_i \pmod{8}$ so that the number in $\{ \}$ in (15) is $\equiv 2z_i \pmod{8}$, where z_i is arbitrary, whence $B_i = 3b_i + b_i\epsilon(z_i + 4u_i)$, where u_i is an integer. Write $\zeta_i = b_iEz_i$, so that ζ_i has a preassigned residue modulo 4, and $B_i = 3b_i + 2^*(\zeta_i + 4b_iEu_i)$.

By choice of y_i , we made $N_i + \epsilon - 6 - 3b_i$ a multiple of ϵ , say $2^* \cdot q_i$. Then

$$N_i + \epsilon - 6 - B_i = 2^*F_i, \quad F_i = q_i - \zeta_i - 4b_iEu_i.$$

We may choose ζ_i so that $F_i \equiv E \pmod{4}$. Then by (16), $\epsilon Q_i \equiv 2^*E \pmod{4 \cdot 2^*}$, whence $Q_i \equiv 1 \pmod{4}$. But Q_i is an integer ≥ 0 . Hence Q_i is a sum of three squares.

8. REDUCTION OF D_i FROM 8 TO 6

The lower limit ν of n obtained from $D_i \geq 8$ may be reduced in certain cases by using $D_i \geq 6$.

First, it suffices to have $D_i \geq 7$. Then (24) holds for 7 consecutive integral values of v_i . If f is the first of them, then the seven together with $f-1$ form

a complete set of residues modulo 8. Two values of v whose sum is $\eta = 3b_3 3^a$ yield the same value of $P = v(\eta - v)$. The value of P which is apparently lacking in view of the missing value $f-1$ of v is actually obtained from the value $u = \eta - f + 1$ of v , and $u \not\equiv f-1$ since $\eta \not\equiv 2f-2 \pmod{8}$, η being odd.

Second, it suffices to have $D_4 \geq 6$. Let f be the first of six consecutive integral values of v . Then the six together with $f-1$ and $f-2$ form a complete set of residues modulo 8. We saw that the two values $f-1$ and $u = \eta - f + 1$ of v give the same value of P ; likewise $f-2$ and $w = \eta - f + 2$. Each of $u \equiv f-2$ and $w \equiv f-1 \pmod{8}$ reduces to

$$(36) \quad 2f - 3 \equiv \eta \pmod{8}.$$

Hence if (36) does not hold, we may employ u and w instead of the missing values $f-1$ and $f-2$ of v , as well as the four residues which together with these four make a complete set of residues modulo 8, and obtain all four even residues of 8 as values of P .

When (36) holds, we modify our proof as follows: We no longer secure the value $P' \equiv (f-1)u \equiv (f-2)w \equiv (f-1)(f-2) \pmod{8}$ of P . But when v ranges over six incongruent residues, no one congruent to $f-1$ or $f-2$ modulo 8, we obtain three incongruent even values of P (viz., all except P'). In the notations at the end of §6, we may assign to ζ_i any one of three values incongruent modulo 4 (viz., any except $\frac{1}{2}P'$), and hence choose ζ_i so that $\epsilon Q_i \equiv \epsilon$ or $2\epsilon \pmod{4}$. Then $Q_i \equiv 1$ or $2 \pmod{4}$ and Q_i is a sum of three squares. Similarly, in the notations at the end of §7, we secure $F_i \equiv E$ or $2E \pmod{4}$, whence $Q_i \equiv 1$ or $2 \pmod{4}$.

9. UNIVERSAL THEOREM $\epsilon = 2$

When $\epsilon = 2$, $\delta = 1$, and we saw that (26) are satisfied if $b_1 = 7$, $b_2 = 11$, $b_3 = 15$, $C = 1025$. The older condition $D_3 \geq 8$ requires $n \geq 9$. The condition $D_3 \geq 7$ barely fails to reduce n from 9 to 8. The best condition $D_3 \geq 6$ holds if $n \geq 8$. Then $D_2 > 14$, $D_1 > 40$. Hence Theorem 2 shows that every integer $\geq 1025 \cdot 3^{24}$ is a sum of nine values of

$$f(x) = x + \frac{1}{3}(x^3 - x) = \frac{1}{3}(x^3 + 2x)$$

for integral values ≥ 0 of x . We employ

THEOREM 3. *Let a polynomial $f(x)$ take integral values ≥ 0 for all integers $x \geq 0$; let $f(x+1) - f(x)$ increase with x . Suppose that every integer n for which $l < n \leq g + f(0)$ is a sum of $k-1$ values of $f(x)$ for integers $x \geq 0$. Let m be the maximum integer for which $f(m+1) - f(m) < g - l$. Then every integer N for which $l + f(0) < N \leq g + f(m+1)$ is a sum of k values of $f(x)$ for integers $x \geq 0$.*

Tables were made showing all integers 1-3000 which are sums of 2, 3, 4, or 5 values of $f(x)$. In particular, all integers 1-3000 except only 42 and 66 are sums of 5 values. This result was proved to hold also for 3000-4000 as follows. A list was made of the integers 2076-3000 which are missing from the table of sums by 4. To this list we added $924=f(14)$ and subtracted $1135=f(15)$ from the sums (actually we subtracted $1135-924=211$ from our list), and noted whether or not each difference is in the table of sums by 4. If not, we subtracted other values of $f(x)$ from the sum.

We may apply Theorem 3 to $f(x)=\frac{1}{3}(x^3+2x)$ since $f(x+1)-f(x)=x^2+x+1$ increases with x . As just proved by tables, every integer n for which $66 < n \leq 4000$ is a sum of 5 values of $f(x)$. The maximum integer m for which

$$(2m+1)^2 < 4(4000-66)-3 = 15733$$

has $2m+1=125$, $m=62$. Since $f(63)=83391$, Theorem 3 shows that all integers from 67 to 87391 are sums of 6 values of $f(x)$. The same is true of 42 and 66. Hence we may apply Theorem 3 with $l=0$, $g=87391$, $k=7$; the maximum m is 295; hence all integers $\leq g+f(296)=8732367$ are sums of 7 values of $f(x)$. The next maximum m is 2954, and all integers ≤ 8609777000 are sums of 8 values of $f(x)$. The next maximum m is 97788, and all integers $\leq 311716 \times 10^9$ are sums of 9 values of $f(x)$. This product exceeds $1025 \cdot 3^{24} = 2894902749 \times 10^5$. This proves

THEOREM 4. *Every integer ≥ 0 is a sum of 9 values of $\frac{1}{3}(x^3+2x)$ for integers $x \geq 0$.*

Besides the functions treated, there arose in Theorem 1 the special pyramidal function $P(x)$. For this case, K. C. Yang proved in his Chicago doctoral dissertation of 1928

THEOREM 5. *Every integer ≥ 0 is a sum of nine pyramidal numbers $\frac{1}{6}(x^3-x)$ for integers $x \geq 0$.*

He verified that every integer < 7240 is a sum of five values.

I have not completed the proof that every integer is a sum of nine values of function (9) for $\epsilon=1$.

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ON THE CONVERGENCE AND OVERCONVERGENCE OF SEQUENCES OF POLYNOMIALS OF BEST SIMULTANEOUS APPROXIMATION TO SEVERAL FUNCTIONS ANALYTIC IN DISTINCT REGIONS*

BY

J. L. WALSH AND HELEN G. RUSSELL

1. Introduction. The purpose of this paper is to present some theorems on the convergence and overconvergence of sequences of polynomials of best approximation to a function $f(z)$ analytic on a closed limited point set whose complement is of multiple (finite or infinite) connectivity. Our main theorem is the following:

THEOREM I. *Let M be an arbitrary closed limited point set of the z -plane whose complement K is connected and possesses a Green's function with pole at infinity.† Let $w = \omega(z)$ be a function which maps K (conformally but not necessarily uniformly) onto the exterior of the unit circle in the w -plane so that the points at infinity in the two planes correspond to each other. Let C_R denote the transform (i.e., in K) of $|w| = R$, $R > 1$, under the mapping function $w = \omega(z)$.*

(1) *If the function $f(z)$ is analytic and single-valued on and within C_R , there exist polynomials $P_n(z)$ of respective degrees n^\ddagger , $n = 1, 2, \dots$, such that the inequalities*

$$(a) \quad |f(z) - P_n(z)| \leq N/R^n, \quad z \text{ on } M, R > 1,$$

where N is dependent on R but not on n or z , are valid for every z on M .

(2) *If there exist polynomials $P_n(z)$ of degree n , $n = 1, 2, \dots$, such that (a) is valid for every z on M , then the sequence $\{P_n(z)\}$ converges interior to C_R , uniformly on any closed point set interior to C_R , and thus $f(z)$ is analytic§ throughout the interior of C_R .*

* Presented to the Society, October 29, 1932; received by the editors February 27, 1933.

† The requirement that K should possess a Green's function is equivalent to the requirement that K should be regular, in the sense that the Dirichlet problem (for arbitrary continuous boundary values) can be solved for K . See Kellogg, *Proceedings of the National Academy of Sciences*, vol. 12 (1926), pp. 397-406.

‡ A polynomial of degree n in z is any expression of the form $a_0 + a_1z + a_2z^2 + \dots + a_nz^n$.

§ If $f(z)$ is not originally assumed to be defined on the entire point set considered, then the definition in the new points is to be made by analytic extension interior to C_R , or, what amounts to the same thing, by means of the convergent sequence of polynomials.

The Green's function $G(x, y)$ with pole at infinity for the region K is (1) harmonic in K except at infinity where $G(x, y) = \log r + G_1(x, y)$, $r = (x^2 + y^2)^{1/2}$, and $G_1(x, y)$ is harmonic at infinity, and (2) $G(x, y)$ is continuous and vanishes on the boundary of K .

It will be noticed that the hypothesis on the point set M is satisfied provided M is closed, limited, without isolated points, and provided K is connected and of finite connectivity.

This theorem is known for the case that the complement of M is simply connected and that M is not a single point. The second part of the theorem for that case is due to Walsh and the formulation of the entire theorem together with detailed references was published by him.* Among the writers to whom various parts of the theorem are due are Faber, S. Bernstein, M. Riesz, Fejér, and Szegő; the theorem for the case that M is a segment of the axis of reals is due to Bernstein. The generalization to be proved here is made possible by the consideration of the equipotential curves for the infinite region K and of approximation by them to the boundary of K , by the approximation to analytic curves by lemniscates, and finally by the use of a sequence of polynomials found by interpolation.

By means of Theorem I we shall derive some results on convergence and overconvergence,—results which are generalizations of results already established by Walsh† in the less general case mentioned. We study also the convergence of sequences of polynomials of best approximation, where *best approximation* is measured (1) in the sense of Tchebycheff, (2) by line integrals taken over rectifiable Jordan curves bounding the point set considered, (3) by surface integrals taken over the region considered. The two latter methods of approximation yield interesting results in regard to polynomials *belonging to a point set*, a problem which has been considered in the case of a simply connected region by Faber, Fejér, Szegő, Bergmann, Bochner, Carleman, and Smirnov.‡ All three methods of approximation yield results on the exact region of uniform convergence of the sequence of polynomials of best approximation and show that this region depends not merely on the singularities of the given function $f(z)$ but also on the monogenic character of the function $f(z)$.

The term *overconvergence* is used in the sense of Walsh to denote that if a sequence of polynomials converges sufficiently rapidly on a point set M of the kind described, then that sequence necessarily converges also on a cer-

* Münchner Berichte, 1926, pp. 223–229.

† These Transactions, vol. 32 (1930), pp. 794–816; these Transactions, vol. 33 (1931), pp. 370–388. We shall refer to these papers as (1) and (2) respectively.

‡ Detailed references are given below.

tain larger point set containing M in its interior.

2. Approximation by analytic curves to the boundary of a given point set. We shall prove several lemmas.

LEMMA I. *Let M be a closed limited point set of the z -plane whose complement K is connected and possesses a Green's function $G(x, y)$ with pole at infinity. Then $w = \omega(z) = e^{G+iH}$, where H is conjugate to G , is a function which maps K onto the exterior of the unit circle in the w -plane so that the points at infinity in the two planes correspond.*

The equipotential lines, $G=c$, $c>0$, take the following forms: (1) the locus $G=c$ consists of a finite number of simple analytic closed curves, mutually exterior, bounding an infinite region T of points $G>c$; or (2) the locus $G=c$ consists of a finite number of mutually exterior closed curves, at least one of which has a multiple point of order $m \geq 2$, bounding an infinite region T of points $G>c$.

Consider the set of points $T: G>c$, in which we count the point at infinity. Because G is continuous in K except at infinity, the boundary points of T all belong to the equipotential $G=c$. Conversely, all points of $G=c$ are boundary points of T . If not, then in a neighborhood of a point P of $G=c$ which is not a boundary point of T , we have only points $G \leq c$. Since G is harmonic in this neighborhood, by Gauss' mean-value theorem G equals c on the circumference of a sufficiently small circle about P , and we have a contradiction.

The set T is a region, that is, every point of the set $G>c$ is an interior point of the set, and any two points of the set can be connected by a Jordan arc all of whose points belong to the set. Otherwise, a region T_1 belonging to the set T exists not including the point at infinity and having $G=c$ on its entire boundary. Since G is harmonic in T_1 , G is identically equal to c in T_1 , which leads to a contradiction.

The locus $G=c$, $c>0$, consists of analytic arcs which fall into a set of closed curves; otherwise the continuity hypothesis is contradicted.*

The locus $G=c$, $c>0$, consists of a finite number of curves. If M is bounded by a finite number of mutually exclusive closed point sets, the statement follows at once from the facts that any curve on which $G=c>0$ contains in its interior points of M and no two loci $G=0$ and $G=c>0$ have a common point. If M is bounded by an infinite number of mutually exclusive point sets, assume the curves $G=c: C^{(1)}, C^{(2)}, \dots$ to be infinite in number and consider a point P_1 on $C^{(1)}$, P_2 on $C^{(2)}$, \dots . Since $G=c$ is a closed limited point set, these points must have a limit point P on $G=c$. If P is not a point at which the gradient of G vanishes, the curve $G=c$ through P is a single

* See for instance Kellogg, *Foundations of Potential Theory*, Berlin, 1929, pp. 273-277.

analytic piece, as the theorem on implicit functions shows.* If P is a point at which the gradient vanishes, it is not a limit point of points at which the gradient vanishes, for such points can occur only on the boundary of K , as is evident from consideration of the derivative of the analytic function $f(z)$ of which G is the real part.† If P is a point at which the gradient of G vanishes, the analytic arcs of which $G=c$ consists in the neighborhood of P are finite in number and they pass through the point P with equally spaced tangents‡, so P cannot be a limit point of points on $C^{(1)}, C^{(2)}, \dots$. Hence we have reached a contradiction; and the statement that any locus $G=c, c>0$, consists of a finite number of curves is true.

The locus $G=c, c>0$, consists either entirely of mutually exterior simple curves, or of mutually exterior curves some of which may be simple but at least one of which, C' , has a multiple point of order $m, m \geq 2$; and C' contains in its interior (i.e. the finite regions bounded by C') at least m mutually exclusive closed sets belonging to M . The proof is similar to that already given and is left to the reader.

If the region K is of connectivity greater than unity, there is at least one value of c for which the locus $G=c$ contains a curve with a multiple point.

The number of curves of which the locus $G=c$ is composed increases monotonically (if at all) as c decreases. The locus $G=c$ consists of a finite number of mutually exterior simple curves, except for a countable set of values of c .

Out of Lemma I follows, as the reader will easily verify,

LEMMA II. *Under the hypotheses of Lemma I, the point sets bounding the infinite region K can be approximated by finite sets of mutually exterior analytic curves $G=c$. More explicitly, the equipotential loci $J^{(i)}: G=c_i, i=1, 2, \dots, c_1 > c_2 > c_3 > \dots \rightarrow 0$, lie in K and are such that the region interior to $J^{(i+1)}$ is contained in the region interior to $J^{(i)}$, $J^{(i)}$ and $J^{(i+1)}$ have no common points, and every point in K lies exterior to some $J^{(i)}$. If the c_i are suitably chosen, each $J^{(i)}$ consists of a finite number of mutually exterior analytic simple curves.*

3. Approximation to several analytic curves by a lemniscate. The locus of a point the product of whose distances from m fixed points is constant is a lemniscate. Thus, if the given points are a_1, a_2, \dots, a_m , the lemniscate is defined by the equation $|P(z)| = c$, where $P(z) = (z-a_1)(z-a_2) \dots (z-a_m)$. For $m=1$, the lemniscate is a circle; for $m=2$, the lemniscate is a Cassinian oval. We note that $|P(z)| = 0$ consists of the points $z=a_i, i=1, 2, \dots, m$;

* Osgood, *Lehrbuch der Funktionentheorie*, vol. I, Leipzig, 1923, p. 675.

† The proof follows that of Kellogg in the case that K is simply connected: loc. cit., pp. 364-365.

‡ See for instance Kellogg, loc. cit., p. 275.

and, in the general case, since $G = \log [|P(z)|/|c|]^{1/m}$ is Green's function with pole at infinity for the region exterior to $|P(z)| = c \neq 0$, the curves $G = \log \epsilon$, $\epsilon > 1$, or $|P(z)| = c\epsilon^m$, for ϵ sufficiently near unity and c sufficiently near zero, are m simple closed analytic curves, each containing one root a_i , $i = 1, 2, \dots, m$, of $P(z) = 0$, if the a_i are all distinct.

The possibility of approximation of analytic curves by lemniscates is the basis of our proof of Theorem I, and Theorem I is the source of all succeeding results in this paper.

LEMMA III. *A finite number k of arbitrary mutually exterior closed analytic curves can be approximated by the same lemniscate; that is to say, given a set C consisting of k mutually exterior closed analytic curves C_j , $j = 1, 2, \dots, k$, and a number $\eta > 0$ such that the η -neighborhoods of C_j are distinct, a lemniscate $\Gamma: |(z-a_1)(z-a_2)\dots(z-a_m)| = c$ exists which lies exterior to C and interior to these η -neighborhoods, and contains C in its interior.**

Let C'_j , $j = 1, 2, \dots, k$, be k curves constructed as follows:

- (1) The curve C'_j is contained in the region swept out by a circle of radius η whose center describes C_j .
- (2) The curve C'_j contains in its interior one and only one of the given curves, say C_j , and lies exterior to C .
- (3) The curves C'_j lie exterior to one another.

Let $s(\zeta)$ measure arc length on the curves C_j whose lengths are d_j , $j = 1, 2, \dots, k$. For $0 \leq s(\zeta) \leq d_1$, ζ shall lie on C_1 ; for $d_1 < s(\zeta) \leq d_1 + d_2$, ζ shall lie on C_2 ; \dots ; for $\sum_{j=1}^{k-1} d_j < s(\zeta) \leq \sum_{j=1}^k d_j$, ζ shall lie on C_k .

Green's function $G(x, y)$ exists† for the region exterior to C : (1) $G(x, y)$ is harmonic exterior to C except at infinity where $G(x, y) = \log r + G_0(x, y)$, and $G_0(x, y)$ is harmonic at infinity and has the value $-\mu$ at infinity, and (2) $G(x, y)$ is continuous and vanishes on C . We define a function $V(x, y)$ so that $V(x, y) = G(x, y) + \mu$; and we now prove that there exists a continuous positive function

$$\phi(s) = \frac{1}{2\pi} \frac{\partial V(x, y)}{\partial n}$$

(n is the exterior normal for C) such that

* This lemma was proved by Hilbert in the case of approximation to one analytic curve and applied to approximation of analytic functions by polynomials: Göttinger Nachrichten, 1897, pp. 63-70.

Simultaneous approximation of several distinct curves by lemniscates has also been used by other writers, especially Faber, Szegő, Fekete, and Pólya, in connection with the approximation of functions by polynomials and related topics, but without detailed proof of the results of the present paper. See particularly Faber, *Münchener Berichte*, 1922, pp. 157-178, and for further references Pólya and Szegő, *Crelle's Journal*, vol. 165 (1931), pp. 4-49.

† Osgood, loc. cit., pp. 687-703.

$$V(x, y) = \int_C \phi(s) \log r \, ds,$$

where now $r = |z - \zeta|$, $ds = |d\zeta|$, and $z = x + iy$ is any point of the z -plane exterior to C .

By a familiar theorem of potential theory, a function $G(x, y)$ which is (1) harmonic in the region S which is bounded by C and a circle C_0 (with center $P: (x, y)$ exterior to C) containing C in its interior, and (2) continuous together with its partial derivatives of the first order on the boundary of S , satisfies the following equation:

$$(a) \quad G(x, y) = \frac{1}{2\pi} \int_{C_0} \left(\log r \frac{\partial G}{\partial n} - G \frac{\partial \log r}{\partial n} \right) ds + \frac{1}{2\pi} \int_C \left(\log r \frac{\partial G}{\partial n} - G \frac{\partial \log r}{\partial n} \right) ds.$$

Here r denotes distance from $P: (x, y)$ and n denotes interior normal with respect to S .

If we use the Green's function $G(x, y) = \log r + G_1(x, y)$, we have from (a)

$$(b) \quad G(x, y) = -\mu + \frac{1}{2\pi} \int_C \log r \frac{\partial G}{\partial n} \, ds,$$

for we have

$$G(x, y) = \frac{1}{2\pi} \int_{C_0} \left(\log r \frac{\partial G_1}{\partial n} - G_1 \frac{\partial \log r}{\partial n} \right) ds + \frac{1}{2\pi} \int_C \log r \frac{\partial G}{\partial n} \, ds, \\ \int_{C_0} \log r \frac{\partial G_1}{\partial n} \, ds = 0, \quad \frac{1}{2\pi} \int_{C_0} G_1 \frac{\partial \log r}{\partial n} \, ds = \frac{1}{2\pi} \int_{C_0} G_0 \frac{\partial \log r}{\partial n} \, ds = \mu.$$

Consequently,

$$G(x, y) + \mu = \frac{1}{2\pi} \int_C \log r \frac{\partial (G + \mu)}{\partial n} \, ds,$$

and

$$V(x, y) = \frac{1}{2\pi} \int_C \log r \frac{\partial V}{\partial n} \, ds.$$

The function $\partial V / \partial n$ is continuous on C since $\partial G / \partial n$ is continuous on C , and $\partial V / \partial n$ is positive since $V(x, y)$ is harmonic exterior to C except at infinity where it is logarithmically infinite. Hence

$$\frac{1}{2\pi} \frac{\partial V}{\partial n} = \phi(s)$$

is the function desired.

Since $V(x, y)$ is harmonic exterior to C except at infinity, $V(x, y)$ takes on all the C'_j a minimum value $\mu_1 > \mu$. We now choose $\epsilon > 0$ such that $\epsilon < (\mu_1 - \mu)/2$. Since $V(x, y) - \mu$ is Green's function for the region exterior to C we may apply Lemma I. If $V = \mu + \epsilon$ is a locus consisting of curves not all of which are simple, some curve of the locus must intersect a C'_j . Since $\mu + \epsilon < \mu_1$, and μ_1 is the minimum value of V on C'_j , the curve cannot cut C'_j . Hence $V = \mu + \epsilon$ consists of a simple closed curve γ_1 in the ring $C_1C'_1$, a simple closed curve γ_2 in the ring $C_2C'_2$, \dots , a simple closed curve γ_k in the ring $C_kC'_k$.

By similar reasoning, $V = \mu_1 - \epsilon$ consists of a simple closed curve γ'_1 in the ring $C_1C'_1$, a simple closed curve γ'_2 in the ring $C_2C'_2$, \dots , a simple closed curve γ'_k in the ring $C_kC'_k$.

We denote by $\gamma_j\gamma'_j$ the ring region bounded by γ_j and γ'_j . We let

$$u_0 = \int_0^{d_1+d_2+\dots+d_k} \phi(s) ds;$$

and we make the change of variable

$$u(s) = \int_0^{s(s)} \phi(s) ds,$$

so that u increases from 0 to u_0 as s increases from 0 to $\sum_{j=1}^k d_j$. Then

$$V(x, y) = \int_0^{u_0} \log r \, du,$$

and

$$V(x, y) = \lim_{n \rightarrow \infty} (u_0/n)(\log r_1 + \log r_2 + \dots + \log r_n),$$

where r_1, r_2, \dots, r_n are distances from z to n points of C_j corresponding to equidistant values $u_1 = u_0/n, u_2 = 2u_0/n, \dots, u_n = u_0$ of u . For simplicity, the dependence of r_i on n is not indicated in the notation.

For z interior to $\gamma_j\gamma'_j$, convergence of the sequence of functions $u_0 \log r_1, (u_0/2) (\log r_1 r_2), \dots, (u_0/N) (\log r_1 r_2 \dots r_n), \dots$, to $V(x, y)$ is uniform.* For n sufficiently large, $n \geq N$, and z interior to $\gamma_j\gamma'_j$, we have

$$-\epsilon' < V(x, y) - (u_0/n)(\log r_1 r_2 \dots r_n) < \epsilon'.$$

* The detailed proof of uniformity offers no difficulty. See for instance Walsh, Bulletin of the American Mathematical Society, vol. 35 (1929), pp. 499-544; Lemma, p. 538.

If we denote by Γ the locus exterior to $C: (u_0/N) (\log r_1 r_2 \cdots r_N) = \lambda$, where $\mu + \epsilon < \lambda < \mu_1 - \epsilon$, and choose ϵ' sufficiently small, we have, on Γ , $\mu + \epsilon < \lambda - \epsilon' < V(x, y) < \lambda + \epsilon' < \mu_1 - \epsilon$. Every Jordan arc joining a point of γ_j to a point of γ_j' must cut Γ . The locus Γ has the following properties:

(1) Γ consists of a curve $\Gamma^{(1)}$ enclosing γ_1 , a curve $\Gamma^{(2)}$ enclosing γ_2, \dots , and a curve $\Gamma^{(k)}$ enclosing γ_k . Otherwise there would exist a region in some $\gamma_j \gamma_j'$ in which the harmonic function $(u_0/N) (\log r_1 r_2 \cdots r_N)$ constant on Γ would be identically constant, which is impossible.

(2) Γ lies in the rings $\gamma_j \gamma_j'$, since λ is such that $\mu + \epsilon < \lambda - \epsilon' < V(x, y) < \lambda + \epsilon' < \mu_1 - \epsilon$.

(3) Γ is a lemniscate, for the equation

$$r_1 r_2 \cdots r_N = e^{N\lambda/u_0}$$

is of the form $|P(z)| = c > 0$.

The proof of Lemma III is now complete.

If we choose η successively $1, 1/2, \dots, 1/n, \dots$, we have lemniscates $\Gamma_1, \Gamma_2, \dots, \Gamma_n, \dots$, exterior to C . From this set can be extracted a subset such that (1) each Γ_{i+1} is interior to Γ_i , (2) Γ_i and Γ_{i+1} have no common point, and (3) every point exterior to C lies exterior to some Γ_i .

4. Lemmas involving the mapping function $\omega(z)$. We prove the following lemmas:

LEMMA IV. Under the hypotheses of Lemma I, a multiple point of order m of the curves $G=c$, $c>0$, occurs at $(x, y) = (x', y')$ if and only if $\omega'(z') = 0$, $\omega''(z') = 0, \dots, \omega^{(m-1)}(z') = 0, \omega^{(m)}(z') \neq 0, z' = (x', y')$, $m \geq 2$, simultaneously, that is, when and only when $z = z'$ is a branch point of the inverse of the mapping function $\omega(z) = e^{G+iH}$.

The proof of this lemma is essentially included in Lemma I.

LEMMA V. Let $M, K, \omega(z)$ be defined as in Theorem I. Let C_R denote* the transform in the z -plane of $|w| = R$, $R > 1$, under the mapping function $\omega(z)$. Let ρ be arbitrary, $1 < \rho < R$. Then there exists a lemniscate (of Lemma III) $\Gamma: |(z-a_1) \cdots (z-a_m)| = c$ such that Γ contains M in its interior and such that $\Gamma_{R/\rho}: |(z-a_1) \cdots (z-a_m)| = cR^m/\rho^m$ lies interior to C_R . Thus for z on and within Γ (hence on M) and for t on or exterior to $\Gamma_{R/\rho}$, we have

$$\left| \frac{(z-a_1) \cdots (z-a_m)}{(t-a_1) \cdots (t-a_m)} \right| \leq \frac{\rho^m}{R^m}.$$

* A symbol of the form C_R denotes henceforth the transform in the z -plane of $|w| = R$, $R > 1$, under the mapping function $\omega(z)$.

Let Γ be a lemniscate contained in K and lying interior to C_ρ . Then the locus $\Gamma_{R/\rho}$ lies interior to $[C_\rho]_{R/\rho} = C_R$, as follows from the study of the Green's functions for the exterior of C_ρ and the exterior of Γ . If these Green's functions are denoted by G_1 and G_2 respectively, their difference $G_1 - G_2$ is negative on C_ρ , hence harmonic and negative exterior to C_ρ even at infinity. Then on $\Gamma_{R/\rho}$ we have $G_1 - \log R/\rho < 0$, so on $\Gamma_{R/\rho}$ we have $G_1 < \log R/\rho$; the curve $C_R: G_1 = \log R/\rho$ lies exterior to $\Gamma_{R/\rho}$.

LEMMA VI. Let $M, K, \omega(z)$ be defined as in Theorem I. If $Q(z)$ is a polynomial of degree n such that $|Q(z)| \leq L, z$ on M , then

$$|Q(z)| \leq LR_0^n, z \text{ on and within } C_{R_0}, R_0 > 1.$$

The special case in which M is a line segment is due to S. Bernstein,* and the method used in proving Lemma VI is a generalization of the method of M. Riesz† for this special case. This lemma was proved by Walsh‡ in case K is simply connected and the possibility of its extension to the more general case was indicated by him. See also Faber (loc. cit.), who proves Lemma VI for a set M bounded by a finite number of Jordan curves.

5. Approximation to an analytic function. We proceed now to the proof of Theorem I.

We first prove (1). Since $f(z)$ is analytic on and within C_R , the function $f(z)$ is also analytic on and within some $C_{R'}$, $R' > R$. Choose the present R' as the R of Lemma V and the present ratio R'/R as the quantity ρ of Lemma V. Then by Lemma V there exists a lemniscate $\Gamma: |(z-a_1) \cdots (z-a_m)| = c$ containing M in its interior, while $\Gamma_R: |(z-a_1) \cdots (z-a_m)| = cR^m$ is interior to $C_{R'}$; for z on and within Γ (in particular on M) and for t on or exterior to Γ_R (in particular on $C_{R'}$), we have

$$\left| \frac{(z-a_1) \cdots (z-a_m)}{(t-a_1) \cdots (t-a_m)} \right| \leq \frac{1}{R^m}.$$

A unique polynomial $P_{mp-1}(z)$ of degree $mp-1$, $p=1, 2, \dots$, exists with the properties

$$P_{mp-1}^{(i)}(a_j) = f^{(i)}(a_j) \quad (i=0, 1, 2, \dots, p-1; j=1, 2, \dots, m). \S$$

* Mémoires de l'Académie Royale de Belgique, Classe des Sciences, (2), vol. 4 (1912), pp. 36-94.

† Acta Mathematica, vol. 40 (1916), pp. 337-347.

‡ Münchner Berichte, loc. cit., p. 225.

§ Hilbert (loc. cit.) has exhibited such polynomials in the case of approximation in a simply connected region. See also Jacobi, Crelle's Journal, vol. 53 (1856-57), pp. 103-126; and Montel, *Leçons sur les Séries de Polynômes*, Paris, 1910, pp. 47-49, 95-97.

Two distinct polynomials $P_{mp-1}(z)$ of degree $mp-1$ surely cannot satisfy these conditions, for their difference would have at least mp roots. We actually exhibit the polynomial $P_{mp-1}(z)$ (necessarily unique):

$$P_{mp-1}(z) = f(z) - \frac{1}{2\pi i} \int_{C_R} \frac{f(t)}{t-z} \left[\frac{(z-a_1) \cdots (z-a_m)}{(t-a_1) \cdots (t-a_m)} \right]^p dt, \quad z \text{ interior to } C_R.$$

Indeed, it is clear by inspection that $P_{mp-1}(z)$ thus defined satisfies the conditions on interpolation to $f(z)$, since this equation is valid for $z=a_j$. Moreover, if $f(z)$ is expressed by Cauchy's integral (which may be taken over the whole of C_R even if C_R consists of several curves):

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(t)}{t-z} dt, \quad z \text{ interior to } C_R,$$

substitution in the previous equation leads to an integrand which has no singularity in z and which is a polynomial in z of degree $mp-1$, so the function $P_{mp-1}(z)$ is seen to be a polynomial of degree $mp-1$.

For z on M , we have

$$|f(z) - P_{mp-1}(z)| \leq \frac{1}{2\pi} \int_{C_R} \frac{|f(t)|}{|t-z|} \left| \frac{(z-a_1) \cdots (z-a_m)}{(t-a_1) \cdots (t-a_m)} \right|^p |dt|.$$

Since $f(z)$ is analytic on and within C_R , there follow the inequalities $|f(t)| \leq N''$; $|(z-a_1) \cdots (z-a_m)| / |(t-a_1) \cdots (t-a_m)| \leq 1/R^m$, and $1/|t-z| \leq 1/\delta$, z on M , t on C_R . Set

$$\int_{C_R} |dt| = L;$$

we have

$$\begin{aligned} |f(z) - P_{mp-1}(z)| &\leq \frac{1}{2\pi} \cdot \frac{1}{R^{mp}} \cdot \frac{N''L}{\delta} \\ &\leq \frac{N'}{R^{mp}}, \quad z \text{ on } M, \end{aligned}$$

where N' is independent of p and z .

The polynomial $P_n(z)$ of degree n , $n=1, 2, \dots$, already defined when n is of the form $mp-1$, is now defined for arbitrary n by the equation

$$P_n(z) = P_{mp-1}(z), \quad mp-1 \leq n < m(p+1)-1.$$

Then we have the inequality

$$|f(z) - P_n(z)| \leq \frac{N'}{R^{n-m+2}} \leq \frac{N}{R^n}, \quad z \text{ on } M,$$

where $N = N'R^{m-2}$, and where N is independent of n and z . The proof of the first part of the theorem is complete.

The proof of the second part of the theorem is the analog of the corresponding proof given by Walsh in the case of a closed limited point set whose complement is simply connected,* and is a direct application of Lemma VI.

The following theorem is simpler but less explicit than Theorem I:

THEOREM. *Let M be an arbitrary closed limited point set whose complement K is connected and possesses a Green's function with pole at infinity. A necessary and sufficient condition that $f(z)$ be analytic on M is that there exist polynomials $P_n(z)$ of respective degrees n such that the inequality*

$$(a) \quad |f(z) - P_n(z)| \leq \frac{N}{R^n}, \quad R > 1,$$

N not dependent on n or z , is valid for every z on M .

The function $f(z)$ of Theorem I is not necessarily a monogenic analytic function; in other words, if we consider the functions defined on various separated pieces of M , the hypotheses of the theorem may well be satisfied where $f(z)$ is not a monogenic analytic function.

Theorem II shows the best degree of approximation (measured like the convergence of a geometric series) possible for a sequence of polynomials $\{P_n(z)\}$:

THEOREM II. *Let M , K , $\omega(z)$ be defined as in Theorem I, and let $f(z)$ be analytic on M . Let R be the largest number for which the following is true: (1) a function $F(z)$ is analytic and single-valued interior to C_R , (2) $F(z) \equiv f(z)$ on M . Then there exists a sequence of polynomials $\{P_n(z)\}$ of respective degrees n , $n=1, 2, \dots$, such that*

$$|f(z) - P_n(z)| \leq \frac{N}{R_0^n}, \quad z \text{ on } M, \quad R_0 \text{ arbitrary} < R,$$

N dependent on R_0 but not on n or z ; but for no sequence of polynomials $\{P_n(z)\}$ do we have

$$|f(z) - P_n(z)| \leq \frac{N}{R_1^n}, \quad z \text{ on } M, \quad R_1 > R,$$

N dependent on R_1 but not on n or z .

* Münchner Berichte, loc. cit., p. 226.

The number R defined by (1) and (2), finite or infinite, exists; the formal proof is left to the reader.

The curve C_R is characterized by the fact that the function $f(z)$ (when suitably extended analytically from M along paths interior to C_R) is analytic and single-valued interior to C_R , but is not analytic or is not single-valued or fails in both particulars interior to every $C_{R'}$, $R' > R$, when extended from M along paths interior to $C_{R'}$. Thus, (a) at some point P of C_R the function $f(z)$ has a singularity for analytic extensions from M along paths interior to C_R terminating in P ; or (b) the curve C_R has at least one multiple point Q , and there is disagreement at Q among the various analytic extensions of $f(z)$ from the various parts of M to Q along paths belonging to the several regions interior to and bounded by C_R ; or (c) both (a) and (b) occur.

As an illustration of (b) let the point set M be the closed interior of the lemniscate $|z^2 - 1| = c$, $c < 1$, and let $f(z) = 1$ in the oval to the right of the origin, and $f(z) = -1$ in the oval to the left of the origin. Then C_R is the lemniscate $|z^2 - 1| = 1$. As an illustration of (c) let $f(z) = 1/(z - 2^{1/2})$ and $1/(z + 2^{1/2})$ in the right and left ovals of the point set M above. Then C_R is again the lemniscate with double point $|z^2 - 1| = 1$.

The first statement of Theorem II has been proved in Theorem I, although the polynomials there exhibited depend on R_0 ; this restriction does not appear for the polynomials of Theorem III below. We shall prove the second statement.

Assume that polynomials $P_n(z)$ of degree n , $n = 1, 2, \dots$, exist such that

$$|f(z) - P_n(z)| \leq \frac{N}{R_1^n}, \quad z \text{ on } M, R_1 > R,$$

N independent of n and z . By Theorem I, the sequence $\{P_n(z)\}$ converges to an analytic function $F(z)$ within C_{R_1} . Then $F(z)$ is analytic interior to C_{R_1} and $F(z) \equiv f(z)$ on M , where R_1 is greater than R , contrary to hypothesis.

6. **The Tchebycheff polynomial.** The Tchebycheff polynomial of degree n for approximation to $f(z)$ on M is the polynomial $\Pi_n(z)$ of degree n such that

$$\max |f(z) - \Pi_n(z)|, \quad z \text{ on } M,$$

is not greater than the corresponding expression for any other polynomial of degree n . The Tchebycheff polynomial exists and is unique,* under the hypotheses of Theorem I.

Theorem III states the exact region of uniform convergence of sequences of polynomials of best approximation in the sense of Tchebycheff. The first

* Tonelli, *Annali di Matematica*, vol. 15 (1908), pp. 47-119.

part of this theorem was proved by Faber* for a point set M bounded by an analytic Jordan curve and the entire theorem was proved by Walsh† in the case of a closed limited point set whose complement is simply connected.

THEOREM III. *Under the hypotheses of Theorem II, the sequence of polynomials $\{\Pi_n(z)\}$ of respective degrees $n, n=1, 2, \dots$, of best approximation in the sense of Tchebycheff to $f(z)$ on M converges interior to C_R , uniformly on any closed point set interior to C_R , and converges uniformly in no region containing a point of C_R in its interior.*

The proof of this theorem is the analog of that given by Walsh in the case of a closed limited point set whose complement is simply connected, and is omitted.

The proof of Theorem III holds for the following theorem:

Any other sequence of polynomials which converges on M like the Tchebycheff polynomials, or, in other words, such that the inequality $|f(z) - P_n(z)| \leq N/R_0^n$ is satisfied for z on M and for R_0 arbitrary less than R , where N depends on R_0 but not on z , converges as in Theorem III.

The following theorem was proved by Walsh‡ in the special case of a point set whose complement is simply connected:

Under the hypotheses of Theorem III, neither the sequence of polynomials $\{\Pi_n(z)\}$ of best approximation to $f(z)$ on M nor any other sequence of polynomials which converges like the sequence of polynomials of best approximation converges like a geometric series in any region or on any Jordan arc exterior to C_R .

The proof follows the method of proof of the last part of Theorem III.

In particular, the discussion holds for simultaneous approximation to real analytic functions on a finite number of intervals on the axis of reals.

7. Other measures of approximation. There are other measures of approximation such as (1) approximation by the Tchebycheff method with a norm function, (2) approximation on M as measured in the sense of least p th powers ($p > 0$) by line integrals in the case that M (closed, limited) is bounded by a finite number of rectifiable Jordan curves, (3) approximation on M as measured in the sense of least p th powers ($p > 0$) by surface integrals where M (closed, limited, consisting of a finite number of regions) is an open set plus its boundary points.

In each of these cases the polynomial of best approximation exists, and is

* Crelle's Journal, vol. 150 (1920), pp. 79-106.

† (1), p. 795; (2), pp. 381-384.

‡ (2), p. 385.

unique if $p > 1$.^{*} In each case, as we shall proceed to indicate, under suitable restrictions on M , the sequence of polynomials of best approximation to $f(z)$ on M converges satisfying the inequality $|f(z) - P_n(z)| \leq N/R_0^n$, $R_0 < R$, z on M , and hence converges interior to C_R , uniformly on any closed set interior to C_R , but converges uniformly in no region containing a point of C_R in its interior.

The proofs in each of these cases are analogous to proofs already given by Walsh.[†] In cases (2) and (3), an inequality of form $|f(z) - P_n(z)| \leq N/R_0^n$, $R_0 < R$, is first proved not for z on M but for z on a suitably chosen closed set M' interior to M . The conclusion follows from the fact that when M' approaches M , then $C_{R'}$ (defined for M' as C_R is defined for M) approaches C_R ; this latter fact is a consequence of the fundamental results of Lebesgue[‡] on harmonic functions and variable domains.

A Tchebycheff polynomial for approximation to $f(z)$ on M with the norm function $p(z)$, where $p(z)$ is given continuous and different from zero on M , is the unique polynomial $\Pi'_n(z)$ of degree n such that

$$\max [|p(z)| |f(z) - \Pi'_n(z)|], z \text{ on } M,$$

is not greater than the corresponding expression for any other polynomial of degree n .

THEOREM IV. *Under the hypotheses of Theorem II, the sequence of Tchebycheff polynomials $\{\Pi'_n(z)\}$ for approximation to $f(z)$ on M with an arbitrary positive continuous norm function $p(z)$ converges interior to C_R , uniformly on an arbitrary closed point set interior to C_R , and converges uniformly in no region containing a point of C_R in its interior.*

A polynomial of best approximation in the sense of least weighted p th powers as measured on $\sum \Gamma_j$, where Γ_j , $j = 1, 2, \dots, k$, are k rectifiable Jordan curves bounding the point set M (satisfying the hypotheses of Theorem I), is a polynomial $\Pi_n(z)$ of degree n such that

$$\sum_{j=1}^k \int_{\Gamma_j} |f(z) - \Pi_n(z)|^p n(z) dz,$$

where $p > 0$ and $n(z)$ is arbitrary, continuous, positive, is not greater than the corresponding expression formed for any other polynomial of degree n .

THEOREM V. *Let the closed limited point set M whose complement is connected be bounded by a finite number k of non-intersecting rectifiable Jordan*

^{*} See for instance Walsh, these Transactions, vol. 33 (1931), pp. 668-689; p. 681.

[†] (1); (2).

[‡] Palermo Rendiconti, vol. 24 (1907), pp. 371-402.

curves Γ_j . Under the hypotheses of Theorem II, the sequence of polynomials $\{\Pi_n(z)\}$ of best approximation to $f(z)$ on M in the sense of least weighted p th powers ($p > 0$) as measured on Γ_j converges throughout the interior of C_R , uniformly on any closed point set interior to C_R , and converges uniformly in no region containing a point of C_R in its interior.

The case $p=2$ is of especial interest. Here the polynomial $\Pi_n(z)$ of best approximation to an arbitrary function $f(z)$ is of the form

$$\Pi_n(z) = a_0 P_0(z) + a_1 P_1(z) + \cdots + a_n P_n(z),$$

where the $P_i(z)$, $i=1, 2, \dots, n$, depend on the Γ_j but not on $f(z)$, and the coefficients a_i ($i \leq n$) are independent of n . The set of polynomials $P_i(z)$ is said to belong to the point set M .

The method of approximation used in Theorem V for $p=2$, $n(z) \equiv 1$, was discussed and the corresponding special case of Theorem V was proved (under an additional restriction) by Szegő* and Smirnov† for the case of a point set whose complement is simply connected.

A polynomial of best approximation in the sense of least weighted p th powers as measured by integration over the areas \bar{R}_j , where \bar{R}_j , $j=1, 2, \dots, k$, are arbitrary closed regions, is a polynomial $\Pi_n(z)$ of degree n , $n=1, 2, \dots$, such that

$$\sum_{j=1}^k \iint_{\bar{R}_j} |f(z) - \Pi_n(z)|^p n(z) dS,$$

where $p > 0$, $n(z)$ is continuous and positive in \bar{R}_j , is not greater than the corresponding expression‡ formed for any other polynomial of degree n .

THEOREM VI. Let \bar{R}_j , $j=1, 2, \dots, k$, be arbitrary closed limited regions no two of which have a common point. Let K denote the region consisting of all points which can be connected with the point at infinity by Jordan arcs which do not contain points of the \bar{R}_j . Let $G(x, y)$ be Green's function with pole at infinity for K . Under the hypotheses of Theorem II, the sequence $\{\Pi_n(z)\}$ of polynomials of best approximation to $f(z)$ in the sense of least weighted p th powers, $p > 0$, over the areas \bar{R}_j , $j=1, 2, \dots, k$, converges interior to C_R , uniformly on any closed point set interior to C_R , and converges uniformly in no region containing a point of C_R in its interior.

It will be noticed that the regions \bar{R}_j are not necessarily Jordan regions, and in fact any region \bar{R}_j may be multiply connected and even if simply con-

* Mathematische Zeitschrift, vol. 9 (1921), pp. 218-270.

† Journal de la Société Physico-Mathématique de Leningrad, vol. 2 (1928), pp. 155-178.

‡ If any of the boundaries of \bar{R}_j , $j=1, 2, \dots, k$, have area, either upper or lower integral may be used here.

nected may separate various regions B from K . The hypothesis of Theorem VI includes the analyticity of $f(z)$ in all such regions B .

The case $p=2$ is again of especial interest. The polynomial $\Pi_n(z)$ of best approximation to an arbitrary function $f(z)$ is of the form

$$\Pi_n(z) = a_0P_0(z) + a_1P_1(z) + \cdots + a_nP_n(z),$$

where the $P_i(z)$, $i=1, 2, \cdots, n$, depend on the \bar{K}_i but not on $f(z)$, and the coefficients a_i ($i \leq n$) are independent of n . The set of polynomials $P_i(z)$ is said to belong to the point set M .

The method of approximation used in Theorem VI was considered by Bergmann,* Bochner,† and Carleman‡ in the case of a single Jordan region, $p=2$, $n(z) \equiv 1$, although without proof of our results on degree of convergence and overconvergence.

As a complement to Theorems IV-VI we add

THEOREM VII. *If M consists of a finite number of mutually exclusive closed Jordan regions and if the function $f(z)$ is analytic in the interior points of M , continuous in the corresponding closed regions, then (1) the sequence of polynomials of best approximation to $f(z)$ on M in the sense of Tchebycheff with a positive continuous norm function converges to $f(z)$ uniformly on M ; (2) if the Jordan curves bounding M are rectifiable, the sequence of polynomials of best approximation to $f(z)$ on M in the sense of least p th powers ($p > 0$) as measured by a line integral with a positive continuous norm function converges to $f(z)$ at every interior point of M , uniformly on any closed set interior to M ; (3) the sequence of polynomials of best approximation to $f(z)$ on M in the sense of least p th powers ($p > 0$) as measured by a surface integral with a positive continuous norm function converges to $f(z)$ at every interior point of M , uniformly on any closed set interior to M .*

In case (3) it is indeed sufficient (see Carleman, loc. cit.) for this conclusion if $f(z)$ is analytic interior to M and if $\iint_M |f(z)|^2 dS$ exists; the restrictions in case (2) can similarly be somewhat lightened (see Smirnov, loc. cit.).

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* Mathematische Annalen, vol. 86 (1922), pp. 238-271.

† Mathematische Zeitschrift, vol. 14 (1922), pp. 180-207.

‡ Arkiv för Matematik, Astronomi, och Fysik, vol. 17 (1922-23).

THE FIRST AND SECOND VARIATIONS OF AN n -TUPLE INTEGRAL IN THE CASE OF VARIABLE LIMITS*

BY

H. A. SIMMONS

1. Introduction. The main purpose of this paper is to generalize except in one detail all of the results which we obtained relative to a double integral in a previous article.[†] The first eight sections here would constitute a complete generalization of those results if the integrand of the $(n-1)$ -tuple integral in equation (6.9) below could be expressed as a linear homogeneous function of ζ and not involve the ζ_{x_i} (cf. equation (49) of the previous article). We have not been able to do this. A special device that was used in obtaining equation (49) of the previous article does not seem capable of generalization here. In §9 we suggest for the fundamental formulas in (4.8), for the integrals $J'(0)$ and $J''(0)$, certain applications that are not made in §§1-8.

In view of certain papers of Lichtenstein[‡] and Reid[§], in which Jacobi's condition is stated in terms of characteristic numbers of boundary value problems somewhat like the problem of §7 below, and also because of recent advances in the theory of elliptic partial differential equations|| our generalization seems desirable.

The theses of Bates[¶] and Powell^{**} are useful in studying curvilinear coordinate systems of the type that we employ here. We take the lines of curvature as the parameter lines.

The legitimacy of the use that we make of an extended Green's theorem is well known.^{††}

In this paper the variables x_1, \dots, x_n are the coordinates of a euclidean space X of n ($n \geq 2$) dimensions in the euclidean space XZ of $(n+1)$ dimensions of coordinates x_1, \dots, x_n, z . An equation $z = z(x)$ or $\phi(x, z) = 0$, where

* Presented to the Society, December 1, 1933; received by the editors March 29, 1933.

† These Transactions, April, 1926, p. 235.

‡ Monatshefte für Mathematik und Physik, vol. 28 (1917), p. 3; Mathematische Zeitschrift, vol. 5 (1919), p. 26.

§ American Journal of Mathematics, vol. 54 (1932), p. 791.

|| Cf. bibliography at the close of Raab's thesis, *Jacobi's condition* . . . , The University of Chicago Press.

¶ These Transactions, vol. 12 (1911), p. 19.

** The University of Chicago Press.

†† Cf., for example, Franklin, *Annals of Mathematics*, 1923, p. 213.

$x \equiv (x_1, \dots, x_n)$, defines an n -dimensional hypersurface in XZ . We let $p_i \equiv \partial z / \partial x_i$ ($i = 1, \dots, n$) and let W stand for a $(2n+1)$ -dimensional open region in the space XZP of the variables x, z , and $p \equiv (p_1, \dots, p_n)$. Then we define an *admissible hypersurface* $z = z(x)$ to be one with its elements in W and having the following four properties: (i) z is a single-valued function of the x 's; (ii) z is of class C'' ; (iii) it has a real, *simply closed* $(n-1)$ -dimensional intersection (that is, a connected $(n-1)$ -dimensional intersection that is bounded, closed, and does not intersect itself) L'_0 , with a fixed hypersurface $\phi(x, z) = 0$, which is of class C'' and has no singular point for x and z in W ; (iv) it is such that the projection, L_0 , of L'_0 on X is met by a line parallel to any one of the coordinate axes, x_i , in a finite number of points and segments.

Property (iii) indicates the sense in which we use the term *variable limits*. The manifold L'_0 is the boundary of the portion of the admissible hypersurface $z = z(x)$ that we consider. On account of property (i), the correspondence between points of L_0 and L'_0 is one-to-one; and so L_0 is also a simply closed $(n-1)$ -dimensional manifold (cf. (iii)). It bounds a simply connected portion of X space. This we call S_0 .

Property (iv) is required to insure that our application to L_0 and S_0 of an extended Green's theorem in §§5, 7 below shall be legitimate.

We consider here the n -tuple integral ($n \geq 2$)

$$(1.1) \quad I = \int_{S_0}^n f(x, z, p) dx,$$

where x, z, p, S_0 are as defined above, and f is of class C''' in W . This integral I is our generalization of the double integral

$$\iint_{A_0} f(x, y, z, p, q) dx dy$$

of the previous article.

Assuming that $z = z(x)$ is a minimizing admissible hypersurface for the integral I , we let $\zeta(x)$ be any function of the x 's with properties (i), (ii), and such that if a is a real parameter sufficiently small numerically, then $z = z(x) + a\zeta(x)$ is admissible. Let L'_a denote the $(n-1)$ -dimensional manifold

$$\phi(x, z(x) + a\zeta(x)) = 0, \quad z = z(x) + a\zeta(x)$$

when x is sufficiently near the x 's that determine points of S_0^* ; let L_a denote the projection on X space of L'_a ; and designate by S_a the hyperarea in the X space that is bounded by L_a . Then in place of the integral (1.1), we have $I(a)$:

* So near that the correspondence between points x and points uv in §2 is (1,1), reversible.

$$(1.2) \quad I(a) = \int_{S_a}^n f(x, z + a\zeta, p + a\zeta_x) dx.$$

Our main problem is to obtain the first and second derivatives $I'(0)$ and $I''(0)$, which are analogous to corresponding integrals of the previous article. We assume that $f \neq 0$ on the hypersurface $z = z(x)$ along its intersection L'_0 with the fixed hypersurface $\phi(x, z) = 0$ for a reason analogous to that which made a similar assumption desirable in the previous article.

In §2, we set up a normal curvilinear coordinate system which plays an important rôle in later sections of this paper; in §3, the equations of Rodriguez* are generalized and the result is used to obtain a simple expansion of a functional determinant of §2; in §4, Theorem 1 of the previous article is generalized to the case of an n -tuple integral; in §5, the results relative to the first variation in the previous article are generalized; in §6, two expressions for $I''(0)$ are given; in §§7 and 8, the boundary value problem and the discussion of the minimal surface, respectively, of the previous article are generalized; and the object of §9 is as we stated above.

We wish to thank Professor L. P. Eisenhart for numerous suggestions that he has given relative to §§1, 2 of this paper.

2. The normal coordinate system.† Let L_0 be a simply closed $(n-1)$ -dimensional manifold with equations

$$x_i = \xi_i(u) \quad (i = 1, \dots, n),$$

where u is $(n-1)$ -partite and the ξ 's are defined for all real values of the u 's, are of class C'' , have

$$\sum_{i=1}^n \xi_{iu_j}^2 \neq 0 \quad (j = 1, \dots, n-1),$$

where the subscript u_j indicates partial differentiation of the ξ_i with respect to u_j (similar subscript notation is used throughout the sequel) and each ξ_i has in u_j a period, say t_j , which is passed through once (exactly) when a point x passes once around the u_j -curve on L_0 . We agree further, as stated above, to take the lines of curvature on L_0 as our parameter lines, so that the u -curves on L_0 are mutually orthogonal.‡

We now introduce near L_0 a uv -coordinate system determined by the equations

$$(2.1) \quad x_i = \xi_i(u) + vA_i \quad (i = 1, \dots, n), \quad 0 \leq u_j \leq t_j, \quad v_1 \leq v \leq v_2,$$

* Cf. Eisenhart's *Differential Geometry*, p. 122.

† Some of the ideas of this section are also expressed in §6 of Powell's thesis, which was referred to in §1.

‡ Cf., for example, Bates, loc. cit., p. 25, Theorem 1.

where

$$(2.2) \quad A_i = (-1)^{i-1} \frac{\partial(\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_n)}{\partial(u_1, \dots, u_{n-1})};$$

$\sum_{i=1}^n A_i^2 = 1$, a condition that can be realized by a suitable choice of parameters; v is one-partite; and $v_1 < 0$, $v_2 > 0$ are sufficiently small numerically that there exist unique functions

$$u_i = U_i(x_1, \dots, x_n), \quad v = V(x_1, \dots, x_n)$$

of class C' satisfying equations (2.1). This is possible* since for x on L_0 , $(u, v) = (u, 0)$ and the functional determinant

$$(2.3) \quad \Delta(v) \Big|_{v=0} = \begin{vmatrix} \xi_{1u_1} + vA_{1u_1} & \dots & \xi_{nu_1} + vA_{nu_1} \\ \xi_{1u_{n-1}} + vA_{1u_{n-1}} & \dots & \xi_{nu_{n-1}} + vA_{nu_{n-1}} \\ A_1 & \dots & A_n \end{vmatrix} \Big|_{v=0} \\ = \begin{vmatrix} \xi_{1u_1} & \dots & \xi_{nu_1} \\ \xi_{1u_{n-1}} & \dots & \xi_{nu_{n-1}} \\ A_1 & \dots & A_n \end{vmatrix} = (-1)^{n-1} \sum_{i=1}^n A_i^2 = (-1)^{n-1} \neq 0.$$

We call our coordinate system *normal* because the $(n-1)$ lines of curvature u_j are mutually orthogonal and, at every point P of L_0 , v measures along the unique normal in the X space to L_0 through P for all values of $n \geq 2$.

After obtaining, in §3, a simple expansion of the determinant $\Delta(v)$, we shall employ, in §4, the coordinate system of this section in differentiating an n -tuple integral with respect to a parameter.

3. Use of generalized equations of Rodriguez† in expanding $\Delta(v)$. The generalization of the equations of Rodriguez may be obtained from the first set of equations that Bates displays on page 24 of his article referred to above. In these equations we let n be the $(n-1)$, number of u 's, of the present paper, and we take his x_j , ζ_j^i as our u_j , A_{iu_j} , respectively. Since we are taking the lines u_j to be lines of curvature (mutually orthogonal), we thus obtain the following generalized equations of Rodriguez:

$$(3.1) \quad \rho_j A_{iu_j} = \xi_{iu_j} \quad (i = 1, \dots, n),$$

where ρ_j is, except possibly for sign, the radius of curvature of the u_j -line of

* Cf. G. A. Bliss, Princeton Colloquium Lectures, p. 20.

† Cf. Eisenhart's *Differential Geometry*, p. 122.

curvature. As a consequence of (3.1), the determinant $\Delta(v)$ in (2.3) may be written

$$\begin{vmatrix} \xi_{1u_1}(1 + v/\rho_1) & \cdots & \xi_{nu_1}(1 + v/\rho_1) \\ \vdots & \ddots & \vdots \\ \xi_{1u_{n-1}}(1 + v/\rho_{n-1}) & \cdots & \xi_{nu_{n-1}}(1 + v/\rho_{n-1}) \\ A_1 & \cdots & A_n \end{vmatrix} \\ = \pm (1 + v/\rho_1) \cdots (1 + v/\rho_{n-1}) \sum_{i=1}^n A_i^2 ;$$

or

$$(3.2) \quad \Delta(v) = \pm (1 + \pi_1 v + \cdots + \pi_{n-1} v^{n-1}),^*$$

where π_i is the elementary symmetric function of i th order of the ρ_j^{-1} ($j=1, \dots, n-1$).

4. The derivatives of an n -tuple integral with respect to a parameter. Consider a family of $(n-1)$ -dimensional manifolds, one of which, L_a (cf. the equation of L_0 in §2), is given by the equations

$$(4.1) \quad x_i = \xi_i + v(u, a)A_i \quad (i = 1, \dots, n),$$

where $v(u, a)$ is defined and of class C'' for all (u, a) having each u_j as it was defined in §2 and a sufficiently near zero ($v_1 \leq v(u, a) \leq v_2$), where $v(u, a)$ has for each u_j a period $T_j(a)$ that reduces to $T_j(0) = t_j$ for $a=0$; and $v(u, 0) \equiv 0$. All of the manifolds L_a are closed on account of this periodicity and each L_a is also simply closed for all values of a sufficiently near zero since L_0 is simply closed. We let S_a denote the hyperarea bounded by L_a with the special understanding that when a satisfies the equation $v(u, a) = v_1$, we are to designate S_a and L_a as S_1 and L_1 , respectively.

Let $g(x, a)$ be a function of a and the x_i ($i=1, \dots, n$) which is of class C'' for all sets (x, a) having x in a sufficiently small neighborhood of the hyperarea S_0 bounded by L_0 and having a such that $v_1 \leq v(u, a) \leq v_2$. Define $J(a)$ by the formula

$$(4.2) \quad J(a) = \int_{S_a} g(x, a) dx.$$

We desire the derivatives $J'(0)$ and $J''(0)$. To obtain them, we first express the integral (4.2) as a sum of two integrals:

* The relative simplicity of this expansion may be observed by comparing it with the one that results if $\Delta(v)$, in (2.3), is expanded by minors in the notation that Bates used in a similar expansion (cf. Bates, loc. cit., equation (34)).

$$(4.3) \quad J(a) = \int_{S_1}^n g(x, a) dx + \int_{\Delta S}^n g(x, a) dx,$$

where S_1 is as it was defined in §2 and ΔS is the hyperarea in the X space that is bounded by the $(n-1)$ -dimensional manifolds L_1 and L_a . The derivative of the first integral in (4.3) has the value

$$(4.4) \quad \int_{S_1}^n g_a(x, a) dx.$$

To differentiate the last integral in (4.3), we first transform it to the uv -coordinate system by means of (2.1). Letting $\Delta(v)^+$ stand for the value of $\Delta(v)$ when the $+$ sign is used before the parenthesis in (3.2), we find

$$(4.5) \quad \int_{\Delta S}^n g(x, a) dx = \int_{L_0}^{n-1} \left[\int_{v_1}^{v(u, a)} g(\xi + vA, a) \Delta(v)^+ dv \right] du,$$

where $\xi + vA$ stands for the n expressions $\xi_1 + vA_1, \dots, \xi_n + vA_n$. Since a occurs only in the upper limit of the inner integral of (4.5) and explicitly in g , we find the derivative in question to be

$$\int_{L_0}^{n-1} -gv_a \Delta(v)^+ du + \int_{L_0}^{n-1} \left[\int_{v_1}^{v(u, a)} g_a \Delta(v)^+ dv \right] du,$$

where in the first integral $v = v(u, a)$. Adding this result to the expression (4.4), we obtain

$$\begin{aligned} J'(a) = \int_{S_1}^n g_a dx + \int_{L_0}^{n-1} \left[\int_{v_1}^{v(u, a)} g_a \Delta(v)^+ dv \right] du \\ + \int_{L_0}^{n-1} v_a g(\xi + vA, a) \Delta(v)^+ du. \end{aligned}$$

Hence, after transforming the second integral to x -coordinates, we find

$$(4.6) \quad J'(a) = \int_{S_a}^n g_a dx + \int_{L_0}^{n-1} v_a g(\xi + vA, a) \Delta(v)^+ du.$$

From the above procedure (perhaps with reference to the previous article), we now find without difficulty that

$$(4.7) \quad J''(a) = \int_{S_a}^n g_{aa} dx + \int_{L_0}^{n-1} [(gv_{aa} + 2g_a v_a + g_v v_a^2) \Delta(v)^+ + g v_a \Delta_a(v)^+] du.$$

Putting $a=0$ in (4.6) and (4.7) and recalling that $v(u, 0) \equiv 0$, we find the desired results, which we express as follows.

THEOREM 4.1. The derivatives $J'(0)$ and $J''(0)$ of the n -tuple integral $J(a)$ defined by (4.2), taken over the n -dimensional region S_a bounded by the manifold L_a , defined by equations (4.1), have the values

$$(4.8) \quad \begin{aligned} J'(0) &= \int_{S_0}^n g_a dx + \int_{L_0}^{n-1} g v_a du, \\ J''(0) &= \int_{S_0}^n g_{aa} dx + \int_{L_0}^{n-1} (g v_{aa} + \pi_1 g v_a^2 + 2g_a v_a + g_a v_a^2) du. * \end{aligned}$$

The derivatives (4.8) have been computed for the family of variations (4.1) of L_0 . However we can obtain from (4.8) analogous formulas for a more general family of variations of L_0 of the form

$$(4.9) \quad x_i = X_i(\tau, a) \quad (i = 1, \dots, n),$$

where τ is $(n-1)$ -partite and $\tau_j = \tau_j(u_1, \dots, u_{n-1}, a)$ ($j = 1, \dots, n-1$), with $\tau_j(u_1, \dots, u_{n-1}, 0) = u_j$. We suppose that (4.9) represents a one-parameter family of simply closed $(n-1)$ -dimensional manifolds containing L_0 for $a=0$. The functions X_i are supposed to be of class C'' for all values of (τ, a) having each τ_j real and a sufficiently near zero. They have a period $\Gamma_j(a)$ for every a that we consider, with $\Gamma_j(0) = t_j$. Such a family is representable in the form (4.1) if we can solve the equations

$$(4.10) \quad X_i(\tau, a) - \xi_i(u) - v A_i(u) = 0$$

for v and the τ_j as functions of a and the u_j . According to the implicit function theorem used in §2, this can be done since the equations (4.10) have the particular solution $(v, \tau, u, a) = (0, u, u, 0)$ for $0 \leq u_j \leq t_j$, on which the functional determinant

$$\begin{vmatrix} X_{1\tau_1}, \dots, X_{1\tau_{n-1}}, X_{1v} \\ \vdots \\ X_{n\tau_1}, \dots, X_{n\tau_{n-1}}, X_{nv} \end{vmatrix} \begin{vmatrix} \xi_{1u_1}, \dots, \xi_{1u_{n-1}}, -A_1 \\ \vdots \\ \xi_{nu_1}, \dots, \xi_{nu_{n-1}}, -A_n \end{vmatrix} = -\Delta(0) = \mp 1 \neq 0.$$

Hence we can obtain v_a and v_{aa} for the general family (4.9). Differentiating (4.10) once, twice, and agreeing that a term in which j appears as a repeated index (even though it be a subscript of a subscript) is to be summed for all integral values of j from 1 to $(n-1)$, and setting $a=0$, we obtain (4.11), (4.12), respectively,

$$(4.11) \quad X_{i\tau_j\tau_{ja}} - A_i v_a + X_{ia} = 0,$$

$$(4.12) \quad X_{i\tau_j\tau_{jaa}} - A_i v_{aa} + X_{iaa} + 2X_{ia\tau_j\tau_{ja}} + X_{i\tau_j\tau_k\tau_{ja}\tau_{ka}} = 0.$$

* In the second of equations (9) of the previous article there is a misprint. The last term of the integrand of the line integral of $J''(0)$ there should be $g_a v_a^2$.

The determinant of the n equations (4.11) in the τ_{ja} and v_a , like that of the n equations (4.12) in the τ_{jaa} and v_{aa} , is $-\Delta(0) = \mp 1$. After a sense is assigned to L_0 , so as to give $\Delta(0)$ a definite value (cf. (2.2)), say $+1$, equations (4.11) define v_a as a polynomial in the X_{ia} and the X_{ir_j} ($=\xi_{ir_j}$ on L_0), while (4.12) define v_{aa} similarly in the X_{iaa} , X_{iar_j} , $X_{ir_j r_k}$, X_{ir_j} , and τ_{ja} . But the τ_{ja} in v_{aa} can be eliminated by means of (4.11); the τ_{ja} are then polynomials in the X_{ia} , X_{ir_j} since the A_i are polynomials in the X_{ir_j} (cf. the functional determinant last displayed above). Hence we have the following corollary of Theorem 4.1.

COROLLARY. *The derivatives $J'(0)$ and $J''(0)$ in equations (4.8) can be generalized to the case where (4.1) is replaced by (4.9). When this is done, v_a is a polynomial in the X_{ia} and the X_{ir_j} , while v_{aa} is a polynomial in the X_{iaa} , X_{iar_j} , $X_{ir_j r_k}$.*

5. The first variation. In the sequel, any term that contains a repeated index other than a, v, z , however it may appear, is to be summed for all integral values of the index from 1 to n . Thus we write $f_{pi}(\phi_{zi} + p_i \phi_z)$ for the sum of the n terms that one obtains from this expression by taking $i=1, \dots, n$. Further, when we use Kronecker δ 's with $\delta_i^k=1$ or 0 according as $k=i$ or $k \neq i$, respectively, we extend customary convention by admitting subscripts of subscripts as summation indices; thus we would write $\xi_{zi} \delta_i^k = \xi_{zi}$.

Proceeding now as we did in §3 of the previous article, we find without difficulty the following equations, of which we number only those that are to be referred to later:

$$I'(0) = \int_{S_0}^n f_a dx + \int_{L_0}^{n-1} f_{va} du, \quad f_a = f_{zi} \xi + f_{pi} \xi_{zi};$$

$$\phi(\xi + vA, z(\xi + vA) + a\xi(\xi + vA)) = 0,$$

$\xi + v(u, a)A$ being as in (4.5), so that ϕ contains the variables u, v, a ;

$$(5.1) \quad v_a = -\phi_a/\phi_v \quad (\phi_v \neq 0, \text{ cf. (5.8)});$$

$$(5.2) \quad \phi_a = \phi_{zi} \xi, \quad \phi_v = (\phi_{zi} + p_i \phi_{zi}) A_i;$$

$$(5.3) \quad v_a = -\phi_{zi} \xi / (\phi_{zi} + p_i \phi_{zi}) A_i;$$

$$(5.4) \quad I'(0) = \int_{S_0}^n (f_{zi} \xi + f_{pi} \xi_{zi}) dx + \int_{L_0}^{n-1} \frac{-\xi f_{pi} du}{(\phi_{zi} + p_i \phi_{zi}) A_i}.$$

Hence we have the following theorem:

THEOREM 5.1. *The first derivative $I'(0)$ of the n -tuple integral (1.1), taken over the portion of the hypersurface $z=z(x)+a\xi(x)$ bounded by its intersection with the hypersurface $\phi(x, z)=0$, has the value given by (5.4).*

From the point of view of the calculus of variations it is desirable to perform an integration by parts on the terms $f_{p_i} \zeta_{x_i}$ of (5.4). Since

$$f_{p_i} \zeta_{x_i} = \frac{\partial}{\partial x_i} (f_{p_i} \zeta) - \zeta \frac{\partial}{\partial x_i} f_{p_i},$$

we can replace the n -tuple integral in (5.4) by

$$\begin{aligned} \int_{S_0}^n \left[\zeta \left(f_z - \frac{\partial}{\partial x_i} f_{p_i} \right) + \frac{\partial}{\partial x_i} f_{p_i} \zeta \right] dx \\ = \int_{S_0}^n \zeta \left(f_z - \frac{\partial}{\partial x_i} f_{p_i} \right) dx + \int_{L_0}^{n-1} \zeta f_{p_i} A_i du. * \end{aligned}$$

Using this result in (5.4), we obtain

$$(5.5) \quad I'(0) = \int_{S_0}^n \zeta \left(f_z - \frac{\partial}{\partial x_i} f_{p_i} \right) dx + \int_{L_0}^{n-1} \zeta \left(\frac{\phi_v A_i f_{p_i} - f \phi_z}{\phi_v} \right) du.$$

Since, along L'_0 , $\phi(\xi_1(u), \dots, \xi_n(u), z(\xi_1, \dots, \xi_n)) \equiv 0$ in the u , we can replace (5.5) by an equivalent equation analogous to equation (32) of the previous article. Differentiating this identity, $\phi \equiv 0$, with respect to each u , we obtain the $(n-1)$ equations

$$(5.6) \quad (\phi_{x_i} + p_i \phi_z) \xi_{iu_j} = 0 \quad (j = 1, \dots, n-1).$$

Hence with (5.6) and the last equation in (5.2), we have n equations which determine the $\phi_{x_i} + p_i \phi_z$:

$$(5.7) \quad \phi_{x_i} + p_i \phi_z = A_i \phi_v \text{ (cf. (2.2))},$$

where

$$(5.8) \quad \phi_v^2 = \sum_{i=1}^n (\phi_{x_i} + p_i \phi_z)^2 \neq 0.$$

That $\phi_v^2 \neq 0$ may be proved as follows. Suppose $\phi_v^2 = 0$, so that

$$\phi_{x_i} + p_i \phi_z = 0 \quad (i = 1, \dots, n).$$

Then the hypersurfaces $z = z(x)$ and $\phi(x, z) = 0$ are tangent to each other, which is impossible when $f \neq 0$ along L'_0 , as we shall see just below Corollary 5.2. In view of (5.7), (5.5) is equivalent to

$$(5.9) \quad I'(0) = \int_{S_0}^n \zeta \left(f_z - \frac{\partial}{\partial x_i} f_{p_i} \right) dx + \int_{L_0}^{n-1} \zeta \frac{[f_{p_i}(\phi_{x_i} + p_i \phi_z) - f \phi_z] du}{(\phi_{x_i} + p_i \phi_z) A_i}.$$

* In obtaining this term we have used the extended Green's theorem referred to above with the A_i as direction cosines of the outer normal to L_0 .

The Euler necessary condition for a minimum value of I in the case of fixed limits (where the hypersurface $\phi(x, z)=0$ is replaced by a bounded, closed, connected $(n-1)$ -dimensional manifold, such as L_0) is

$$(5.10) \quad f_z - \frac{\partial}{\partial x_i} f_{p_i} = 0^*$$

at every point of S_0 . This is surely a necessary condition for the case of variable limits. Since $I'(0)=0$ is a necessary condition for a minimum value of I , it now follows that if $z=z(x)$ minimizes I the second integral in (5.9) vanishes, and indeed that the numerator, N , of the integrand of this integral is zero at every point of L_0 , $0 \leq u_i \leq t_j$, as we presently prove. Suppose N does not so vanish. Either N has one sign on the entire manifold L_0 or there is a non-zero $(n-1)$ -dimensional subregion of L_0 on which N has one sign. We may take $\zeta(u_1, \dots, u_{n-1})$ to be of the same sign as N on one such subregion and zero elsewhere. Then since $\phi_z \neq 0$, $I'(0) \neq 0$ (contradiction). Hence we have the transversality condition

$$(5.11) \quad f_{p_i}(\phi_{z_i} + p_i \phi_z) - f \phi_z = 0$$

at every point of L_0 .

We now have the following corollaries of Theorem 5.1.

COROLLARY 5.1. *The first derivative of the n -tuple integral $I(a)$, of (1.2), taken over the portion of the hypersurface $z=z(x)+a\zeta(x)$ bounded by its intersection with the fixed hypersurface $\phi(x, z)=0$, has the value given by (5.9).*

COROLLARY 5.2. *In case $z=z(x)$ is a minimizing hypersurface for the n -tuple integral (1.1), the Euler equation (5.10) must hold at every point of the portion of the hypersurface $z=z(x)$ inside L'_0 , and the transversality condition (5.11) must hold at every point of the boundary, L'_0 , which is the manifold of intersection of the hypersurfaces $z=z(x)$ and $\phi(x, z)=0$.*

Since we have assumed in §1 that $f \neq 0$ along L'_0 , it follows from (5.11) that the hypersurface $z=z(x)$ is not tangent to the hypersurface $\phi(x, z)=0$ at any point of L'_0 . In the case of the minimal hypersurface for which $f=(1+p_i p_i)^{1/2}$, (5.11) reduces to $p_i \phi_{z_i} - \phi_z = 0$, which shows that the hypersurfaces $z=z(x)$ and $\phi(x, z)=0$ meet at right angles.

6. The second variation. To get $I''(0)$, we apply to (1.2) the result in the second of equations (4.8). Replacing g in that equation by f , we obtain

$$(6.1) \quad I''(0) = \int_{S_0}^n f_{aa} dx + \int_{L_0}^{n-1} M du,$$

* Cf. page 5 of Powell's thesis, loc. cit.

where $M \equiv f v_{aa} + f \pi_1 v_a^2 + f_v v_a^2 + 2 f_a v_a$ and

$$(6.2) \quad f_{aa} = f_{zz} \zeta^2 + 2 f_{zp} \zeta \zeta_{z_i} + f_{p_i p_j} \zeta_{z_i} \zeta_{z_j}.$$

By differentiating the equation $\phi_v v_a + \phi_a = 0$ (cf. (5.1)), with respect to a , we find, as in the previous article, that

$$(6.3) \quad v_{aa} = -\frac{1}{\phi_v} (\phi_{vv} v_a^2 + 2 \phi_{av} v_a + \phi_{aa}),$$

where ϕ_v is defined in (5.2), and ϕ_{vv} , ϕ_{av} , ϕ_{aa} , at $a=0$, are obtained by differentiating ϕ as a function of the arguments

$$\xi_i + v A_i \quad (i = 1, \dots, n), \quad z(\xi + v A) + a \zeta(\xi + v A),$$

$\xi + v A$ standing for the set of n expressions $\xi_i + v A_i$. We find, for $a=0$,

$$\phi_{vv} = (\phi_{z_i z_j} + 2 \phi_{z_i z} p_j + \phi_{zz} p_i p_j + \phi_z r_{ij}) A_i A_j, \quad r_{ij} \equiv z_{z_i z_j};$$

$$\phi_{av} = (\phi_{z_i z} \zeta + \phi_{zz} \zeta p_i + \phi_z \zeta_{z_i}) A_i; \quad \phi_{aa} = \phi_{zz} \zeta^2.$$

Using these derivatives together with (5.2) and (5.3), we find (cf. (6.3))

$$(6.4) \quad \begin{aligned} v_{aa} = & -\frac{1}{\phi_v^3} \{ \phi_z^2 \zeta^2 [\phi_{z_i z_j} + 2 \phi_{z_i z} p_j + \phi_{zz} p_i p_j + \phi_z r_{ij}] \\ & - 2 \phi_z \zeta [\zeta (\phi_{z_i z} + p_i \phi_{zz}) (\phi_{z_j} + p_j \phi_z) \\ & + \phi_z (\phi_{z_j} + p_j \phi_z) \zeta_{z_i}] + \phi_{zz} \zeta^2 (\phi_{z_i} + p_i \phi_z) (\phi_{z_j} + p_j \phi_z) \} A_i A_j. \end{aligned}$$

Hence if we collect in (6.4) the terms involving the second derivatives of ϕ , those involving the second derivatives of z , and those free of second derivatives, we find

$$(6.5) \quad \begin{aligned} v_{aa} = & -\frac{\zeta^2}{\phi_v^3} (\phi_z^2 \phi_{z_i z_j} - 2 \phi_z \phi_{z_j} \phi_{z_i z} + \phi_{zz} \phi_{z_i} \phi_{z_j}) A_i A_j - \frac{\phi_z^2 \zeta^2}{\phi_v^3} \cdot r_{ij} A_i A_j \\ & + \frac{2 \zeta \zeta_{z_i}}{\phi_v^3} (\phi_z^2 \phi_{z_j} + \phi_z^3 p_j) A_i A_j. \end{aligned}$$

Now using the notation

$$\begin{aligned} q_i &= -\phi_{z_j} / \phi_z, \quad \Delta = [(p_i - q_i)(p_i - q_i)]^{1/2}, \\ s_{ij} &= -\frac{\phi_z^2 \phi_{z_i z_j} - 2 \phi_z \phi_{z_j} \phi_{z_i z} + \phi_{zz} \phi_{z_i} \phi_{z_j}}{\phi_z^3}, \end{aligned}$$

so that $(\partial q_i / \partial x_j) = (s_{ij} + s_{ji})/2$, we find from (6.5) that

$$(6.6) \quad v_{aa} = \frac{1}{\Delta^3} [\zeta^2 (s_{ij} - r_{ij}) + 2 \zeta \zeta_{z_i} (p_j - q_i)] A_i A_j.$$

The introduction of the q_i and the s_{ij} (whose denominators involve $\phi_{,2}$) does not require that $\phi(x, z) = 0$ be representable in the form $z = z_1(x)$ (cf. the $(n-1)$ -tuple integral in (6.9), in which $\Delta \neq 0$ since $0 \neq \phi_{,2} = \phi_{,2}\Delta$).

The other three terms of M are

$$\begin{aligned} f v_a^2 \pi_1 &= \frac{f \zeta^2}{\Delta^2} \left(\frac{1}{\rho_1} + \cdots + \frac{1}{\rho_{n-1}} \right), \\ (6.7) \quad f v_a^2 &= \frac{\zeta^2}{\Delta^2} (f)_{z_i} (p_i - q_i), \\ 2f_a v_a &= - (2\zeta/\Delta) (f_{z_i} \zeta + f_{p_i} \zeta_{z_i}), \end{aligned}$$

where

$$(6.8) \quad (f)_{z_i} = f_{z_i} + f_{z_j} p_i + f_{p_j} r_{ij}.$$

Collecting the terms of M , as they are given by (6.6), (6.7), (6.8), and using the value (6.2) of f_{aa} , we obtain the following theorem.

THEOREM 6.1. *The second derivative $I''(0)$ of the n -tuple integral $I(a)$ of equation (1.2), taken over the portion of the hypersurface $z = z(x) + a\zeta(x)$ bounded by its intersection with the hypersurface $\phi(x, z) = 0$, has the value*

$$(6.9) \quad I''(0) = \int_{S_0}^n 2\Omega dx + \int_{L_0}^{n-1} (\zeta/\Delta^2) (B\zeta + C\zeta_{z_i}) du,$$

where

$$\begin{aligned} 2\Omega &\equiv f_{aa}\zeta^2 + 2f_{z p_i} \zeta \zeta_{z_i} + f_{p_i p_j} \zeta_{z_i} \zeta_{z_j}, \\ (6.10) \quad B &\equiv f(s_{ij} - r_{ij}) A_i A_j + f \Delta \pi_1 + (f)_{z_i} (p_i - q_i) - 2f_a \Delta^2, \\ C_i &\equiv 2[f(p_i - q_i) A_i A_i - f_{p_i} \Delta^2]. \end{aligned}$$

7. A boundary value problem associated with the second variation. We generalize the boundary value problem of the previous article. By Euler's theorem on homogeneous functions, the n -tuple integral of (6.9) can be written in the form

$$\int_{S_0}^n (\zeta \Omega_i + \zeta_{z_i} \Omega_i) dx; \quad \Omega_i \equiv \Omega_i \zeta_{z_i}.$$

Then, after performing a customary integration by parts, we find

$$\int_{S_0}^n 2\Omega dx = \int_{S_0}^n \left[\zeta \psi(\zeta) + \frac{\partial}{\partial x_i} \zeta \Omega_i \right] dx, \quad \psi(\zeta) \equiv \Omega_i - \frac{\partial}{\partial x_i} \Omega_i.$$

Applying the extended Green's theorem heretofore used, we now find

$$(7.1) \quad \int_{S_0}^n 2\Omega dx = \int_{S_0}^n \xi \psi(\xi) dx + \int_{L_0}^{n-1} \xi A_i \Omega_i du.$$

From (6.9) and (7.1), we now obtain

$$I''(0) = \int_{S_0}^n \xi \psi(\xi) dx + \int_{L_0}^{n-1} \xi (B\xi + C_i \xi_{z_i} + A_i \Omega_i) du;$$

or since

$$\Omega_i = \xi f_{z_i p_i} + \xi_{z_j} f_{p_i p_j},$$

we have

$$(7.2) \quad I''(0) = \int_{S_0}^n \xi \psi(\xi) dx + \int_{L_0}^{n-1} \xi (D\xi + E_i \xi_{z_i}) du,$$

where

$$(7.3) \quad D \equiv B + A_i f_{z_i p_i}, \quad E_i \equiv C_i + A_j f_{p_j p_i} \text{ (cf. (6.10)).}$$

From (7.2) we can now state a new necessary condition in order that the hypersurface $z = z(x)$ shall minimize the n -tuple integral (1.1).

THEOREM 7.1. *In order that the hypersurface $z = z(x)$ shall minimize the n -tuple integral (1.1), it is necessary that for negative values of λ the boundary value problem*

$$\psi(\xi) - \lambda \xi = 0 \text{ in the region } S_0,$$

$$D\xi + E_i \xi_{z_i} = 0 \text{ on the boundary } L_0 \text{ of } S_0$$

have no solution except $\xi \equiv 0$, D and the E_i being defined through (6.10) and (7.3).

8. The minimal hypersurface. Here we define $f = (1 + p_i p_i)^{1/2}$. We shall compute the $I''(0)$ of (6.9) for the present case. Since $f_z = 0$, the only derivatives needed here are

$$f_{p_i} = p_i/f, \quad f_{p_i p_j} = (f^2 \delta_{ij} - p_i p_j)/f^3, \quad (f)_{z_i} = p_i r_{ij}/f \text{ (cf. (6.8)),}$$

$$f_{zz} = 2\Omega = \xi \Omega_\xi + f_{p_i p_j} \xi_{z_i} \xi_{z_j} = \xi_{z_i} \xi_{z_j} (f^2 \delta_{ij} - p_i p_j)/f^3.$$

One now finds that the B , C_i of (6.10) reduce to B' , C'_i , respectively, where

$$(8.1) \quad \begin{aligned} B' &= f(s_{ij} - r_{ij})A_i A_j + f\left(\frac{1}{\rho_1} + \cdots + \frac{1}{\rho_{n-1}}\right)\Delta + p_i r_{ij}(p_i - q_i)/f, \\ C'_i &= (2/f)[f^2(p_i - q_i)A_i A_i - p_i \Delta^2]. \end{aligned}$$

Hence we have the following corollary of Theorem 6.1.

COROLLARY 8.1. *In the case of the minimal hypersurface, the $I''(0)$ of Theorem 6.1 reduces to*

$$I''(0) = \int_{S_0}^n (\zeta_{z_i} \zeta_{z_j} / f^3) (f^2 \delta_i^j - p_i p_j) dx + \int_{L_0}^{n-1} \zeta (B' \zeta + C'_i \zeta_{z_i}) du,$$

where B' and the C'_i are as defined in (8.1).

To make a similar specialization of Theorem 7.1, we observe that in the present case

$$(8.2) \quad \begin{aligned} \psi(\zeta) &= -\frac{\partial}{\partial x_i} \Omega_i = -\frac{\partial}{\partial x_i} \zeta_{z_j} (f^2 \delta_i^j - p_i p_j) / f^3, \\ D &= B', \text{ of (8.1),} \\ E_i &= C'_i + A_j (f^2 \delta_i^j - p_i p_j) / f^3 \text{ (cf. (7.3) and (8.1)).} \end{aligned}$$

Consequently we have the following corollary to Theorem 7.1.

COROLLARY 8.2. *In order that the hypersurface $z=z(x)$ shall minimize the n -tuple integral (1.1) in the case where $f=(1+p_i p_i)^{1/2}$, it is necessary that for negative values of λ the following boundary value problem have no solution except $\zeta=0$:*

$$-\frac{\partial}{\partial x_i} \Omega_i + \lambda \zeta = 0 \text{ on } S_0,$$

$$B' \zeta + E_i \zeta_{z_i} = 0 \text{ on } L_0,$$

where

$$-\frac{\partial}{\partial x_i} \Omega_i = \psi(\zeta)$$

by (8.2), B' is defined by (8.1), and the E_i are given in (8.2).

9. Further applications of Theorem 4.1. Since g in §4 is merely required to be a function of class C'' in the x 's and the parameter a , there may be numerous applications of Theorem 4.1, even to more complicated variation problems than the problem associated with §§1-8 above. We have made one such application. We have used the first equation of (4.8) to compute the first variation of the integral

$$K = \int_{S_0}^n f(x, z, p, r) dx$$

where S_0 , n , x , z , p have the meaning relative to K which they had for I in §§1-8; r is the set of all of the derivatives

$$\frac{\partial p_i}{\partial x_j} = r_{ij} \quad (i, j = 1, \dots, n);$$

and f is supposed to have suitable continuity in a region W of the space $XZPR$ in which a minimum value of K is desired. A fixed hypersurface $\phi(x, z) = 0$, with suitable continuity, is employed as in §§1-8.

We state without proof that the analogs of (5.4) and (5.9) here are (9.1) and (9.2) below, respectively:

$$(9.1) \quad K'(0) = \int_{S_0}^n (f_z \zeta + f_{p_i} \zeta_{x_i} + f_{r_{ij}} \zeta_{x_i x_j}) dx - \int_{L_0}^{n-1} \frac{f \zeta \phi_z du}{(\phi_{x_i} + p_i \phi_z) A_i};$$

$$(9.2) \quad K''(0) = \int_{S_0}^n \zeta \left(f_{zz} - \frac{\partial}{\partial x_i} f_{p_i} + \frac{\partial^2 f_{r_{ij}}}{\partial x_i \partial x_j} \right) dx \\ + \int_{L_0}^{n-1} \left[(f_{p_i} \zeta + f_{r_{ij}} \zeta_{x_j}) (\phi_{x_i} + p_i \phi_z) - \zeta (\phi_{x_j} + p_j \phi_z) \frac{\partial}{\partial x_i} f_{r_{ij}} \right. \\ \left. - \phi_z \zeta f \right] \cdot \frac{du}{(\phi_{x_i} + p_i \phi_z) A_i}.$$

By methods that were used in preceding sections, one could compute $K''(0)$.

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CONSECUTIVE COVARIANT CONFIGURATIONS AT A POINT OF A SPACE CURVE*

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I. INTRODUCTION

A systematic study of the projective differential geometry of space curves was first made by Halphen† in a memoir of 1880. Wilczynski‡ in 1905 and 1906 and Sannia§ in 1926 made important additions to the subject.

The projective differential theory of a curve involves many configurations associated covariantly with the curve. The purpose of this paper is to make some contributions to the theory of a space curve, which are based upon the study of consecutive configurations. The work follows the lines of a similar investigation made by Lane|| for the case of a plane curve.

We now make precise the meaning of the word "consecutive" as used in the present paper. Let us consider an analytic curve C in projective space of three dimensions. The equations of such a curve, in non-homogeneous projective coordinates x, y, z , can be written in the form of two power series expansions,

$$(1) \quad y = a_0 + a_1x + a_2x^2 + \cdots, \quad z = c_0 + c_1x + c_2x^2 + \cdots,$$

which represent C in the neighborhood of the ordinary point P with coordinates $0, a_0, c_0$; the neighborhood is supposed to be sufficiently small so that the series converge. If Q is a point with coordinates h, k, l on C near P , then h, k, l must satisfy the foregoing equations when substituted in place of x, y, z , respectively:

$$k = a_0 + a_1h + a_2h^2 + \cdots, \quad l = c_0 + c_1h + c_2h^2 + \cdots.$$

If, now, h is regarded as an infinitesimal, and if we are considering problems

* Presented to the Society, June 22, 1933; received by the editors June 24, 1933.

† G. H. Halphen, *Sur les invariants différentiels des courbes gauches*, Journal de l'Ecole Polytechnique, vol. 28 (1880), p. 1.

‡ E. J. Wilczynski, *General projective theory of space curves*, these Transactions, vol. 6 (1905), p. 99.

§ E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, B. G. Teubner, 1906.

§ G. Sannia, *Nuova trattazione della geometria proiettivo-differenziale delle curve sghembe* (memoria 2), Annali di Matematica, (4), vol. 3 (1926), p. 1.

|| E. P. Lane, *On the projective differential geometry of plane curves*, Tôhoku Mathematical Journal, vol. 37 (1933), p. 423.

in which the powers of h higher than the first are negligible in comparison with the first power, we can drop all terms after the second in each series and write

$$k = a_0 + a_1 h, \quad l = c_0 + c_1 h.$$

In such a case the point Q is said to be "consecutive" to the point P . Moreover, the tangent line of the curve at Q is said to be "consecutive" to the tangent line at P , and similarly for other corresponding covariant configurations associated with the points P and Q .

For the development of the theory of consecutive configurations we need first to call to mind some fundamental facts from the projective differential geometry of a space curve. This is done in §II. In this section are described certain configurations covariantly associated with a point P of a space curve. These determine geometrically the vertices and unit point of a local coordinate system at the point P . Equations (1), when referred to this local coordinate system, reduce to the canonical form with which the section opens.

In §III we consider the configurations associated with a point Q consecutive to the point P as those of §II are associated with P . Their equations referred to the local coordinate system at the point P are determined. From these can be found the coordinates, in the same coordinate system at P , of the vertices and unit point of the local coordinate system associated with the point Q , that is, the consecutive local coordinate system. This makes it possible to set up the equations of transformation between the two local coordinate systems as is also done in the section. Lastly, the equations of the curve C referred to the consecutive coordinate system are deduced.

The application of this theory is made in §IV. There we consider three main types of problems. All of these have in common, however, the feature that the point or curve to be determined depends upon two consecutive curves, or upon two consecutive points or surfaces, as the case may be.

II. FUNDAMENTALS OF SPACE CURVE THEORY

In the theory of consecutive covariant configurations associated with a space curve, we restrict ourselves, as we have already stated, to the case of an analytic curve C . It is provable in projective differential geometry that for such a curve equations (1), when referred to a particular covariant local coordinate system, assume the simple form

$$(2) \quad \begin{aligned} y &= x^2 + ax^7 + bx^8 + cx^9 + \dots, \\ z &= x^3 + x^6 + cx^7 + dx^8 + gx^9 + \dots. \end{aligned}$$

It is the purpose of this section to recall to the reader's mind the facts which

give geometric significance to the vertices of the tetrahedron of reference and to the unit point of the particular coordinate system involved.*

We consider on the curve C the point P which is the vertex $(0, 0, 0)$ of the tetrahedron of reference. The tangent line to C at P has the equations $y = z = 0$, and the osculating plane at P has the equation $z = 0$. The equations of the osculating twisted cubic at the point P , that is, the twisted cubic having six-point contact with the curve C at P , has the equations

$$(3) \quad y = x^2, \quad z = x^3,$$

and its osculating conic at P , which by definition is the osculating conic of C at P , is given in homogeneous form by the equations

$$(4) \quad 4x_1x_3 - 3x_2^2 = x_4 = 0,$$

where

$$x_2/x_1 = x, \quad x_3/x_1 = y, \quad x_4/x_1 = z.$$

The null system of the osculating cubic has the equations

$$(5) \quad \xi_1 = y_4, \quad \xi_2 = -3y_3, \quad \xi_3 = 3y_2, \quad \xi_4 = -y_1.$$

The bundle of quadric surfaces having seven-point contact with the curve C at the point P is represented by the equation

$$\alpha(y - x^2) + \beta(y^2 - zx) + \gamma(z - xy - z^2) = 0,$$

in which the coefficients α, β, γ are arbitrary constants. These quadrics have as an eighth point of intersection, the point of Sannia, which, in homogeneous coordinates, is the point $(1, 0, 0, 1)$. In this bundle of quadrics there is one cone, called the osculating quadric cone, whose vertex is at the point P . Its equation is

$$(6) \quad y^2 - zx = 0.$$

All the quadric surfaces which pass through the osculating cubic of the curve C at the point P form another bundle with the equation

$$\alpha(y - x^2) + \beta(y^2 - zx) + \gamma(z - xy) = 0,$$

in which again α, β, γ are quite arbitrary. Two cones of this bundle are also seven-point cones. One of these is the osculating quadric cone; the other has the equation

$$(7) \quad y - x^2 = 0;$$

* For a convenient reference for the derivation of the results mentioned in this section (with the exception of the surface of Calapso) see E. P. Lane, *Projective Differential Geometry of Curves and Surfaces*, University of Chicago Press, 1932, pp. 20-25.

its vertex is at the point $(0, 0, 0, 1)$ called the *Halphen point* corresponding to the point P of the curve C .

We next recall the definitions of the *principal plane* and of the *principal point* of the tangent. Let the curve C and its osculating cubic at the point P be projected onto their common osculating plane $z=0$ from a point not on that plane. In general, the projections have six-point contact. If, however, the center of projection lies in the plane $y=0$, they have seven-point contact. This plane $y=0$ is called the *principal plane* at the point P of the curve C . The point $(0, 1, 0, 0)$, which corresponds to the principal plane in the null system of the osculating cubic given by equations (5), is the *principal point* of the tangent line at P .

The polar line of this principal point with respect to the osculating conic of the curve C at the point P meets the osculating conic in P and in the point $(0, 0, 1, 0)$. This point together with the principal point and the point of Sannia $(1, 0, 0, 1)$ determines the plane $x_1 - x_4 = 0$, one of whose three intersections with the osculating cubic is the unit point $(1, 1, 1, 1)$.

Lastly, a configuration associated with a point P of the curve C , but not assisting in the characterization of our coordinate system, is the *surface of Calapso*.^{*} It is the locus of the vertices of the six-point quadric cones at P of the curve C . Its equation, in homogeneous coordinates since that is the form in which we shall make use of it, is known to be

$$(8) \quad x_1x_4^2 - 3x_2x_3x_4 + 2x_3^3 = 0.$$

This surface is, in fact, a cubic ruled surface of the Cayley type, sometimes called a Cayley cubic scroll.

III. TRANSFORMATION OF CONSECUTIVE LOCAL COORDINATES

With every point on the curve C there may be associated a coordinate system which is related to the point in the same geometric manner as the coordinate system just described is related to the point P . In particular, there is such a coordinate system, which we shall refer to as the *consecutive coordinate system*, associated with the point Q consecutive to P . In this section we wish to find the equations of transformation between the original coordinate system at P and the consecutive one at Q .

To accomplish our purpose, we make use of a suitable auxiliary transformation of the coordinates x, y, z into new coordinates ξ, η, ζ in an auxiliary coordinate system having the point Q as origin $(0, 0, 0)$. When the equations of the curve C have been transformed to these new coordinates, we are ready

^{*} R. Calapso, *Sulle superficie gobbe di terzo grado (del tipo di Cayley) legate al punto di una data superficie*, Rendiconti dei Lincei, (6), vol. 13 (1931), p. 495.

to obtain in the coordinates ξ, η, ζ the equations of the configurations related to the point Q as those of the preceding section are related to the point P . The inverse of the auxiliary transformation already used now permits us to transform these equations back to the original coordinate system, and hence we can find the coordinates in the xyz -system of the vertices of the consecutive tetrahedron of reference and of the consecutive unit point. Since five points, no four of which are coplanar, whose coordinates in each of two systems are known, are sufficient to determine the transformation between the two systems, the transformation between the original and the consecutive coordinate systems can now be established.

We introduce first the auxiliary transformation and then apply it to the equations of the curve C . Let us impose on the new coordinate system, besides the condition that the point Q consecutive to the point P be the origin $(0, 0, 0)$, the further conditions that the line $\eta = \zeta = 0$ be the tangent to C at Q , and that the plane $\zeta = 0$ be the osculating plane. We consider what these conditions imply for the coordinates x, y, z . For a point with coordinates h, k, l near the point P on the curve C , equations (2) tell us that

$$\begin{aligned} k &= h^2 + ah^7 + bh^8 + ch^9 + \dots, \\ l &= h^3 + h^6 + ch^7 + dh^8 + gh^9 + \dots \end{aligned}$$

If, now, the point is Q , the point consecutive to P on C , so that we can neglect powers of h higher than the first, the coordinates x, y, z of Q become $h, 0, 0$. The general equations of the tangent to C at a point $(\bar{x}, \bar{y}, \bar{z})$ are

$$y - \bar{y} - (x - \bar{x})\bar{y}' = 0, \quad z - \bar{z} - (x - \bar{x})\bar{z}' = 0,$$

where \bar{y}' and \bar{z}' mean dy/dx and dz/dx , respectively, evaluated at the point $(\bar{x}, \bar{y}, \bar{z})$. Hence, the equations of the tangent to C at $Q(h, 0, 0)$ are

$$y - 2hx = z = 0,$$

since we neglect powers of h higher than the first. The general equation of the osculating plane to C at a point $(\bar{x}, \bar{y}, \bar{z})$ can be written in the form

$$z - \bar{z} - (x - \bar{x})\bar{z}' - \frac{[y - \bar{y} - (x - \bar{x})\bar{y}']\bar{z}''}{\bar{y}''} = 0.$$

For the point Q , this becomes

$$z - 3hy = 0.$$

Summarizing results, we find that *the point $(h, 0, 0)$ in the original coordinate system becomes the point $(0, 0, 0)$ in the new; the line $y - 2hx = z = 0$ in the old, becomes the line $\eta = \zeta = 0$; and the plane $z - 3hy = 0$ becomes the plane $\zeta = 0$. The*

auxiliary transformation is completely determined by these relationships; its equations are

$$(9) \quad \xi = x - h, \quad \eta = y - 2hx, \quad \zeta = z - 3hy,$$

or, in homogeneous form,

$$(10) \quad \sigma\xi_1 = x_1, \quad \sigma\xi_2 = x_2 - hx_1, \quad \sigma\xi_3 = x_3 - 2hx_2, \quad \sigma\xi_4 = x_4 - 3hx_3,$$

where σ is a proportionality factor. The inverse of the non-homogeneous form is

$$(11) \quad x = \xi + h, \quad y = 2h\xi + \eta, \quad z = 3h\eta + \zeta.$$

When equations (2) are subjected to transformation (11), we obtain the equations of C referred to the new coordinate system, namely,

$$(12) \quad \begin{aligned} \eta &= \xi^2 + 7ah\xi^6 + (a + 8bh)\xi^7 + \dots, \\ \zeta &= \xi^3 + 6h\xi^5 + (1 + 7ch)\xi^6 + [c + h(8d - 3a)]\xi^7 + \dots. \end{aligned}$$

Our next problem is to determine the covariant configurations at the point Q , by means of which the vertices of the consecutive tetrahedron of reference and the consecutive unit point may be characterized. We first consider the consecutive osculating cubic. Its parametric equations in homogeneous coordinates are

$$(13) \quad \xi_1 = 1 + 12ht^2, \quad \xi_2 = t + 6ht^3, \quad \xi_3 = t^2, \quad \xi_4 = t^3.$$

We shall verify this by showing that these equations satisfy the equations (12) of the curve C through terms in ξ^5 . By setting

$$\xi = \xi_2/\xi_1, \quad \eta = \xi_3/\xi_1, \quad \zeta = \xi_4/\xi_1,$$

we obtain the non-homogeneous form of equations (13), namely,

$$(14) \quad \xi = t - 6ht^3, \quad \eta = t^2 - 12ht^4, \quad \zeta = t^3 - 12ht^5.$$

If we invert the first of these we get

$$t = \xi + 6h\xi^3.$$

When this expression for t is substituted in the last two of equations (14), we have the non-homogeneous equations of the osculating cubic:

$$\eta = \xi^2, \quad \zeta = \xi^3 + 6h\xi^5,$$

which obviously coincide with equations (12) through terms in ξ^5 . Referred to the original coordinate system we may write the equations of the consecutive osculating cubic in the form

$$(15) \quad y = x^2, \quad z = x^3 + 6hx^5.$$

To introduce the consecutive osculating conic, let us first consider the tangent line to the cubic at a point ξ . It intersects the osculating plane $\xi_4 = 0$ of the cubic at the point Q in one point. The locus of this point of intersection as the point ξ varies over the cubic is by definition the osculating conic of the cubic at the point Q ; it is also called the osculating conic of the curve C at the point Q , or the consecutive osculating conic. Since the tangent line is determined by the points ξ and ξ' whose coordinates are given respectively by equations (13) and by the equations obtained by differentiating (13) with respect to t , its parametric equations are

$$\begin{aligned}\xi_1 &= 1 + 12ht^2 + 24h\lambda t, & \xi_2 &= t + 6ht^3 + \lambda(1 + 18ht^2), \\ \xi_3 &= t^2 + 2\lambda t, & \xi_4 &= t^3 + 3\lambda t^2.\end{aligned}$$

It meets the plane $\xi_4 = 0$ in the point with coordinates

$$\xi_1 = 1 + 4ht^2, \quad \xi_2 = 2t/3, \quad \xi_3 = t^2/3, \quad \xi_4 = 0.$$

The locus of this point as ξ varies along the cubic is found, by eliminating t from these equations, to be

$$4\xi_1\xi_3 - 48h\xi_2^2 - 3\xi_2^2 = \xi_4 = 0.$$

Making use of equations (10), we obtain the equation of the consecutive osculating conic referred to the original coordinate system:

$$(16) \quad 4x_1x_3 - 2hx_1x_2 - 48hx_3^2 - 3x_2^2 = x_4 - 3hx_3 = 0.$$

We shall next interest ourselves in the determination of the bundle of quadric surfaces having seven-point contact with the curve C at the point Q and of their eighth point of intersection, which is the consecutive point of Sannia. Let us write the general equation of the second degree in ξ, η, ζ and impose the condition that it be satisfied identically in ξ , through terms of the sixth degree, by the power series (12) for η and ζ . This gives for a general one of the seven-point quadrics, the equation

$$(17) \quad \begin{aligned}\alpha(\eta - \xi^2 - 7ah\xi^2) + \beta(\eta^2 - \xi\zeta + 6h\xi^2) \\ + \gamma[\zeta^2 + \xi\eta - \zeta + h(7c\xi^2 + 6\eta\zeta)] = 0,\end{aligned}$$

in which α, β, γ are arbitrary constants. After making use of transformation (9) we have the result that the equation of the bundle of seven-point quadrics referred to the original coordinate system is

$$\begin{aligned}\alpha(y - x^2 - 7ahz^2) + \beta[y^2 - xz + h(z + 6z^2 - xy)] \\ + \gamma[xy + z^2 - z + h(7cz^2 + 6yz - 4y - 2x^2)] = 0.\end{aligned}$$

The eighth point of intersection of all the quadrics of this bundle can be deter-

mined as the eighth intersection point of the three particular seven-point quadrics whose equations are, respectively,

$$\begin{aligned} & y - x^2 - 7ahz^2 = 0, \\ (18) \quad & y^2 - xz + h(z + 6z^2 - xy) = 0, \\ & xy + z^2 - z + h(7cz^2 + 6yz - 4y - 2x^2) = 0. \end{aligned}$$

It is easy to solve these equations if we notice that, since for $h=0$ the eighth solution must be $(0, 0, 1)$, we can suppose that our required solution will be of the form $(hm, hn, 1+hr)$. When we substitute these expressions for x, y, z in equations (18) and neglect powers of h higher than the first, we obtain simple equations which readily give $(7h, 7ah, 1-7ch)$ as the eighth solution of equations (18). In homogeneous coordinates, then, *the consecutive point of Sannia referred to the original coordinate system is the point*

$$(1, 7h, 7ah, 1 - 7ch).$$

Among the seven-point quadrics there is one cone having its vertex at the point Q . It is found by making equation (17) homogeneous and imposing the condition that the four first partial derivatives of the left member be zero at the point $(1, 0, 0, 0)$. In this way we get the conditions $\alpha = \gamma = 0$. When these values are substituted in equation (17), the equation

$$\eta^2 - \xi\zeta + 6h\zeta^2 = 0$$

is obtained. Hence, *the equation of the osculating quadric cone referred to the original coordinate system is*

$$(19) \quad y^2 - xz + h(z + 6z^2 - xy) = 0.$$

We next consider the bundle of quadrics through the consecutive osculating cubic, and determine in it the cone different from the osculating quadric cone which is also a seven-point cone. Its vertex will be the consecutive Halphen point. We write again the general equation in ξ, η, ζ of a quadric surface. This time we demand that it be identically satisfied in t by equations (14) of the consecutive osculating cubic. The result is

$$(20) \quad \alpha(\eta - \xi^2) + \beta(\eta^2 - \xi\zeta + 6h\zeta^2) + \gamma(\xi\eta - \zeta + 6h\eta\zeta) = 0,$$

wherein α, β, γ are arbitrary constants. To find the condition that one of these quadrics be also a seven-point quadric, we compare the left members of this equation and of equation (17), thus obtaining the relation

$$\gamma = 7ah\alpha$$

between the arbitrary constants. If, moreover, this quadric is to be a cone,

we have a further relation

$$\alpha\beta = -12\alpha^2h,$$

obtained by setting the discriminant of (20) equal to zero and simplifying by means of the first relation. If $\alpha=0$, then $\gamma=0$ and the cone is the osculating quadric cone, with which we are not at the moment concerned. Hence, for our desired cone we know that $\beta = -12\alpha h$, since $\alpha \neq 0$. Therefore, the equation of the cone is

$$\eta - \xi^2 + 12h(\xi\zeta - \eta^2) + 7ah(\xi\eta - \zeta) = 0,$$

or, in the coordinates x, y, z ,

$$(21) \quad y - x^2 + 12h(xz - y^2) + 7ah(xy - z) = 0.$$

To find the coordinates of its vertex we write this last equation in homogeneous form and set the four first partial derivatives of the left member equal to zero. The solution $(0, 6h, 7ah, 1)$ of the four equations so obtained is the vertex of the cone or the consecutive Halphen point. Summarizing, we can state that *the seven-point cone through the consecutive osculating cubic, which is not the osculating quadric cone, is given in the original coordinates by equation (21). Its vertex, the consecutive Halphen point, has the coordinates*

$$0, 6h, 7ah, 1.$$

Our next concern will be with the consecutive principal plane. To find it we first determine the equations of the projections of the curve C and of the consecutive osculating cubic onto the osculating plane $\zeta=0$, the center of projection being a general point not on the plane. Then we find the condition that these projected curves have seven-point contact. The equations of a line joining the point $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ of C and any point (α, β, γ) not in the plane $\zeta=0$ are

$$\xi = \alpha + (\bar{\xi} - \alpha)\rho,$$

$$\eta = \beta + (\bar{\eta} - \beta)\rho,$$

$$\zeta = \gamma + (\bar{\zeta} - \gamma)\rho,$$

where ρ is a parameter. By setting $\rho = -\gamma/(\bar{\zeta} - \gamma)$ the equations of the projection of the curve C from the point (α, β, γ) onto the plane $\zeta=0$ are found to be

$$\xi = \alpha - \gamma(\bar{\xi} - \alpha)/(\bar{\zeta} - \gamma), \quad \eta = \beta - \gamma(\bar{\eta} - \beta)/(\bar{\zeta} - \gamma).$$

Upon expanding the right members of these equations and replacing $\bar{\eta}$ and $\bar{\zeta}$ by their values in terms of $\bar{\xi}$ given by equations (12) of the curve C , we

obtain the form

$$\xi = \bar{\xi} - \alpha \bar{\xi}^3/\gamma + \bar{\xi}^4/\gamma - 6h\alpha \bar{\xi}^5/\gamma - [\alpha + \gamma(\alpha + 7ach - 6h)]\bar{\xi}^6/\gamma^2 - \dots,$$

$$\eta = \bar{\xi}^2 - \beta \bar{\xi}^3/\gamma - (6h\beta - 1)\bar{\xi}^5/\gamma - [\beta + \beta\gamma(1 + 7ch) - 7ah\gamma^2]\bar{\xi}^6/\gamma^2 - \dots.$$

When the first of these is inverted and the power series for $\bar{\xi}$ in terms of ξ so obtained is substituted for $\bar{\xi}$ in the second, the resulting equation of the projection of C onto the osculating plane $\zeta=0$ is

$$\eta = \xi^2 - \beta \xi^3/\gamma + 2\alpha \xi^4/\gamma - [(1 + 6h\beta)\gamma + 3\alpha\beta]\xi^5/\gamma^2 \\ + [7\alpha^2 + 2\beta - \beta\gamma + h(7a\gamma^2 + 12\alpha\gamma - 7c\beta\gamma)]\xi^6/\gamma^2 + \dots.$$

By a similar procedure, the projection of the osculating cubic onto the plane $\zeta=0$ is found. It proves to be the same as the projection of C through terms of the sixth degree in ξ , except that in the coefficient of ξ^6 the terms

$$[-\beta\gamma + h(7a\gamma^2 - 7c\beta\gamma)]/\gamma^2$$

are missing. Hence, if the two projections are to have seven-point contact, this expression must vanish. This means that

$$\beta + 7ch\beta - 7ah\gamma$$

must equal zero, for we know $\gamma \neq 0$ since the center of projection does not lie in the plane $\zeta=0$. This relation between the coordinates β and γ merely implies that the center of projection must lie in the plane with the equation

$$(1 + 7ch)\eta - 7ah\zeta = 0.$$

This plane is by definition the consecutive principal plane. After applying transformation (9), we have the equation of the consecutive principal plane in the coordinates x, y, z , namely,

$$2hx - (1 + 7ch)y + 7ahz = 0.$$

As the consecutive principal point is the point corresponding to the consecutive principal plane in the null system of the consecutive osculating cubic, it will now be necessary to determine this null system. The osculating plane at any point ξ of the cubic is found, by differentiating (13) twice and writing the equation of the plane determined by ξ, ξ', ξ'' , to be

$$(22) \quad t^3\xi_1 - 3t^2\xi_2 + (3t - 12ht^3)\xi_3 - (1 - 18ht^2)\xi_4 = 0,$$

where t is the value of the parameter corresponding to the particular ξ chosen. Since this is a cubic equation in t there are in general three values of t which will satisfy it for any arbitrarily chosen values $\xi_1, \xi_2, \xi_3, \xi_4$. Therefore, through

any point η of S_3 there pass three osculating planes of the osculating cubic. If we let t_1, t_2, t_3 be the parametric values corresponding to the three points of osculation, we can easily write the equation of the plane determined by these points. When it is simplified by means of the values obtained from equation (22) of the elementary symmetric functions of t_1, t_2, t_3 , it takes the form

$$\eta_4 \xi_1 - 3\eta_3 \xi_2 + (3\eta_2 - 30h\eta_4)\xi_3 - (\eta_1 - 30h\eta_3)\xi_4 = 0.$$

By applying transformation (10) we reach the corresponding equation in the old coordinates, namely,

$$y_4 x_1 - 3y_3 x_2 + (3y_2 - 30hy_4)x_3 - (y_1 - 30hy_3)x_4 = 0.$$

This shows us that *the equations of the null system of the osculating cubic are*

$$\xi_1 = y_4, \quad \xi_2 = -3y_3, \quad \xi_3 = 3y_2 - 30hy_4, \quad \xi_4 = -y_1 + 30hy_3,$$

where ξ_1, \dots, ξ_4 are now the coordinates of the plane corresponding to the point with the coordinates y . From these equations it is easy to derive the result that *the coordinates of the consecutive principal point referred to the original system are*

$$21ah, 1 + 7ch, 2h, 0.$$

There is one more vertex of the consecutive tetrahedron of reference whose coordinates relative to the xyz -system are to be determined. The polar line of the point $(21ah, 1+7ch, 2h, 0)$ with respect to the conic given by equation (16) has the equation

$$hx_1 - x_2 + 14ahx_3 = 0.$$

Solution of this equation with (16) gives

$$12h, 14ah, 1, 3h,$$

as the coordinates referred to the original system of the point distinct from the point Q , in which the polar line of the consecutive principal point with respect to the consecutive osculating conic meets the conic. This is the required vertex.

Finally, in order to determine the equations of transformation between the original coordinate system and the consecutive one, it is sufficient, with the information we already have, to know the coordinates in the first system of the unit point. The plane determined by the consecutive point of Sannia, the consecutive principal point, and the point distinct from the point Q in which the polar line of the consecutive principal point with respect to the consecutive osculating conic meets the conic is found to be

$$x_1 - 21ahx_2 - 9hx_3 - (1 + 7ch)x_4 = 0,$$

or in non-homogeneous coordinates,

$$1 - 21ahx - 9hy - (1 + 7ch)z = 0.$$

Solution of this equation with equations (15) for the consecutive osculating cubic is made simple by assuming the value of x to be of the form $1 + rh$, an assumption which is permissible since we already know the required solution has $x=1$ for the case $h=0$. When the result is expressed in homogeneous coordinates we have for the consecutive unit point referred to the original coordinate system the point

$$[1, 1 - h(5 + 7a + 7c/3), 1 - h(10 + 14a + 14c/3), 1 - h(9 + 21a + 7c)].$$

We are now ready to derive the equations of transformation between the coordinates x_1, x_2, x_3, x_4 in the original system and the coordinates, which we shall denote by X_1, X_2, X_3, X_4 , in the consecutive system, since we have found the coordinates, in each of the two systems, of five points no four of which are coplanar, namely, the four vertices of the consecutive tetrahedron and its unit point. We write the general linear equations of transformation of the coordinates x into the coordinates X . Substitution in these equations of the coordinates of each of the five pairs of corresponding points yields twenty equations homogeneous in the sixteen constants of the transformation and in five proportionality factors. When the coefficients of the transformation are determined from these equations, we obtain, after simplification, the following result:

The equations of transformation from the original system to the consecutive system are

$$\begin{aligned} \rho x_1 &= X_1 + 21ahX_2 + 12hX_3, \\ (23) \quad \rho x_2 &= hX_1 + (1 - 7ch/3)X_2 + 14ahX_3 + 6hX_4, \\ \rho x_3 &= 2hX_2 + (1 - 14ch/3)X_3 + 7ahX_4, \\ \rho x_4 &= 3hX_3 + (1 - 7ch)X_4, \end{aligned}$$

where ρ is a proportionality factor.

In non-homogeneous coordinates they become

$$\begin{aligned} x &= X + h(1 - 7cX/3 + 14aY + 6Z - 21aX^2 - 12XY), \\ (24) \quad y &= Y + h(2X - 14cY/3 + 7aZ - 21aXY - 12Y^2), \\ z &= Z + h(3Y - 7cZ - 21aXZ - 12YZ). \end{aligned}$$

By interchanging x_1, x_2, x_3, x_4 with X_1, X_2, X_3, X_4 , respectively, in equations (23), and changing the sign of h , the inverse transformation in homogeneous coordinates is readily found to be given by the equations

$$\begin{aligned}
 \sigma X_1 &= x_1 - 21ahx_2 - 12hx_3, \\
 \sigma X_2 &= -hx_1 + (1 + 7ch/3)x_2 - 14ahx_3 - 6hx_4, \\
 \sigma X_3 &= -2hx_2 + (1 + 14ch/3)x_3 - 7ahx_4, \\
 \sigma X_4 &= -3hx_3 + (1 + 7ch)x_4,
 \end{aligned}
 \tag{25}$$

where σ is a factor of proportionality.

It is now possible to determine the equations of the curve C referred to the consecutive coordinate system. In equations (2) we substitute for the variables x, y, z their values in terms of X, Y, Z as given by equations (24). In the resulting equations we replace Y and Z by power series in X with undetermined coefficients. Since we now have two identities in X , the coefficient of each power of X can be equated to zero. Solution of the equations so obtained, for the coefficients of the power series representing Y and Z , permits us to write for the equations of C in the coordinates X, Y, Z ,

$$Y = X^2 + AX^7 + BX^8 + \dots, \quad Z = X^3 + X^8 + CX^7 + DX^8 + \dots,$$

where A, B, C, D are defined by

$$\begin{aligned}
 A &= a + h(12 + 8b - 56ac/3), \\
 B &= b + h(12c - 7ad - 14bc + 9e), \\
 C &= c + h(8d - 24a - 28c^2/3), \\
 D &= d + h(9g - 28ac - 6 - 35cd/3 - 3b).
 \end{aligned}
 \tag{26}$$

By means of transformation (25) and the relations (26) we can now find the equation, or equations, of the locus consecutive to a given locus, not only from the geometric definition of the locus, as we have been doing throughout this section, but also by a more direct method. We write equations identical with the homogeneous form of those of the given locus, except that x_1, x_2, x_3, x_4 are replaced by X_1, X_2, X_3, X_4 , and a, b, c, d, \dots by A, B, C, D, \dots , and then apply transformations (25) and (26). It is obvious that the method to be used to obtain the point consecutive to one whose coordinates are known, is to substitute the given coordinates in equations (23).

IV. APPLICATIONS OF THE TRANSFORMATION

§IV is concerned with some applications which can be made of the results obtained in the preceding section. The problems we shall consider are of three types. The first of these is the determination of the tangents of the loci of various covariant points associated with a point P on the curve C . We have already found the consecutive points to a number of covariant points, as, for example, the vertices of the original tetrahedron of reference. Each covariant

point with its consecutive point determines a line which is the tangent at the point to its locus as the point P moves along C . Such covariant curves as intersect their consecutive curves have envelopes generated by the intersection points. The envelopes are the edges of regression of the surfaces generated by the curves, and the intersection points are the focal points on the edges of regression. Our second problem is to find out which covariant curves that we have discussed have envelopes and to determine their contact or focal points. In the third kind of problem we determine the characteristic curve of a covariant surface, that is, the curve of intersection of a surface with its consecutive surface. We go yet farther than this and find the focal point of the edge of regression of the envelope of the surface, that is, the point in which a characteristic curve intersects its consecutive curve.

We turn now to the determination of the tangents of the loci of some covariant points. Obviously, the locus of the point $(1, 0, 0, 0)$ is the curve C itself and its tangent at any point is the tangent to C at that point. The points consecutive to the other three vertices

$$(0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$$

of the tetrahedron of reference are, respectively,

$$(21ah, 1 + 7ch, 2h, 0), (12h, 14ah, 1, 3h), (0, 6h, 7ah, 1).$$

Hence we find for the tangents to the respective loci of the points $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, the lines whose equations are

$$\begin{aligned} 2x_1 - 21ax_3 &= x_4 = 0, \\ 7ax_1 - 6x_2 &= x_1 - 4x_4 = 0, \\ 7ax_2 - 6x_3 &= x_1 = 0. \end{aligned}$$

The point

$$[1, 1 - h(5 + 7a + 7c/3), 1 - h(10 + 14a + 14c/3), 1 - h(9 + 21a + 7c)]$$

is consecutive to the unit point $(1, 1, 1, 1)$; together these points determine, as tangent of the locus of the unit point, the line

$$x_1 - 2x_2 + x_3 = (2R - 6)x_1 - (3R - 6)x_2 + Rx_4 = 0,$$

where R is defined by

$$R = 5 + 7a + 7c/3.$$

The point consecutive to the point of Sannia $(1, 0, 0, 1)$ is the point $(1, 7h, 7ah, 1 - 7ch)$. This gives us that the tangent to the locus of the point $(1, 0, 0, 1)$ is the line

$$ax_2 - x_3 = ax_1 - cx_3 - ax_4 = 0.$$

As an example of the second type of problem we consider first the case of the osculating conic, whose equations are given by (4). Solution of these equations with equations (16) of the consecutive osculating conic gives the point $(1, 0, 0, 0)$ for the only intersection of the two curves. Therefore, *the osculating conic has no envelope but the curve C itself*. The case of the osculating cubic proves more fruitful. If we make equations (3) and (15) homogeneous and solve them simultaneously we find that *the contact points of the osculating cubic with its envelope are the points $(1, 0, 0, 0)$ and $(0, 0, 0, 1)$* .

We now investigate the situation for some covariant lines. It is obvious that the envelope of the tangent line is the curve C. Let us consider some other edge of the tetrahedron of reference, as the line $x_1 = x_2 = 0$. Its consecutive line is $X_1 = X_2 = 0$, or by applying (25),

$$x_1 - 21ahx_2 - 12hx_3 = -hx_1 + (1 + 7ch/3)x_2 - 14ahx_3 - 6hx_4 = 0.$$

There is no common solution of these four equations, and hence the line $x_1 = x_2 = 0$ does not intersect its consecutive line. In a similar way we find that the other edges of the tetrahedron of reference are skew to the corresponding edges of the consecutive tetrahedron. There are many other covariant lines we might consider, for example, the lines joining the point of Sannia to the vertices $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$, and to the unit point, and the lines joining the unit point to the vertices of the tetrahedron. In each case we write the equations of the line determined by the two points considered, then write identical equations only with coordinates X instead of coordinates x and apply transformation (25). In every instance we find that the four equations so obtained have no common solution. We may sum up these findings as follows:

The tangent line has the curve C for envelope, but the other edges of the tetrahedron of reference, the lines joining the point of Sannia or the unit point to a vertex of the tetrahedron, and the line joining the point of Sannia with the unit point do not generate developable surfaces and hence have no envelopes.

Our third kind of problem is concerned with the characteristic curves and edges of regression of the envelopes of covariant surfaces. Let us take first the plane $x_1 = 0$. The equation of the consecutive plane is $X_1 = 0$ or, after making use of (25),

$$x_1 - 21ahx_2 - 12hx_3 = 0.$$

The characteristic line of the plane $x_1 = 0$ is then the intersection of these two planes, which is the same as the intersection of the two planes

$$(27) \quad x_1 = 0, \quad 7ax_2 + 4x_3 = 0.$$

To find the point at which the plane $x_1=0$ touches the edge of regression we must know the consecutive characteristic line. Since the characteristic line is a covariant configuration and is determined by the two planes (27), its consecutive characteristic line is determined by the two consecutive planes

$$X_1 = 0, \quad 7AX_2 + 4X_3 = 0.$$

The first of these gives us nothing new, but when the second has been subjected to transformations (25) and (26) we find that the point where the plane $x_1=0$ touches the edge of regression is determined by the system of planes with the equations

$$\begin{aligned} x_1 &= 0, & 7ax_2 + 4x_3 &= 0, \\ (76 + 56b - 147ac)x_2 - 98a^2x_3 - 70ax_4 &= 0. \end{aligned}$$

After solving these equations, we state our conclusions: *the characteristic line of the plane $x_1=0$ has the equations (27); the focal point of the edge of regression of the developable of this plane has the coordinates*

$$0, 140a, -245a^2, M,$$

where M is defined by

$$(28) \quad M = 152 + 112b - 294ac + 343a^3.$$

In a similar way we find the characteristic lines and points of contact with the edges of regression of the developables of other covariant planes. *The characteristic lines of the planes $x_2=0$, $x_3=0$, $x_4=0$, and of the plane $x_1-x_4=0$ determined by the point of Sannia and the two vertices $(0, 0, 1, 0)$ and $(0, 1, 0, 0)$ have, respectively, the equations*

$$\begin{aligned} x_2 &= x_1 + 14ax_3 + 6x_4 = 0, & x_3 &= 2x_2 + 7ax_4 = 0, & x_3 &= x_4 = 0, \\ x_1 - x_4 &= 21ax_2 + 9x_3 + 7cx_4 = 0. \end{aligned}$$

The respective focal points of the developables of these planes are

$$\begin{aligned} &(3M - 42 - 343a^3, 0, 21c - 49a^2, N), \\ &(3 - N, -7a, 0, 2), & (1, 0, 0, 0), \\ &(3M - S, 49c^2 - 343a^2c - 84d + 567a, 196ad - 1323a^2 - 273c \\ &\quad - 196bc + 343a^2, 3M - S), \end{aligned}$$

where M is as defined in equation (28) and N and S are given by

$$N = 98ac - 56b - 69, \quad S = 105 + 84b - 294ac.$$

The characteristic curve of the osculating quadric cone is the curve of

intersection of the two quadric surfaces whose non-homogeneous equations are (6) and (19). This is equivalent to the curve of intersection of the first surface and the surface with homogeneous equation

$$x_2x_3 - x_1x_4 - 6x_1^2 = 0;$$

it is composed of the tangent line $x_3 = x_4 = 0$ and the cubic curve

$$(29) \quad x_1 = t^3 - 6, \quad x_2 = t^2, \quad x_3 = t, \quad x_4 = 1.$$

Hence, we state the conclusion: *the characteristic curve of the osculating quadric cone of the curve C at the point P consists of the tangent line to the curve C at the point P and the cubic (29). This cubic has no contact with the curve C at the point P. The cone has seven-point contact with its edge of regression at (1, 0, 0, 0) and touches it again at the point*

$$(343c^3 - 750, 245c^2, 175c, 125).$$

From equations (7) and (21) made homogeneous we find that the intersection of the two quadric surfaces

$$x_1x_3 - x_2^2 = 0, \quad 12(x_2x_4 - x_3^2) - 7a(x_1x_4 - x_2x_3) = 0$$

forms the characteristic of the seven-point cone with vertex at the Halphen point. As we should expect since this cone also contains the osculating cubic, the osculating cubic makes up a part of the characteristic, the remainder being the line whose equations are

$$(30) \quad 7ax_1 - 12x_2 = 0, \quad 7ax_2 - 12x_3 = 0.$$

So we state the following result:

The characteristic of the seven-point cone with vertex at the Halphen point corresponding to the point P of the curve C consists of the osculating cubic to C at P and of the line with equations (30). This cone has five-point contact with its edge of regression at (1, 0, 0, 0), two-point contact at

$$(1728, 1008a, 588a^2, 343a^3),$$

and single contact at

$$(1728, 1008a, 588a^2, 1372a^3 - 24T),$$

where T is defined by

$$T = 96 + 56b - 147ac.$$

The characteristic curve of the surface of Calapso whose equation is

given by (8) is the intersection of the two cubic surfaces

$$\begin{aligned}x_1x_4^3 - 3x_2x_3x_4 + 2x_3^3 &= 0, \\2x_2^3x_4 - x_2x_3^3 + 2x_3x_4^3 - x_1x_3x_4 &= 0;\end{aligned}$$

it, also, includes the tangent line $x_3 = x_4 = 0$ as a part.

A second procedure for determining characteristic curves is suggested by the fact, which we recall from differential geometry, that the characteristic curve of a surface belonging to a one-parameter family has for its equations the equation of the surface and the derivative of that equation with respect to the parameter. To use this method we see at once that we need differentiation formulas for x_1, x_2, x_3, x_4 with respect to the parameter h . These are deduced from equations (25) in which σ is taken to be 1. In each equation the term on the right which is free of h is transposed to the left; then both members of the equation are divided by h . Taking the limits as h approaches 0, we obtain the formulas of differentiation for x_1, x_2, x_3, x_4 with respect to h :

$$\begin{aligned}(31) \quad x_1' &= -21ax_2 - 12x_3, \\x_2' &= -x_1 + 7cx_3/3 - 14ax_3 - 6x_4, \\x_3' &= -2x_2 + 14cx_3/3 - 7ax_4, \\x_4' &= -3x_3 + 7cx_4,\end{aligned}$$

where x_i' denotes the derivative of x_i with respect to h , and so on. As an instance of the application of these formulas to the solution of our problem we shall consider the osculating quadric cone with equation

$$x_3^2 - x_2x_4 = 0.$$

Differentiation of this equation gives

$$2x_3x_3' - x_2x_4' - x_2'x_4 = 0.$$

Substitution from (31), followed by simplification by means of the equation of the osculating quadric cone, corroborates our conclusion of a previous paragraph, that the characteristic curve of this quadric is its intersection with the quadric $x_1x_4 - x_2x_3 + 6x_4^2 = 0$.

Finally, we mention the cross ratio of four lines in the plane $x_1 = 0$ through the Halphen point $(0, 0, 0, 1)$. The two edges of the tetrahedron of reference

$$x_1 = x_2 = 0, \quad x_1 = x_3 = 0,$$

the tangent to the locus of the point $(0, 0, 0, 1)$ with equations

$$x_1 = 7ax_2 - 6x_3 = 0,$$

and the characteristic line of the plane $x_1=0$ with equations

$$x_1 = 7ax_2 + 4x_3 = 0,$$

have cross ratio equal to $-3/2$.

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ANALYTIC EXTENSIONS OF DIFFERENTIABLE FUNCTIONS DEFINED IN CLOSED SETS*

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I. DIFFERENTIABLE FUNCTIONS IN CLOSED SETS

1. Introduction. Let A be a closed set, bounded or unbounded, in euclidean n -space E , and let $f(x)$ be a function defined and continuous in A . It is well known that this function can be extended so as to be continuous throughout E .‡ If A satisfies certain conditions, the solution of the Dirichlet problem is a function harmonic in $E-A$ and taking on the given boundary values in A . Two questions which arise are the following: Is there always a function differentiable, or perhaps analytic, in $E-A$, and taking on the given values in A ? If the given function $f(x)$ is in some sense differentiable in A , can the extension $F(x)$ be made differentiable to the same order throughout E ?

These questions are answered in the affirmative in Theorem I. We use a definition of the derivatives of a function in a general set which arises naturally from a consideration of Taylor's formula. In Part II, a differentiable extension of $f(x)$ is found, whether $f(x)$ is differentiable to finite or infinite order. Part III is devoted to some general approximation theorems. It is well known that a continuous function in a bounded closed set can be approximated uniformly (together with any finite number of derivatives) by polynomials; we show that functions defined in *open* sets may be approximated (together with derivatives) by *analytic* functions, the approximation being closer and closer as we approach the boundary of the set. This theorem, together with the results of Part II, furnish an immediate proof of Theorem I. In Part IV we give some extensions of Theorem I; in particular, we show that

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‡ See references in a paper by P. Urysohn, *Mathematische Annalen*, vol. 94 (1925), p. 293, footnote 51.

A continuous extension the author has not seen in the literature may be given as follows; we assume for simplicity that A is bounded. Let $h(r)$ ($r \geq 0$) be a continuous and monotone increasing function such that $h(0) = 0$, and if x and y are any two points of A whose distance apart is r_{xy} , then $|f(x) - f(y)| \leq h(r_{xy})$. For any points x of E and y of A , set $H(x, y) = f(y) - h(r_{xy})$; then if x is in A , $H(x, y) \leq f(x)$. The continuous extension of $f(x)$ is $F(x)$, which at each point x of E equals the maximum of $H(x, y)$ as y varies over A .

the extension of $f(x)$ may be made analytic at the isolated points of A . Theorem III includes all preceding results but Lemma 7.

2. Notations. We shall write all equations involving n variables as if there were but a single variable present. For instance, we write

$$f_0(x) \quad \text{for} \quad f_0 \dots 0(x_1, \dots, x_n),$$

$$D_k f(x') \quad \text{for} \quad \frac{\partial^{k_1+\dots+k_n}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} f(x'_1, \dots, x'_n),$$

$$\binom{k}{l} \quad \text{for} \quad \binom{k_1}{l_1} \dots \binom{k_n}{l_n},$$

etc. For any n -fold subscript k , we put

$$\sigma_k = k_1 + \dots + k_n.$$

Note that $\sigma_{k+l} = \sigma_k + \sigma_l$. r_{xy} will always denote the distance between x and y (unless x and y are complex). As an example, (3.1) below is short for

$$f_{k_1 \dots k_n}(x'_1, \dots, x'_n) = \sum_{\substack{l_1+\dots+l_n \\ \leq m-(k_1+\dots+k_n)}} \frac{f_{k_1+l_1, \dots, k_n+l_n}(x_1, \dots, x_n)}{l_1! \dots l_n!} (x'_1 - x_1)^{l_1} \dots (x'_n - x_n)^{l_n} \\ + R_{k_1 \dots k_n}(x'_1, \dots, x'_n; x_1, \dots, x_n).$$

3. Differentiable functions in subsets of E . Let $f(x)$ be defined in the set A , and let m be an integer ≥ 0 . We say $f(x) = f_0(x)$ is of class C^m in A in terms of the functions $f_k(x)$ ($\sigma_k \leq m$) if the functions $f_k(x)$ are defined in A for all n -fold subscripts k with $\sigma_k \leq m$, and

$$(3.1) \quad f_k(x') = \sum_{\sigma_l \leq m - \sigma_k} \frac{f_{k+l}(x)}{l!} (x' - x)^l + R_k(x'; x)$$

for each $f_k(x)$ ($\sigma_k \leq m$), where $R_k(x'; x)$ has the following property. Given any point x^0 of A and any $\epsilon > 0$, there is a $\delta > 0$ such that if x and x' are any two points of A with $r_{xx^0} < \delta$ and $r_{x'x^0} < \delta$, then

$$(3.2) \quad |R_k(x'; x)| \leq r_{xx'}^{m-\sigma_k} \epsilon.$$

One might define the derivatives of a function at the points of a set B , when the function is defined in a larger set A . We shall not do this here.

If $m=0$, (3.1) and (3.2) state merely that $f(x)$ is continuous. Note that the conditions are satisfied automatically at all isolated points of A , no matter how the $f_k(x)$ are defined there.

It is easily seen that the $f_k(x)$ are continuous in a neighborhood of each point of A , and are thus bounded there. From this we prove that if $f(x)$ is of class C^m in A in terms of the $f_k(x)$ ($\sigma_k \leq m$), then it is of class $C^{m'}$ in A ($m' < m$) in terms of the $f_k(x)$ ($\sigma_k \leq m'$).

Any function we shall say is of class C^{-1} in A . $f(x)$ is of class C^∞ in A in terms of the $f_k(x)$ (defined for all k) if it is of class C^m in A in terms of the $f_k(x)$ ($\sigma_k \leq m$) for each m .

Suppose $f(x)$ is defined throughout the region R , and is of class C^m in terms of the $f_k(x)$ ($\sigma_k \leq m$). Then putting $x = (x_1, \dots, x_n)$, $x' = (x_1, \dots, x_h + \Delta x_h, \dots, x_n)$, (3.1) gives

$$(3.3) \quad f_{k_1 \dots k_n}(x') = f_{k_1 \dots k_n}(x) + f_{k_1, \dots, k_h+1, \dots, k_n}(x) \Delta x_h \\ + R_{k_1 \dots k_n}^{(h)}(x'; x)$$

(provided $\sigma_k < m$), where $R_{k_1 \dots k_n}^{(h)}(x'; x)/\Delta x_h \rightarrow 0$ as $\Delta x_h \rightarrow 0$, which shows that

$$(3.4) \quad \frac{\partial}{\partial x_h} f_{k_1 \dots k_n}(x) = f_{k_1, \dots, k_h+1, \dots, k_n}(x) \quad (\sigma_k < m)$$

in R ; thus in this case, $f(x)$ is of class C^m in the ordinary sense, and the $f_k(x)$ are the partial derivatives of $f(x)$. The converse is true, by Taylor's Theorem.

4. The main theorem of the present paper is the following:

THEOREM I. *Let A be a closed subset of E , and let $f(x) = f_0(x)$ be of class C^m (m finite or infinite) in A in terms of the $f_k(x)$ ($\sigma_k \leq m$). Then there is a function $F(x)$ of class C^m in E in the ordinary sense, such that*

- (1) $F(x) = f(x)$ in A ,
- (2) $D_k F(x) = f_k(x)$ in A ($\sigma_k \leq m$),
- (3) $F(x)$ is analytic in $E - A$.†

Of course (2) includes (1).

No such theorem holds if we leave out the uniformity condition on $R_k(x'; x)$, i.e. if we assume merely that for any x and $\epsilon > 0$ there is a $\delta > 0$ such that if $r_{xx'} < \delta$, then $|R_k(x'; x)| < r_{xx'}^{m-\sigma_k} \epsilon$. The following example shows this. Let A be the set of points (using one variable) $x = 0, 1/2^s$ and $1/2^s + 1/2^{2s}$ ($s = 1, 2, \dots$). Set $f(x) = 0$ at $x = 0$ and $1/2^s$, and $f(x) = 1/2^{2s}$ at the remaining points. Set $f_1(x) \equiv 0$ in A . The above condition is satisfied, but there is no extension of $f(x)$ which has a continuous first derivative.

5. The following lemma will be needed; its proof is elementary.

† It is seen from the proof in §16 that $F(x)$ is analytic in a complex region with the following property. If x is a point of $E - A$ distant 3ρ from A , then the region contains all points within a distance ρ of x .

LEMMA 1. Let $w(z)$ be a continuous function of one variable defined throughout an interval containing z_0 , let A^* be a closed set in this interval, and let w'_0 be a fixed number. Suppose that for every $\epsilon > 0$ there is a $\delta > 0$ such that

(1) if z is in A^* and $|z - z_0| < \delta$, then

$$|w(z) - w(z_0)/(z - z_0) - w'_0| < \epsilon;$$

(2) if z is not in A^* and $|z - z_0| < \delta$, then the derivative $w'(z)$ exists and

$$|w'(z) - w'_0| < \epsilon.$$

Then $w(z)$ has a derivative at z_0 , and $w'(z_0) = w'_0$.

II. DIFFERENTIABLE EXTENSIONS

6. The functions $\psi_k(x'; x)$. We shall make use of functions defined as follows for x in A and x' in E (m finite):

$$(6.1) \quad \psi_k(x'; x) = \sum_{\sigma \leq m - \sigma_k} \frac{f_{k+\sigma}(x)}{\sigma!} (x' - x)^\sigma \quad (\sigma_k \leq m);$$

$\psi_k(x'; x)$ is the value at x' of the polynomial of degree at most $m - \sigma_k$ which approximates the function $f_k(x)$ to the $(m - \sigma_k)$ th order at x . Keeping x fixed, it is a polynomial in x' , given by Taylor's formula in terms of its value and derivatives at x . In terms of these functions, (3.1) becomes

$$(6.2) \quad f_k(x') = \psi_k(x'; x) + R_k(x'; x) \quad (\sigma_k \leq m).$$

The l th derivative of the function of x' $\psi_k(x'; x)$ at x' is $\psi_{k+l}(x'; x)$; if we express $\psi_k(x''; x)$ by Taylor's formula in terms of its value and derivatives at x' , we obtain

$$\begin{aligned} \psi_k(x''; x) &= \sum_l \frac{\psi_{k+l}(x'; x)}{l!} (x'' - x')^l \\ &= \sum_l \frac{(x'' - x')^l}{l!} \sum_j \frac{f_{k+l+j}(x)}{j!} (x' - x)^j. \end{aligned}$$

The definition of $\psi_k(x''; x')$ in conjunction with this identity gives, for any points x and x' in A and x'' in E ,

$$\begin{aligned} \psi_k(x''; x') &= \sum_l \frac{f_{k+l}(x')}{l!} (x'' - x')^l \\ &= \sum_l \frac{(x'' - x')^l}{l!} \left[\sum_j \frac{f_{k+l+j}(x)}{j!} (x' - x)^j + R_{k+l}(x'; x) \right] \\ &= \psi_k(x''; x) + \sum_l \frac{R_{k+l}(x'; x)}{l!} (x'' - x')^l. \end{aligned} \quad (6.3)$$

7. The function $\Theta(x)$. Let R be the region given by the inequalities $|x_h| < 1$ ($h=1, \dots, n$), let R' be R minus the origin, and let R^* be the boundary of R . Define the functions θ, θ', Θ as follows:

$$(7.1) \quad \theta(x) = 2(1 - x_1^2) \cdots (1 - x_n^2) - 1 \text{ in } R',$$

$$(7.2) \quad \theta'(x) = \frac{\theta(x)}{1 - \theta^2(x)} \quad \text{in } R',$$

$$(7.3) \quad \Theta(x) = \begin{cases} e^{\theta'(x)} & \text{in } R', \\ 0 & \text{in } E - R. \end{cases}$$

It is seen that $-1 < \theta(x) < +1$, $\theta(x) \rightarrow +1$ as $x \rightarrow 0$, and $\theta(x) \rightarrow -1$ as $x \rightarrow R^*$; hence $\theta'(x) \rightarrow +\infty$ as $x \rightarrow 0$ and $\theta'(x) \rightarrow -\infty$ as $x \rightarrow R^*$. Consequently $\Theta(x) \rightarrow +\infty$ to infinite order as $x \rightarrow 0$ and $\Theta(x) \rightarrow 0$ to infinite order as $x \rightarrow R^*$; also $\Theta(x)$ is of class C^∞ for $x \neq 0$. If $\Theta'(x) = 1/\Theta(x)$ in R' and $\Theta'(x) = 0$ for $x = 0$, then $\Theta'(x)$ is of class C^∞ in R .

8. The subdivision of $E - A$. Divide E into n -cubes of side 1, and let K_0 be the set of all these cubes whose distances from A are at least $6n^{1/2}$ (if there are any). In general, having constructed the cubes of K_{s-1} , divide each cube which is now present but is not in $K_0 + \dots + K_{s-1}$ into 2^s cubes of side $1/2^s$, and let K_s be the set of all these cubes whose distances from A are at least $6n^{1/2}/2^s$ (if there are any).

The distance from any cube C of K_s to A is $< 18n^{1/2}/2^s$ ($s \geq 1$); for it lies in a cube C' of the previous subdivision which does not belong to K_{s-1} , and whose distance from A is therefore $< 6n^{1/2}/2^{s-1}$.

Any cube C of K_s is separated from any cube C' of K_{s+2} by at least four cubes of K_{s+1} . For the distance from C to A is $\geq 12n^{1/2}/2^{s+1}$, the distance from any point of C' to A is $< 9n^{1/2}/2^{s+1}$, and the diameter of any cube of K_{s+1} is $n^{1/2}/2^{s+1}$.

9. The functions $\phi_\nu(x)$. We introduce the following definitions:

y^1, y^2, \dots is the set of all vertices of cubes of $K_0 + K_1 + \dots$, arranged in a sequence.

r_ν is the distance from y_ν to A ($\nu = 1, 2, \dots$).

x^* is a fixed point of A whose distance from y^* is r_* .

b_* is the length of side of the largest cube of $K_0 + K_1 + \dots$ with y^* as a vertex.

I_* is the set of points x for which $|x_h - y_h^*| \leq b_*$ ($h = 1, \dots, n$); B_* is its boundary.

$$\begin{aligned}\pi_\nu(x) &= \Theta\left(\frac{x_1 - y_1^\nu}{b_\nu}, \dots, \frac{x_n - y_n^\nu}{b_\nu}\right) \text{ in } E - y^\nu; \\ \pi_\nu'(x) &= \Theta\left(\frac{x_1 - y_1^\nu}{b_\nu}, \dots, \frac{x_n - y_n^\nu}{b_\nu}\right) \text{ in } I_\nu - B_\nu. \\ \phi_\nu(x) &= \begin{cases} \frac{\pi_\nu(x)}{\sum_\lambda \pi_\lambda(x)} & \text{in } E - A, x \neq y^1, y^2, \dots, \\ 1, & x = y^\nu, \\ 0, & x = y^\mu (\mu \neq \nu). \end{cases}\end{aligned}$$

Suppose y^* is a given point of $E - A$, distant δ_* from A (or from a given point x^0 of A), and suppose y^* lies in the cube C of K_* . Then if I_ν , with center y^ν , has points in common with C , and y^ν is distant d_ν from A (or from x^0),

$$(9.1) \quad \delta_*/2 \leq d_\nu < 2\delta_*.$$

To prove this, say C' is a largest cube with y^ν as a vertex, and C' is in K_* ; then $t \geq s - 1$. The diameter of C' is $n^{1/2}/2^t$; hence y^ν is distant at most $n^{1/2}/2^t \leq 2n^{1/2}/2^s$ from any point of I_ν . As the diameter of C is $n^{1/2}/2^s$, y^ν is distant at most $3n^{1/2}/2^s$ from y^* . But $\delta_* \geq 6n^{1/2}/2^s$, and the inequalities follow.

Each function $\pi_\nu(x)$ is >0 in $I_\nu - B_\nu - y^\nu$ and only there; it approaches ∞ and 0 to infinite order as x approaches y^ν and B_ν , respectively. Each point x of $E - A$ is interior to some cube I_ν , hence $\pi_\nu(x) > 0$ for some ν , and $\sum \pi_\lambda(x) > 0$ in $E - A$, justifying the definition of $\phi_\nu(x)$. Note that $\phi_\nu(x)$ is $\neq 0$ in $I_\nu - B_\nu$ and only there; also

$$(9.2) \quad \sum_\nu \phi_\nu(x) = 1 \text{ in } E - A.$$

We shall show that $\phi_\nu(x)$ is of class C^∞ in $E - A$. This is obvious at points $x \neq y^\nu$. Consider a small neighborhood U_λ of y^λ , $\lambda \neq \nu$. $\pi_\lambda'(x)$ is of class C^∞ in U_λ ; hence the same is true of $\phi_\nu = \pi_\lambda' \pi_\nu / (1 + \pi_\lambda' \sum_{\mu \neq \lambda} \pi_\mu)$ in U_λ . Similarly $\phi_\nu = 1 / (1 + \pi_\nu' \sum_{\mu \neq \nu} \pi_\mu)$ is of class C^∞ in a small neighborhood U_ν of y^ν ; the statement follows.

10. The derivatives of the $\phi_\nu(x)$. Consider two (closed) cubes C and C' of $K_0 + K_1 + \dots$, and let J and J' be those sets I_ν with points in C and C' respectively. We shall say C and C' are of the same type if the sets in J' can be brought into coincidence with the sets in J by a translation and stretching of the axes, that is, if the structure of the subdivision about C' is the same as that about C . There are but a finite number, say d , of possible types of cubes, and for some number c , there are at most c sets I_ν with points in any given cube C .

Take a fixed cube C of K_0 and a fixed k . As each $\phi_\nu(x)$ is of class C^∞ , $D_k \phi_\nu(x)$ is bounded in C ; there are only a finite number of these functions $\neq 0$ in C , and hence they are uniformly bounded:

$$|D_k \phi_\nu(x)| < N_k(C) \quad \text{in } C \quad (\nu = 1, 2, \dots).$$

Consider now any cube C' of any K_s , and let C be a (perhaps hypothetical) cube of K_0 of the same type as C' . If $I_{\lambda_1'}, \dots, I_{\lambda_{l'}}'$ are the sets I_λ with points in C' , let $I_{\lambda_1}, \dots, I_{\lambda_l}$ be the corresponding sets with points in C ; the latter set of sets is carried into the former by a translation of the axes and a stretching by a factor $1/2^s$. Each function ϕ_{λ_q} corresponding to I_{λ_q} goes thereby into the function

$$\phi_{\lambda_q'}(x) = \phi_{\lambda_q}[y^{\lambda_q} + 2^s(x - y^{\lambda_q})]$$

corresponding to $I_{\lambda_q'}$. Therefore, differentiating σ_k times with respect to x ,

$$D_k \phi_{\lambda_q'}(x) = 2^{s\sigma_k} D_k \phi_{\lambda_q}[y^{\lambda_q} + 2^s(x - y^{\lambda_q})]$$

for x in C' , and hence

$$|D_k \phi_\nu(x)| < 2^{s\sigma_k} N_k(C) \quad \text{in } C' \quad (\nu = 1, 2, \dots),$$

as $\phi_\nu(x) = 0$ in C' for $\nu \neq \lambda_1', \dots, \lambda_{l'}'$. Now the constants $N_k(C)$ take on at most d distinct values for a fixed k ; if we let N_k be the largest of these, we can state: *Given any n -fold set of numbers k , there is a number N_k such that if C is any cube of K_s , then*

$$(10.1) \quad |D_k \phi_\nu(x)| < 2^{s\sigma_k} N_k \quad \text{in } C \quad (\nu = 1, 2, \dots).$$

11. A differentiable extension of $f(x)$, m finite. We are now in a position to prove, for m finite,

LEMMA 2. *Under the conditions of Theorem I, there is a function $g(x)$ of class C^∞ in $E - A$, having the properties (1) and (2) of Theorem I.*

For each ν ($\nu = 1, 2, \dots$) there are functions $\phi_\nu(x)$ and $\psi(x; x^\nu) = \psi_0(x; x^\nu)$; we put

$$(11.1) \quad g(x) = \begin{cases} \sum_\nu \phi_\nu(x) \psi(x; x^\nu) & \text{in } E - A, \\ f(x) & \text{in } A. \end{cases}$$

As the $\phi_\nu(x)$ and $\psi(x; x^\nu)$ are of class C^∞ in $E - A$, the same is true of $g(x)$. The function $g(x) = f(x)$ is of class C^m at all inner points of A , by §3. It remains to show that $D_k g(x)$ exists, equals $f_k(x)$, and is continuous, at all boundary points of A , for $\sigma_k \leq m$.

Take a fixed boundary point x^0 of A , and any ϵ , $0 < \epsilon < 1$. Take

$$\eta < \epsilon / \{2c[(m+2)!]^n(108n^{1/2})^m N\} \text{ and } \eta < \epsilon/6,$$

where N is the largest of the numbers N_k for $\sigma_k \leq m$. Take $M > |f_k(x)|$ ($\sigma_k \leq m$, x in A and $r_{xx^0} \leq 1$), and take

$$\delta < \epsilon / \{6(m+1)^n M\} \text{ and } \delta < 1$$

so small that (3.2) holds at the point x^0 with ϵ replaced by η . Take now any point y^* of $E-A$ within a distance $\delta/4$ of x^0 ; we shall show that

$$(11.2) \quad |D_k g(y^*) - f_k(x^0)| < \epsilon \quad (\sigma_k \leq m).$$

Say the distance from y^* to A is $\delta_*/4$ (then $\delta_* < \delta$), and let x^* be a point of A distant $\delta_*/4$ from y^* . Consider the sum in (6.1) with x' and x replaced by x^* and x^0 respectively; as each l_h is $\leq m$, it contains at most $(m+1)^n$ terms. If we take the term with $l_1 = \dots = l_n = 0$ to the other side, there is in each remaining term a factor $(x_h^* - x_h^0)^{l_h}$ with $l_h > 0$. As each $|x_h^* - x_h^0|$ is $< \delta < 1$, we find

$$|\psi_k(x^*; x^0) - f_k(x^0)| < (m+1)^n M \delta < \epsilon/6.$$

But also $|R_k(x^*; x^0)| < \eta < \epsilon/6$; hence, using (6.2),

$$|f_k(x^*) - f_k(x^0)| < \epsilon/3.$$

Similarly we see that $|\psi_k(y^*; x^*) - f_k(x^*)| < \epsilon/6$; therefore

$$(11.3) \quad |\psi_k(y^*; x^*) - f_k(x^0)| < \epsilon/2 \quad (\sigma_k \leq m).$$

Say y^* lies in the cube C of K_* , and let $I_{\lambda_1}, \dots, I_{\lambda_t}$ be those sets I_λ with points in C . Each corresponding point y^{λ_i} is distant $< \delta/2$ from x^0 , by (9.1), and hence each corresponding point x^{λ_i} is distant $< \delta$ from x^0 . As the same is true of x^* , (3.2) gives

$$(11.4) \quad |R_k(x^*; x^*)| \leq r_{x^*x^*}^{m-\sigma_k} \eta \quad (\nu = \lambda_1, \dots, \lambda_t).$$

Set

$$\zeta_{\nu,k}(x) = \psi_k(x; x^\nu) - \psi_k(x; x^*) \quad (\nu = \lambda_1, \dots, \lambda_t);$$

then as $r_{x^*x^*} < \delta_*$ and $|x_h - x_h^*| < \delta_*$ for x in C , $|(x - x^*)^i| < \delta_*^i$, and (6.3) and (11.4) give

$$(11.5) \quad |\zeta_{\nu,k}(x)| < (m+1)^n \delta_*^{m-\sigma_k} \eta \text{ in } C \quad (\nu = \lambda_1, \dots, \lambda_t).$$

Using (9.2), we see that

$$(11.6) \quad g(x) = \psi(x; x^*) + \sum_{s=1}^t \phi_{\lambda_s}(x) \zeta_{\lambda_s,0}(x) \text{ in } C.$$

As $D_k\psi(x; x^*) = \psi_k(x; x^*)$ and therefore $D_k\zeta_{r;0}(x) = \zeta_{r;k}(x)$,

$$D_k g(x) = \psi_k(x; x^*) + \sum_{s=1}^t \sum_l \binom{k}{l} D_l \phi_{\lambda_s}(x) \zeta_{\lambda_s; k-l}(x) \text{ in } C.$$

(10.1) and (11.5) give, as $t \leq c$ (see §10) and

$$\binom{k_h}{l_h} \leq m!,$$

$$(11.7) \quad |D_k g(x) - \psi_k(x; x^*)| < \sum_l c[(m+1)!]^n 2^{\sigma_l} N \delta_*^{m-\sigma_k+\sigma_l} \eta \text{ in } C.$$

Now the distance from C to A is $> \delta_*/6$; also, as C is in K_* , this distance is $< 18n^{1/2}/2^*$. Hence $18n^{1/2}/2^* > \delta_*/6$, or, $2^* < 108n^{1/2}/\delta_*$. This gives, as $\sigma_k \leq m$ and $\delta_* < 1$,

$$(11.8) \quad |D_k g(x) - \psi_k(x; x^*)| < c[(m+2)!]^n (108n^{1/2})^m N \delta_*^{m-\sigma_k} \eta < \epsilon/2$$

in C , and in particular, at y^* . This inequality together with (11.3) gives (11.2), as required.

The proof can now be completed with the aid of Lemma 1. (11.2) with $k=0$ shows that $g(x)$ is continuous throughout E . Take any number $k = (k_1, \dots, k_n)$ with $\sigma_k < m$, and put $k' = (k_1, \dots, k_h+1, \dots, k_n)$. Assuming that $D_k g(x)$ is continuous in E , we shall show that $D_{k'} g(x)$ exists and is continuous in E . Take any boundary point $x^0 = (x_1^0, \dots, x_n^0)$ and put $x_0 = x_h^0$, $w(x) = w(x_h) = D_k g(x_1^0, \dots, x_h, \dots, x_n^0)$, $w'_0 = f_{k'}(x^0)$. Let A^* be the set of points of A for which $x_p = x_p^0$ ($p \neq h$). (3.3) with $x = x^0$ and $\Delta x_h = x_h - x_h^0$, and (11.2) with k replaced by k' , show that the conditions of the lemma are fulfilled; hence $\partial w(x_0)/\partial x_h = D_{k'} g(x^0)$ exists and equals $f_{k'}(x^0)$. (11.2) shows that $D_{k'} g(x^0)$ is continuous at x^0 . Therefore $g(x)$ is of class C^m in E .

12. A differentiable extension of $f(x)$, m infinite. We now prove Lemma 2 for the case $m = \infty$. For any given m , let $\psi_{m;k}(x'; x)$ ($\sigma_k \leq m$) be the function given by the right hand side of (6.1). Choose the axes so that the origin falls on a point of A . Let S_p be the set of all points of E whose distances from the origin are $\leq 2^p$, $p = 1, 2, \dots$. Let M_p be the maximum of $|f_k(x)|$ for $\sigma_k \leq p$ and x in $A \cdot S_p$, and let $N^{(p)}$ be the maximum of N_k for $\sigma_k \leq p$. Choose for each positive integer p a number δ_p such that

$$\delta_p < 1/\{2^{2p+1}c[(p+2)!]^n(36n^{1/2})^p N^{(p)} M_{p+1}\}, \delta_p < \delta_{p-1}/2.$$

The extension $g^{(\infty)}(x)$ of $f(x)$ is determined as follows. Given any number r , determine the number γ_r so that $\delta_{\gamma_r+1} \leq r, < \delta_{\gamma_r}$ (see §9); set $\gamma_r = 0$ if $r, > \delta_1$.

Put

$$(12.1) \quad g^{(\infty)}(x) = \begin{cases} \sum \phi_\nu(x) \psi_{\gamma_\nu;0}(x; x^0) & \text{in } E - A, \\ f(x) & \text{in } A. \end{cases}$$

Given any fixed k , we shall find an inequality similar to (11.2) for $D_k g^{(\infty)}(x)$. Let $g^{(m)}(x)$ be the extension of $f(x)$ of class C^m given by Lemma 2 ($m=1, 2, \dots$). Given any boundary point x^0 of A and any $\epsilon > 0$, choose $p \geq \sigma_k + 2$ so that x^0 lies in S_p and so that $1/2^p < \epsilon$. Take $\delta < \delta_p$ so that (11.2) with g replaced by $g^{(\sigma_k)}$ will hold for our given k and any y^* of $E - A$ within δ of x^0 ; we show next that for any such y^* ,

$$(12.2) \quad |D_k g^{(\infty)}(y^*) - D_k g^{(\sigma_k)}(y^*)| < \epsilon.$$

Choose q so that $\delta_{q+1} \leq \delta_* < \delta_q$, where δ_* is the distance from y^* to A ; then $q \geq p$. Define $C, K_s, I_{\lambda_1}, \dots, I_{\lambda_l}$ as in §11. Note that for $\nu = \text{any } \lambda_h, \delta_{\gamma_\nu+1} \leq r, < 2\delta < 2\delta_p < \delta_{p-1}$, hence $\gamma_\nu + 1 > p - 1$, and thus $\gamma_\nu > p - 2 \geq \sigma_k$. Set

$$(12.3) \quad \xi_\nu(x) = \psi_{\gamma_\nu;0}(x; x^0) - \psi_{\sigma_k;0}(x; x^0) \quad (\nu = \lambda_1, \dots, \lambda_l);$$

using (12.1) and (11.1), we see that

$$(12.4) \quad g^{(\infty)}(x) = g^{(\sigma_k)}(x) + \sum_{u=1}^l \phi_{\lambda_u}(x) \xi_{\lambda_u}(x) \text{ in } C.$$

Now $\bar{D}_j \xi_\nu(x) = \psi_{\gamma_\nu;j}(x; x^0) - \psi_{\sigma_k;j}(x; x^0)$. If we replace k by j in (6.1), then those and only those terms in the sum with $\sigma_i \leq m - \sigma_j$ occur. Replacing m by γ_ν and σ_k successively and subtracting, we have

$$(12.5) \quad D_j \xi_\nu(x) = \sum_{\sigma_i = \sigma_k - \sigma_j + 1}^{\gamma_\nu - \sigma_j} \frac{f_{j+i}(x^0)}{i!} (x - x^0)^i \text{ in } C.$$

Now $r_\nu > \delta_*/2$, by (9.1), hence $r_\nu > \delta_{q+2}$, and thus $\gamma_\nu \leq q+1$ ($\nu = \lambda_1, \dots, \lambda_l$); there are therefore less than $(q+2)^n$ terms in the sum, and in each term, $\sigma_j + \sigma_i \leq q+1$. It follows that $|f_{j+i}(x^0)| < M_{q+1}$ in each term. Also $|x_\lambda - x_{\lambda^*}| < 2\delta_* < 2\delta_q$ and $\sigma_i \geq \sigma_k - \sigma_j + 1$ in each term; hence

$$|D_j \xi_\nu(x)| < (q+2)^n M_{q+1} 2^{q+1} \delta_*^{\sigma_k - \sigma_j} \delta_q$$

in C . This with (12.4) gives

$$\begin{aligned} |D_k g^{(\infty)}(x) - D_k g^{(\sigma_k)}(x)| &\leq \sum_{u=1}^l \sum_j \binom{k}{j} |D_{k-j} \phi_{\lambda_u}(x)| |D_j \xi_{\lambda_u}(x)| \\ &< c \sum_j \binom{k}{j} 2^{s(\sigma_k - \sigma_j)} N^{(\sigma_k)} (q+2)^n M_{q+1} 2^{q+1} \delta_*^{\sigma_k - \sigma_j} \delta_q \end{aligned}$$

in C . Now the distance from C to A is $> \delta_*/2$ and is $< 18n^{1/2}/2^q$; hence $2^q < 36n^{1/2}/\delta_*$. Also $\sigma_k < p \leq q$; therefore

$$(12.6) \quad |D_k g^{(\infty)}(x) - D_k g^{(\sigma_k)}(x)| < c[(q+2)!]^{n(36n^{1/2})^q N^{(q)} M_{q+1} 2^{q+1} \delta_q} < 1/2^q < \epsilon$$

in C , and in particular, at y^* , proving (12.2). Using (11.2), we find $|D_k g^{(\infty)}(y^*) - f_k(x^0)| < 2\epsilon$ for any point y^* of $E-A$ within δ of x^0 . Again we can apply Lemma 1 and show that $D_k g^{(\infty)}(x)$ exists and is continuous throughout E . As this is true for every k , the proof is complete.

13. We prove next a combined extension and approximation theorem.

LEMMA 3. Let $f(x)$ be of class C^m (m finite) in E , with $D_k f(x) = f_k(x)$ ($\sigma_k \leq m$) there, and let $f_k(x)$ ($m < \sigma_k \leq m'$, $m' > m$ finite or infinite) be defined in the closed set A so that $f(x)$ (considered now only in A) is of class $C^{m'}$ there. Then for an arbitrary $\epsilon > 0$ there is a function $g(x)$ which is of class C^m in E , of class $C^{m'}$ in a neighborhood of A , and equals $f(x)$ outside another neighborhood of A , such that

$$(13.1) \quad D_k g(x) = f_k(x) \text{ in } A \quad (\sigma_k \leq m'),$$

and

$$(13.2) \quad |D_k g(x) - D_k f(x)| < \epsilon \text{ in } E \quad (\sigma_k \leq m).$$

Let $f'(x)$ be the extension of class $C^{m'}$ of the values of $f(x)$ in A given by the last lemma, and put $\zeta(x) = f'(x) - f(x)$; then $\zeta(x)$ is of class C^m in E , and

$$D_k \zeta(x) = 0 \text{ in } A \quad (\sigma_k \leq m).$$

Set $\eta = \epsilon / \{c[(m+1)!]^{n(36n^{1/2})^m N}\}$ ($N = \max N_k$ for $\sigma_k \leq m$). As $\zeta(x)$ is of class C^m and $D_k \zeta(x)$ vanishes in A ($\sigma_k \leq m$), we can find an open set R containing A so that if y is any point of $R-A$, at a distance δ from A , then

$$|D_k \zeta(y)| < \eta \delta^{m-\sigma_k} \quad (\sigma_k \leq m).$$

Let ν_1, ν_2, \dots be those numbers such that I_{ν_p} lies wholly in R ($p = 1, 2, \dots$). We set

$$(13.3) \quad g(x) = f(x) + \zeta(x) \sum_{p=1}^{\infty} \phi_{\nu_p}(x) \text{ in } E-A,$$

and $g(x) = f(x)$ in A . As $\sum \phi_{\nu_p}(x) = 1$ in an open set surrounding A , $g(x) = f'(x)$ there. As $\sum \phi_{\nu_p}(x) = 0$ in $E-R$, $g(x) = f(x)$ there. The statements about the class of $g(x)$ are true. To show that (13.2) holds, let y be a point of $R-A$, distant δ from A ; then, defining $C, K, I_{\lambda_1}, \dots, I_{\lambda_t}$ as in the previous lemma, we have

$$\begin{aligned}
 |D_k g(y) - D_k f(y)| &\leq \sum_p \sum_l \binom{k}{l} |D_l \phi_p(y)| |D_{k-l} f(y)| \\
 &< c \sum_l (m!)^n 2^{n\sigma_l} N \delta^{m-\sigma_k+\sigma_l} \eta < \epsilon \quad (\sigma_k \leq m).
 \end{aligned}$$

14. We close this section with a theorem concerning the isolated points of A . Define α_p as follows:

$$(14.1) \quad \alpha_p = \begin{cases} m & \text{if } m \text{ is finite} \\ p & \text{if } m \text{ is infinite} \end{cases} \quad (p = 1, 2, \dots).$$

LEMMA 4. Consider the closed set $A = A' + a_1 + a_2 + \dots$, where a_1, a_2, \dots are isolated points (then A' is closed), and let m be finite or infinite. Let $f_k(x)$ be defined in A' for $\sigma_k \leq m$ and at each a_s for all k , so that $f(x)$ is of class C^m in A in terms of the $f_k(x)$ ($\sigma_k \leq m$). Then there is a function $g'(x)$ of class C^∞ in $E - A'$ and of class C^m in E , such that

$$(14.2) \quad D_k g'(x) = f_k(x) \text{ in } A' \text{ for } \sigma_k \leq m \text{ and at each } a_s \text{ for all } k.$$

Let $g(x)$ be the extension of $f(x)$ of class C^m given by Lemma 2. Let U_1, U_2, \dots be neighborhoods of a_1, a_2, \dots , chosen so that each is at a positive distance from each other and from A' . If m is finite, we alter $g(x)$ in U_1 , next in U_2 , etc., by means of the last lemma†, so that the new function $g'(x)$ will take on the required derivatives at $a_1 + a_2 + \dots$, and so that

$$(14.3) \quad |D_k g'(x) - D_k g(x)| < 1/p \text{ in } U_p \quad (\sigma_k \leq \alpha_p, p = 1, 2, \dots).$$

(14.2) is an immediate consequence of this inequality and Lemma 1.

III. APPROXIMATION THEOREMS

15. We prove first the following extension of the Weierstrass approximation theorem.‡

LEMMA 5. Let $g(x)$ be of class C^m in E (m finite), and let S be a bounded closed set in E .§ Then for each $\epsilon > 0$ there exists a function $G(x)$ analytic in E and such that

$$(15.1) \quad |D_k G(x) - D_k g(x)| < \epsilon \text{ in } S \quad (\sigma_k \leq m).$$

Let R_b be the set of points distant at most b from the origin ($b \geq 0$). Consider the n -tuple integral

† We use the last lemma with A replaced by a_s and m' by ∞ .

‡ Compare de la Vallée Poussin, *Cours d'Analyse*, vol. II, 2d edition, 1912, pp. 126-137.

§ It is sufficient that $g(x)$ be defined over S , for we can then extend its definition over E , by Lemma 2.

$$(15.2) \quad \Phi(b) = T \int_{R_b} e^{-r_{0y}^2} dy = T \int \dots \int e^{-(u^2 + \dots + v_n^2)} dy_1 \dots dy_n,$$

where T is chosen so that $\Phi(\infty) = 1$; then $0 \leq \Phi(b) \leq 1$ for all b . If we replace y by κy and b by κb , we see that

$$(15.3) \quad \Phi(\kappa b) = T \kappa^n \int_{R_b} e^{-\kappa^2 r_{0y}^2} dy.$$

Let $v(x)$ be a function $\equiv 1$ in S , $\equiv 0$ outside some neighborhood of S , and of class C^∞ in E , such that $D_k v(x) = 0$ in S for all k . (Such a function may be found for instance by the aid of Lemma 2.) Put $g'(x) = v(x)g(x)$, and

$$(15.4) \quad G(x) = T \kappa^n \int_E g'(y) e^{-\kappa^2 r_{xy}^2} dy,$$

where κ will be chosen later; $G(x)$ is analytic in E . As r_{xy} is a function of $y - x$ alone, differentiating under the integral sign gives

$$D_k G(x) = T \kappa^n \int_E g'(y) D_k^{(x)} e^{-\kappa^2 r_{xy}^2} dy = (-1)^{\sigma_k} T \kappa^n \int_E g'(y) D_k^{(y)} e^{-\kappa^2 r_{xy}^2} dy,$$

where $D_k^{(x)}$ and $D_k^{(y)}$ denote differentiation with respect to x and y respectively. Integrating by parts σ_k times gives

$$(15.5) \quad D_k G(x) = T \kappa^n \int_E D_k g'(y) e^{-\kappa^2 r_{xy}^2} dy.$$

As $\Phi(\infty) = 1$, we see that

$$(15.6) \quad D_k G(x) - D_k g'(x) = T \kappa^n \int_E [D_k g'(y) - D_k g'(x)] e^{-\kappa^2 r_{xy}^2} dy.$$

Take M so large that

$$(15.7) \quad |D_k g'(x)| < M \text{ in } E \quad (\sigma_k \leq m).$$

The functions $D_k g'(x)$ are uniformly continuous in E ; hence there is a $\delta > 0$ such that

$$(15.8) \quad |D_k g'(y) - D_k g'(x)| < \epsilon/2 \quad (r_{xy} < \delta, \sigma_k \leq m).$$

Take κ so large that

$$(15.9) \quad 1 - \Phi(\kappa\delta) < \epsilon/(4M).$$

For a given x , let U consist of all points within δ of x ; then if J_1 and J_2 are formed by replacing the domain of integration on the right hand side of

(15.6) by U and $E-U$ respectively, we have, using (15.3),

$$|J_1| < T\kappa^n \int_U \frac{\epsilon}{2} e^{-\kappa^2 r_{zv}^2} dy = \frac{\epsilon}{2} \Phi(\kappa\delta) < \frac{\epsilon}{2},$$

$$|J_2| < T\kappa^n \int_{E-U} 2Me^{-\kappa^2 r_{zv}^2} dy = 2M[1 - \Phi(\kappa\delta)] < \frac{\epsilon}{2},$$

and hence $|D_k G(x) - D_k g'(x)| < \epsilon$ in $E(\sigma_k \leq m)$, which gives (15.1).

$G(x)$ may of course be replaced by a polynomial if desired.

16. The above lemma can be generalized as follows.

LEMMA 6. *Let R be an open set and let R_1, R_2, \dots be bounded open sets (some of which may be void) whose sum is R , such that each $\bar{R}_p = R_p$ plus boundary is in R_{p+1} . Then if $g(x)$ is defined and of class C^m (m finite or infinite) in R , and $\epsilon_1 \geq \epsilon_2 \geq \dots$ are given positive numbers, there is an analytic function $G(x)$ defined in R such that*

$$(16.1) \quad |D_k G(x) - D_k g(x)| < \epsilon_p \text{ in } R - R_p \quad (\sigma_k \leq \alpha_p, p = 1, 2, \dots).$$

α_p is defined in (14.1). Note that, if R_1, \dots, R_q are void, then

$$(16.2) \quad |D_k G(x) - D_k g(x)| < \epsilon_q \text{ in } R \quad (\sigma_k \leq \alpha_q).$$

Consider the closed set $\bar{R}_{p-1} + (\bar{R}_{p+1} - R_p) + (E - R_{p+1}) = Q'_p + Q_p + Q''_p$; if in Lemma 2 we replace A by this set and $f(x)$ by a function $\equiv 1$ in Q_p and $\equiv 0$ in $Q'_p + Q''_p$, we find a function $u_p(x)$ for each p , of class C^∞ in E , such that

$$(16.3) \quad u_p(x) = \begin{cases} 1 & \text{in } Q_p, \\ 0 & \text{in } Q'_p + Q''_p; \end{cases} \quad D_k u_p(x) = 0 \text{ in } Q'_p + Q_p + Q''_p (\sigma_k > 0).$$

(If R_{p+1} is void, we put $u_p(x) \equiv 0$; if R_{p+1} is not void but R_{p-1} is void, we have $u_p(x) = 0$ in Q''_p and $= 1$ in \bar{R}_{p+1} .) Let $Z_p \geq 1$ be such a number that

$$(16.4) \quad |D_k u_p(x)| < Z_p \text{ in } E \quad (\sigma_k \leq \alpha_p, p = 1, 2, \dots).$$

We define successively analytic functions $G_1(x), G_2(x), \dots$, by the following formula:

$$(16.5) \quad G_p(x) = T\kappa_p^n \int_E u_p(y) [g(y) - \{G_1(y) + \dots + G_{p-1}(y)\}] e^{-\kappa_p^2 r_{zv}^2} dy.$$

(For $p=1$, the factor in brackets is simply $g(y)$.) κ_p is chosen so that, if we set

$$(16.6) \quad H_p(x) = u_p(x) [g(x) - \{G_1(x) + \dots + G_{p-1}(x)\}],$$

then

$$(16.7) \quad |D_k G_p(x) - D_k H_p(x)| < \beta'_p = \epsilon_{p+1} / \{2^{p+2} [(\alpha_{p+1} + 1)!]^n Z_{p+1}\} \\ \text{in } \bar{R}_{p+1} \quad (\sigma_k \leq \alpha_{p+1})$$

(see Lemma 5); we shall restrict κ_p further later. Remembering the definition of $u_p(x)$, we see that (16.7) with (16.6) gives

$$(16.8) \quad |D_k g(x) - D_k \{G_1(x) + \cdots + G_p(x)\}| < \beta'_p < \epsilon_p/2 \text{ in } Q_p(\sigma_k \leq \alpha_{p+1}).$$

Differentiating $H_p(x)$ and using (16.4) and (16.8) with p replaced by $p-1$, we see that (compare the derivation of (11.7))

$$|D_k H_p(x)| < [(\alpha_p + 1)!]^n Z_p \beta_{p-1}' = \epsilon_p/2^{p+1} \text{ in } Q_{p-1} \quad (\sigma_k \leq \alpha_p).$$

As $u_p(x)$ and its derivatives are 0 in R_{p-1} , this holds in R_{p-1} also; hence, using (16.7), we have

$$(16.9) \quad |D_k G_p(x)| < \epsilon_p/2^p \text{ in } \bar{R}_p \quad (\sigma_k \leq \alpha_p).$$

We set now

$$(16.10) \quad G(x) = G_1(x) + G_2(x) + \cdots;$$

this is the desired approximation to $g(x)$. To prove this, we see first from (16.9) that $D_k [G_1(x) + \cdots + G_p(x)]$ converges uniformly in any bounded closed subset of R ($\sigma_k \leq m$); hence $G(x)$ is defined in R , and

$$(16.11) \quad D_k G(x) = D_k G_1(x) + D_k G_2(x) + \cdots \text{ in } R \quad (\sigma_k \leq m).$$

Next (16.9) shows that

$$(16.12) \quad |D_k G_{p+1}(x) + D_k G_{p+2}(x) + \cdots| < \epsilon_{p+1}/2^{p+1} + \epsilon_{p+2}/2^{p+2} + \cdots \\ \leq \epsilon_p(1/2^2 + 1/2^3 + \cdots) = \epsilon_p/2 \text{ in } \bar{R}_{p+1} \quad (\sigma_k \leq \alpha_{p+1});$$

this with (16.8) gives $|D_k G(x) - D_k g(x)| < \epsilon_p$ in $Q_p(\sigma_k \leq \alpha_p)$, proving (16.1).

It remains to be shown that $G(x)$ is analytic in R . To this end we extend the definition of each $G_p(x)$ to complex values of $x = (x'_1 + ix''_1, \cdots, x'_n + ix''_n)$, using (16.5) still. Consider the analytic function of x

$$r_{xy}^2 = \sum (y_h - x_h)^2 = \sum [(y'_h - x'_h) + i(y''_h - x''_h)]^2;$$

as $y''_h = 0$ in (16.5), the domain of integration being real,

$$\Re(r_{xy}^2) = \sum [(y'_h - x'_h)^2 - x''_h{}^2].$$

Take any point x^0 of R and let U be the complex region of radius ρ about x^0 , where ρ is so small that the real points in the complex region of radius 3ρ about x^0 lie in some R_q ; we take q so that $3\rho^2 > 1/2^q$. Now if $p > q$, x is in U , and y is in $R - R_{p-1}$, then $\sum x''_h{}^2 < \rho^2$ and $\sum (y'_h - x'_h)^2 \geq 4\rho^2$, and hence

$$\Re(r_{xy}^2) > 3\rho^2.$$

Also $H_p(y)$ vanishes in R_q and in $E - R_{p+2}$ for $p > q$; therefore if M'_p is the maximum of $|H_p(y)|$ (note that $H_p(y)$ is determined before we determine κ_p) and V_p is the volume of R_p ($p = 1, 2, \dots$),

$$(16.13) \quad \begin{aligned} |G_p(x' + ix'')| &< T\kappa_p^n \int_{R_{p+2}-R_{p-1}} M'_p e^{-\kappa_p^2 p^2 y^2} dy \\ &< T\kappa_p^n e^{-\kappa_p^2/2^p} M'_p V_{p+2} \end{aligned}$$

for x in U and $p > q$. Hence if we choose κ_p successively for $p = 1, 2, \dots$, so that this quantity is $< 1/2^p$ (and so that (16.7) holds), then the series in (16.10), when defined for complex values of x , converges uniformly in a complex neighborhood of any point of R . Therefore the function $G(x)$ is analytic in R , completing the proof.

17. The numbers κ_p as chosen above depend not only on the functions $u_p(x)$ but also on the function $g(x)$. Under certain restrictions, we can take them independent of $g(x)$, as follows.

LEMMA 7. Let the open sets R_1, R_2, \dots , the numbers $\epsilon_1, \epsilon_2, \dots$, and the functions $u_1(x), u_2(x), \dots$ be given as in Lemma 6; let $\Delta_1(r), \Delta_2(r), \dots$ be a sequence of positive continuous functions defined for $r > 0$, such that $\Delta_p(r) \rightarrow 0$ as $r \rightarrow 0$ and $\Delta_{p+1}(r) \geq \Delta_p(r)$; let a be a point of R , and M a positive number. Then there is a sequence of numbers $\kappa_1, \kappa_2, \dots$, with the following property. If $g(x)$ is any function of class C^m defined in R such that $|g(a)| \leq M$ and

$$(17.1) \quad |D_k g(x') - D_k g(x)| < \Delta_p(r_{xx'}) \text{ in } \bar{R}_p \quad (\sigma_k \leq \alpha_p, p = 1, 2, \dots),$$

and if $G(x)$ is defined in terms of $g(x)$ as in the previous lemma, using the above numbers κ_p , then $G(x)$ is analytic in R and (16.1) holds.

As the u 's and their derivatives are uniformly continuous in E , there are functions $\Gamma_p(x)$ of the same sort as the Δ 's above such that

$$(17.2) \quad |D_k u_p(x') - D_k u_p(x)| < \Gamma_p(r_{xx'}) \text{ in } E \quad (\sigma_k \leq \alpha_{p+1}, p = 1, 2, \dots).$$

The conditions on $g(x)$ imply that for some M_1'' , $|D_k g(x)| < M_1''$ in \bar{R}_3 ($\sigma_k \leq \alpha_3$).† Say

$$|D_k u_p(x)| < Z_p' \text{ in } E \quad (\sigma_k \leq \alpha_{p+1}, p = 1, 2, \dots).$$

Then as $u_1(x) = 0$ in $R - R_3$, we have

$$|D_j u_1(x') D_l g(x') - D_j u_1(x) D_l g(x)| \leq \Gamma_1(r_{xx'}) M_1'' + Z_1' \Delta_3(r_{xx'})$$

† If d_3 is the diameter of \bar{R}_3 , then $|g(x)| < M + \Delta_1(d_3)$ in \bar{R}_3 . Now take any $k' = (k_1, \dots, k_{k-1}, \dots, k_n)$ and $k = (k_1, \dots, k_n)$ ($0 < \sigma_k \leq \alpha_3$). Let $x'x''$ be a line segment parallel to the x_k -axis and lying wholly in \bar{R}_3 ; set $r = |x_k'' - x_k'|$. As $|D_{k'} g(x'') - D_{k'} g(x')| < \Delta_1(r)$, the law of the mean gives, for some point x^* of $x'x''$, $|D_k(x^*)| < \Delta_1(r)/r$. Hence $|D_k(x)| < \Delta_1(r)/r + \Delta_3(d_3)$ in \bar{R}_3 ($0 < \sigma_k \leq \alpha_3$).

for $\sigma_j \leq \alpha_2$ and $\sigma_l \leq \alpha_2$ and any x and x' in E . Hence if we put

$$\Delta_1^*(r) = [(\alpha_2 + 1)!]^n [\Gamma_1(r)M_1'' + Z_1'\Delta_3(r)],$$

we shall have, on differentiating $H_1(x) = u_1(x)g(x)$,

$$(17.3) \quad |D_k H_1(x') - D_k H_1(x)| \leq \Delta_1^*(r_{xx'}) \text{ in } E \quad (\sigma_k \leq \alpha_2).$$

Also $|D_k H_1(x)| < [(\alpha_2 + 1)!]^n Z_1' M_1''$ in \bar{R}_3 and $= 0$ in $E - R_3$ ($\sigma_k \leq \alpha_2$); thus inequalities corresponding to (15.7) and (15.8) hold for $H_1(x)$. Hence if we take $\delta_1 > 0$ so that

$$\Delta_1^*(r) < \beta_1'/2 \quad (r < \delta_1),$$

and take κ_1 so that

$$1 - \Phi(\kappa_1 \delta_1) < \beta_1' / \{4[(\alpha_2 + 1)!]^n Z_1' M_1''\},$$

then if we form $G_1(x)$ for any admissible $g(x)$ by means of (16.5), (16.7) will hold with $p = 1$; we take κ_1 large enough so that the right hand side of (16.13) with $p = 1$ will be $< 1/2$.

If we differentiate (16.5) with $p = 1$ σ_k times ($\sigma_k \leq m$), we derive an equation similar to (15.5); forming this for $x = x$ and $x = x'$ and subtracting, we find (changing y to $y + x' - x$ in one equation)

$$(17.4) \quad D_k G_1(x') - D_k G_1(x) = T \kappa_1^n \int_E [D_k H_1(y + x' - x) - D_k H_1(y)] e^{-\kappa_1^2 r_{xy}^2} dy.$$

This with (17.3), (15.3), and the definition of $\Phi(\infty)$ gives

$$(17.5) \quad |D_k G_1(x') - D_k G_1(x)| \leq \Delta_1^*(r_{xx'}) \text{ in } E \quad (\sigma_k \leq \alpha_2).$$

Assume now we have defined functions $\Delta_p^*(r)$ and have chosen numbers κ_p so that

$$(17.6) \quad |D_k G_p(x') - D_k G_p(x)| \leq \Delta_p^*(r_{xx'}) \text{ in } E \quad (\sigma_k \leq \alpha_{p+1}),$$

so that (16.7) holds, and so that the quantity in (16.13) is $< 1/2^p$, for $p < q$. Then for any admissible $g(x)$, $g(x) - \{G_1(x) + \cdots + G_{q-1}(x)\}$ satisfies the same kind of conditions as $g(x)$; hence, just as before, we find a function $\Delta_q^*(r)$ so that an inequality similar to (17.3) holds for $D_k H_q(x)$ in E ($\sigma_k \leq \alpha_{q+1}$). Also $H_q(x)$ is bounded properly; hence we can choose κ_q so that (16.7) holds for any admissible $g(x)$ with p replaced by q , and so that (16.13) with $p = q$ is $< 1/2^q$. From this we show, as before, that (17.6) holds with p replaced by q . We can thus continue finding functions $\Delta_p^*(r)$ and numbers κ_p indefinitely. We put finally $G(x) = G_1(x) + G_2(x) + \cdots$, and show, just as in Lemma 6, that $G(x)$ has the required properties. This ends the proof.

IV. ANALYTIC EXTENSIONS

18. **Proof of Theorem I.** Let $g(x)$ be the extension of $f(x)$ of class C^m given by Lemma 2. Set $R = E - A$ and define $R_1, R_2, \dots, \alpha_1, \alpha_2, \dots$, and numbers $\epsilon_1, \epsilon_2, \dots$, approaching zero as in §16. Define $G(x)$ in $E - A$ as in Lemma 6, and set $F(x) = G(x)$ in $E - A$, $F(x) = f(x)$ in A . That $F(x)$ is of class C^m in E and property (2) holds follows from (16.1) and Lemma 1, just as in §11; the other facts are obvious.

19. **The functions $\omega_{rk}(x)$.** In the next sections we shall discuss the analyticity of the extension of $f(x)$ at the isolated points of A . Let R be an open set, let a_1, a_2, \dots be points of R having no limit point in R , let m_1, m_2, \dots be corresponding integers ≥ 0 , and let m be an integer ≥ -1 or ∞ . We assume that if a_r, a_{r_1}, \dots is any sequence of points a_r approaching the boundary of R , then

$$(19.1) \quad \liminf_{s \rightarrow \infty} m_s \geq m.$$

Choose about each a_r a neighborhood U_r lying, with its boundary, in R , so that no two have common points. Define the numbers $\rho(\nu; k)$ so that when $(\nu; k)$ runs through the values $(1; k), \sigma_k \leq m_1; (2; k), \sigma_k \leq m_2; \dots$; then $\rho(\nu; k)$ runs through the values $1, 2, 3, \dots$. Let $\rho'(\nu; k)$ equal one plus the largest of the numbers $m_1, \dots, m_r, \rho(\nu; k)$.

Take any positive integer s , and consider all neighborhoods U_r such that $\rho'(\nu k) \geq s$ for some k ($\sigma_k \leq m_r$); let R_s be the set of all points of R whose distances from these neighborhoods and from the boundary of R are $> 1/s$, and whose distances from the origin are $< s$. Then R_s is a bounded open set, \bar{R}_s lies in R_{s+1} ($s = 1, 2, \dots$), $R_1 + R_2 + \dots = R$, and U_r lies in $R - R_{\rho'(\nu k)}$ ($\sigma_k \leq m_r$). By Lemma 2, there are functions $\omega_{rk}(x)$ of class C^∞ in E , defined for $\sigma_k \leq m_r, \nu = 1, 2, \dots$, such that

$$(19.2) \quad D_l \omega_{rk}(a_r) = \begin{cases} 1, & l = k \\ 0, & l \neq k \end{cases} \quad (\sigma_l \leq m_r); \quad \omega_{rk}(x) = 0 \text{ in } E - U_r.$$

Choose for each ν a positive number $\beta_\nu < 1/\nu$ so that $\beta_\nu \geq \beta_{\nu+1}$, and

$$(19.3) \quad \beta_\nu |D_l \omega_{rk}(x)| < 1/\nu [(m_r + 1)^\nu] \text{ in } E \\ (\sigma_k \leq m_r, \sigma_l \leq m_r, \nu = 1, 2, \dots).$$

Now let f_{rk} be any set of numbers, defined for $\sigma_k \leq m_r, \nu = 1, 2, \dots$, satisfying the condition

$$(19.4) \quad |f_{rk}| \leq \beta_\nu \quad (\sigma_k \leq m_r, \nu = 1, 2, \dots).$$

Set

$$(19.5) \quad c_{p(\nu k)} = f_{\nu k}, \quad \omega'_{p(\nu k)}(x) = \omega_{\nu k}(x) \quad (\sigma_k \leq m_\nu, \nu = 1, 2, \dots).$$

Take any $s = p(\nu k)$. As $\omega'_s(x) = \omega_{\nu k}(x) = 0$ in $R - U_\nu$, and $R_s = R_{p(\nu k)}$ is in $R_{p'(\nu k)}$ which has no points in common with U_ν ,

$$(19.6) \quad \omega'_s(x) = 0 \text{ in } R_s \quad (s = 1, 2, \dots).$$

20. The transformation L . Define functions $u_1(x)$, $u_2(x)$, \dots as in Lemma 6. Consider any function

$$(20.1) \quad g(x) = \lambda_1 \omega'_1(x) + \lambda_2 \omega'_2(x) + \dots \quad (|\lambda_s| \leq 1, s = 1, 2, \dots);$$

such functions and λ 's we shall call *admissible*. Set

$$(20.2) \quad \epsilon_s = \beta_s / 2^{s+1} \quad (s = 1, 2, \dots).$$

There are, obviously, functions $\Delta_1(r)$, $\Delta_2(r)$, \dots , so that (17.1) holds for any such $g(x)$; hence, by Lemma 7, we can define numbers κ_1 , κ_2 , \dots , so that if $G(x)$ is defined in terms of $g(x)$ as in Lemma 6, then $G(x)$ is analytic in R and (16.1) holds. *In using Lemma 6, we replace α_p by p .*

We note here a certain property of $G(x)$: *If $g(x)$ is admissible and*

$$(20.3) \quad \text{if } g(x) = 0 \text{ in } R_s, \text{ then } |D_k G(x)| < \epsilon_{s-1} / 2^{s-2} \text{ in } \bar{R}_{s-1} \quad (\sigma_k \leq s-1).$$

As $u_p(x) = 0$ in $R - R_s$ ($p \leq s-2$), $u_p(x)g(x) = 0$ in E ($p \leq s-2$). Using (16.5), we see in succession that $G_1(x) = 0, \dots, G_{s-2}(x) = 0$. This with (16.9) and (16.11) gives

$$|D_k G(x)| \leq \sum_{p=k-1}^{\infty} |D_k G_p(x)| < \sum_{p=k-1}^{\infty} \epsilon_p / 2^p \leq \epsilon_{s-1} / 2^{s-2}$$

in \bar{R}_{s-1} ($\sigma_k \leq s-1$), as required.

Given any admissible $g(x)$, let $Lg(x)$ be the corresponding function $G(x)$. It follows easily from the definition of $G(x)$ that L is linear:

$$(20.4) \quad L[\lambda_1 g_1(x) + \lambda_2 g_2(x)] = \lambda_1 Lg_1(x) + \lambda_2 Lg_2(x).$$

We show now that for admissible numbers λ ,

$$(20.5) \quad L \sum_{s=1}^{\infty} \lambda_s \omega'_s(x) = \sum_{s=1}^{\infty} \lambda_s L\omega'_s(x).$$

To prove this, take any point x^0 of R , in the set R_q , and any $\epsilon > 0$. Take $q' \geq q$ so that $1/2^{q'-2} < \epsilon$. (19.6) and (20.3), for $s = q'+1, q'+2, \dots$, give, as $\lambda_s \omega'_s(x)$ is admissible,

$$\left| \sum_{s=q'+1}^{\infty} \lambda_s L\omega'_s(x) \right| = \left| \sum_{s=q'+1}^{\infty} L\lambda_s \omega'_s(x) \right| < \sum_{s=q'+1}^{\infty} 1/2^{s-2} = 1/2^{q'-2} < \epsilon/2$$

in $R_{q'}$, and in particular, at x^0 . As

$$\sum_{s=q'+1}^{\infty} \lambda_s \omega'_s(x) = 0 \text{ in } R_{q'+1}$$

and is admissible,

$$\left| L \sum_{s=q'+1}^{\infty} \lambda_s \omega'_s(x^0) \right| < 1/2^{q'-1} < \epsilon/2.$$

Moreover

$$L \sum_{s=1}^{\infty} \lambda_s \omega'_s(x) = \sum_{s=1}^{q'} \lambda_s L \omega'_s(x) + L \sum_{s=q'+1}^{\infty} \lambda_s \omega'_s(x);$$

hence

$$\begin{aligned} & \left| L \sum_{s=1}^{\infty} \lambda_s \omega'_s(x^0) - \sum_{s=1}^{\infty} \lambda_s L \omega'_s(x^0) \right| \\ & \leq \left| L \sum_{s=q'+1}^{\infty} \lambda_s \omega'_s(x^0) \right| + \left| \sum_{s=q'+1}^{\infty} \lambda_s L \omega'_s(x^0) \right| < \epsilon, \end{aligned}$$

which proves (20.5).

We prove two inequalities. Take any $(\nu; k)$, $(\mu; l)$ ($\sigma_k \leq m_\nu$, $\sigma_l \leq m_\mu$); then

$$(20.6) \quad |D_k L \omega_{\mu l}(a_\nu) - D_k \omega_{\mu l}(a_\nu)| < \epsilon_{\rho(\nu k)},$$

$$(20.7) \quad |D_k L \omega_{\mu l}(a_\nu) - D_k \omega_{\mu l}(a_\nu)| < \epsilon_{\rho(\mu l)}.$$

The first follows from (16.1) when we note that a_ν is in $R - R_{\rho'(\nu k)}$, and $\epsilon_{\rho'(\nu k)} < \epsilon_{\rho(\nu k)}$, and $\sigma_k \leq m_\nu < \rho'(\nu k)$ (recall that α_p was replaced by p in using Lemma 6). We now prove the second. As $\omega_{\mu l}(x) = 0$ in $R_{\rho'(\mu l)}$ and $\rho'(\mu l) \geq \rho(\mu l) + 1$, (20.3) gives

$$(a) \quad |D_k L \omega_{\mu l}(x) - D_k \omega_{\mu l}(x)| < \epsilon_{\rho(\mu l)} \text{ in } \bar{R}_{\rho'(\mu l)-1} \quad (\sigma_k \leq \rho'(\mu l) - 1).$$

Also (16.1) gives

$$(b) \quad |D_k L \omega_{\mu l}(x) - D_k \omega_{\mu l}(x)| < \epsilon_{\rho(\mu l)} \text{ in } R - R_p \quad (\sigma_k \leq p, p \geq \rho'(\mu l) - 1).$$

Say a_ν is in $R_{p+1} - R_p$. As a_ν is not in $R_{\rho'(\nu k)}$, $\rho'(\nu k) \leq p$, and $\sigma_k \leq m_\nu \leq \rho'(\nu k) - 1 \leq p - 1$. If $p \geq \rho'(\mu l) - 1$, (20.7) follows directly from (b). If $p < \rho'(\mu l) - 1$, then a_ν is in $R_{\rho'(\mu l)-1}$, and $\sigma_k \leq p - 1 < \rho'(\mu l) - 1$, and (a) applies.

21. An infinite system of linear equations. We prove here

LEMMA 8. Suppose η_s and c_s ($s = 1, 2, \dots$), and γ_{st} ($s, t = 1, 2, \dots$), are given, so that $1 \geq \eta_s \geq \eta_{s+1} > 0$ ($s = 1, 2, \dots$), $|c_s| \leq 1$, and

$$(21.1) \quad |\gamma_{st}| < \eta_s/2^{t+1} \quad (s, t = 1, 2, \dots).$$

Then there are numbers λ_s ($s=1, 2, \dots$) such that

$$(21.2) \quad \sum_{t=1}^{\infty} (\gamma_{st} + \delta_{st}) \lambda_t = \sum_{t=1}^{\infty} \gamma_{st} \lambda_t + \lambda_s = c_s \quad (s=1, 2, \dots),$$

and

$$(21.3) \quad |\lambda_s - c_s| \leq \eta_s \quad (s=1, 2, \dots).$$

Using the method of successive approximations, put

$$(21.4) \quad \lambda_{1s} = c_s, \quad \lambda_{ps} = - \sum_{t=1}^{\infty} \gamma_{st} \lambda_{p-1,t} \quad (p=2, 3, \dots).$$

It is readily proved by induction that

$$(21.5) \quad |\lambda_{ps}| < \eta_s / 2^{p-1} \quad (p=2, 3, \dots).$$

Hence the series $\lambda_{1s} + \lambda_{2s} + \dots$ converges to a limit λ_s ($s=1, 2, \dots$), and

$$\begin{aligned} \sum_{t=1}^{\infty} (\gamma_{st} + \delta_{st}) \lambda_t &= \sum_{p=1}^{\infty} \sum_{t=1}^{\infty} (\gamma_{st} + \delta_{st}) \lambda_{pt} = \sum_{p=1}^{\infty} (\lambda_{ps} - \lambda_{p+1,s}) = c_s, \\ |\lambda_s - c_s| &= \left| \sum_{p=2}^{\infty} \lambda_{ps} \right| \leq \sum_{p=2}^{\infty} \eta_s / 2^{p-1} = \eta_s. \end{aligned}$$

22. We are now ready to prove

LEMMA 9. Let R, m, a_r, m_r ($r=1, 2, \dots$) be defined as in §19. Then there are numbers $\beta_r > 0$ ($r=1, 2, \dots$) with the following property. Given any set of numbers f_{rk} defined for $\sigma_k \leq m_r, r=1, 2, \dots$, such that (19.4) holds, there exists a function $G(x)$ analytic in R , such that

$$(22.1) \quad D_k G(a_r) = f_{rk} \quad (\sigma_k \leq m_r, \quad r=1, 2, \dots),$$

and such that if we set $G(x) \equiv 0$ in $E-R$, then $G(x)$ is of class C^m in E , and

$$(22.2) \quad D_k G(x) = 0 \text{ in } E-R \quad (\sigma_k \leq m).$$

We define the $\omega_{rk}(x)$ and the β_r as in §19. Now take any f_{rk} satisfying (19.4), and define the $c_{r(\sigma k)}$ by (19.5). Define the ϵ_r and the transformation L as in §20. Set

$$(22.3) \quad \eta_{r(\sigma k)} = \beta_r \quad (\sigma_k \leq m_r, \quad r=1, 2, \dots),$$

and

$$(22.4) \quad \gamma_{st} = \gamma_{r(\sigma k)\rho(\mu l)} = D_k L \omega_{\mu l}(a_r) - \delta_{st} = D_k L \omega_{\mu l}(a_r) - D_k \omega_{\mu l}(a_r).$$

Let $u = \rho(\theta j)$ be the larger of the two numbers $s = \rho(\nu k), t = \rho(\mu l)$. Then using (20.6) or (20.7) according as $u = s$ or $u = t$, we find (as $\beta_{r(\theta j)} \leq \beta_{r(\nu k)} \leq \beta_r$)

$$|\gamma_{st}| < \epsilon_{p(\theta)} \leq \beta_r/2^{u+1} \leq \eta_r/2^{t+1}.$$

Also $|c_s| = |f_{rk}| \leq \beta_r < 1$. Therefore the equations (21.2) have a solution $\lambda_1, \lambda_2, \dots$, and

$$(22.5) \quad \sum_{i=1}^{\infty} (\gamma_{st} + \delta_{st}) \lambda_i = \sum_{\mu, l} \lambda_{p(\mu l)} D_k L \omega_{\mu l}(a_r) = c_s = c_{p(vk)} = f_{rk}.$$

By (22.8) below, the λ 's are admissible (§20), and we can define the analytic function $G(x)$ in R by the equation

$$(22.6) \quad G(x) = L \sum_{i=1}^{\infty} \lambda_i \omega'_i(x).$$

(20.5) and (22.5) give

$$(22.7) \quad D_k G(a_r) = D_k \sum_{i=1}^{\infty} \lambda_i L \omega'_i(a_r) = \sum_{\mu, l} \lambda_{p(\mu l)} D_k L \omega_{\mu l}(a_r) = f_{rk}.$$

(19.6) and (20.3) show that the last sum above is uniformly convergent in any \bar{R}_p ; hence the termwise differentiation is permissible.

Set $G(x) \equiv 0$ in $E - R$; we must show that $G(x)$ is of class C^m in E . (This is trivial if $m = -1$.) First note that, by (19.4) and (21.3),

$$(22.8) \quad |\lambda_{p(vk)}| \leq |c_{p(vk)}| + \eta_{p(vk)} = |f_{rk}| + \beta_r \leq 2\beta_r;$$

this with (19.3) gives (replacing v, k and l by μ, l and k)

$$(22.9) \quad |\lambda_{p(\mu l)} D_k \omega_{\mu l}(x)| < 2/[\mu(m_\mu + 1)^n] \text{ in } E$$

$$(\sigma_k, \sigma_l \leq m_\mu, \mu = 1, 2, \dots).$$

Now take any boundary point x^0 of R , any integer $m' \leq m$, and any $\epsilon > 0$. Take $q \geq m'$ so that $\epsilon_q < \epsilon/2$. Take $\delta > 0$ so that R_q has no points within δ of x^0 , and so that if v is any number such that U_v has points within δ of x^0 , then $m_v \geq m'$ and $2/v < \epsilon/2$ (see (19.1)). Consider any point y of R within δ of x^0 , and take any $k, \sigma_k \leq m'$. Either $D_k \omega_{\mu l}(y) = 0$ for all μ, l , or else for some μ, y lies in U_μ , in which case there are at most $(m_\mu + 1)^n$ such numbers $\neq 0$, and $2/\mu < \epsilon/2$, and $m_\mu \geq m'$. Thus if we replace x by y in (22.9) and sum over μ and l , we find

$$|D_k \sum_{i=1}^{\infty} \lambda_i \omega'_i(y)| = |\sum_{\mu, l} \lambda_{p(\mu l)} D_k \omega_{\mu l}(y)| < \epsilon/2 \quad (\sigma_k \leq m').$$

As y is in $R - R_q$, replacing α_q by $q \geq m'$ in (16.1) gives

$$|D_k L \sum_{i=1}^{\infty} \lambda_i \omega'_i(y) - D_k \sum_{i=1}^{\infty} \lambda_i \omega'_i(y)| < \epsilon_q < \epsilon/2 \quad (\sigma_k \leq m').$$

This with the last inequality gives

$$|D_k G(y)| < \epsilon \text{ in } R \text{ if } r_{xy} < \delta \quad (\sigma_k \leq m'); \quad (22.3)$$

the proof is completed with the aid of Lemma 1.

23. Functions analytic at the isolated points of A . Lemmas 4, 6 and 9 lead directly to the following theorem.

THEOREM II. Let A be a closed set in E , and let a_1, a_2, \dots be isolated points of A . Set $A' = A - (a_1 + a_2 + \dots)$. Let m be an integer ≥ -1 or ∞ , and let the integers $m_\nu \geq 0, \nu = 1, 2, \dots$, satisfy (19.1). Let $f_k(x)$ be defined for x in $A' (\sigma_k \leq m)$, and for $x = a_\nu (\sigma_k \leq m_\nu)$, so that $f(x)$ is of class C^m in A . Then there is a function $F(x)$ of class C^m in E such that

- (1) $F(x) = f(x)$ in A ,
- (2) $D_k F(x) = f_k(x)$ in A' for $\sigma_k \leq m$ and at each a_ν for $\sigma_k \leq m_\nu$,
- (3) $F(x)$ is analytic in $E - A'$.

We asked that $f(x)$ be of class C^m in A , while $f_k(a_\nu)$ may not be defined for certain values of ν and k ($\sigma_k \leq m$). We require merely that after setting $f_k(a_\nu) = 0$ ($\sigma_k > m_\nu$), $f(x)$ shall be of class C^m in A .

A special case of interest is $m = -1$. The m_ν and the $f_k(a_\nu)$ are then unrestricted. A' may be void, in which case $f(x)$ is analytic throughout E . A' may of course contain isolated points.

To prove the theorem, set $R = E - A'$ and determine the open sets R_ν and the numbers β_ν ($\nu = 1, 2, \dots$), as in §19. Let $g'(x)$ be the extension of $f(x)$ of class C^m in E and of class C^∞ in $E - A'$ given by Lemma 4 (setting $f_k(a_\nu) = 0$ for $\sigma_k > m_\nu$). Let $G'(x)$ be the analytic function in R given by Lemma 6 (with α_p replaced by β) such that

$$(23.1) \quad |D_k G'(x) - D_k g'(x)| < \beta_p \text{ in } R - R_p \quad (\sigma_k \leq p),$$

and set $G'(x) = f(x)$ in A' . $G'(x)$ is of class C^m in E , and

$$(23.2) \quad D_k G'(x) = f_k(x) \text{ in } A' \quad (\sigma_k \leq m)$$

(see §18). Set

$$(23.3) \quad f_{\nu k} = D_k g'(a_\nu) - D_k G'(a_\nu) \quad (\sigma_k \leq m_\nu, \nu = 1, 2, \dots).$$

As a_ν lies in $R - R_{\rho'(\nu k)}$ and $\rho'(\nu k) > m_\nu$, (23.1) gives $|f_{\nu k}| < \beta_{\rho'(\nu k)} < \beta_\nu (\sigma_k \leq m_\nu)$. Thus the conditions of Lemma 9 are satisfied, and there is a function $G(x)$ analytic in R , $\equiv 0$ in A' , of class C^m in E , and such that (22.1) and (22.2) hold. Set

$$(23.4) \quad F(x) = G'(x) + G(x);$$

then $F(x)$ is our required function. It is of class C^m in E as the same is true of $G'(x)$ and $G(x)$; it is analytic in $R = E - A'$ as the same is true of $G'(x)$ and $G(x)$; it equals $f(x)$ in A' as $G'(x) = f(x)$ and $G(x) = 0$ there. (22.1), (23.3) and (14.2) show that $D_k F(a_\sigma) = D_k g'(a_\sigma) = f_k(a_\sigma)$ ($\sigma_k \leq m_\sigma$); (22.2) and (23.2) show that $D_k F(x) = f_k(x)$ in A' , completing the proof.

24. An extension-approximation theorem. We prove here

THEOREM III.[†] Let A be closed, and let A_{-1}, A_0, A_1, \dots be closed subsets of A such that each A_s lies in A_{s+1} . Let a_{s1}, a_{s2}, \dots be points of $A_s - A_{s-1}$ which are isolated points of A , and set $A' = A - \sum a_{si}$. Let B_{-1} be void, and let B_0, B_1, \dots be sets whose sum B lies in $E - A$, such that each B_s lies in B_{s+1} , such that each set $B_s - B_{s-1}$ has limit points in $B - B_{s-1} + A_s$ only, and such that each set $A + B - B_s$ is closed. Let $f_k(x)$ be defined in each set $\Gamma_s = A + B - (A_{s-1} + B_{s-1})$ for $\sigma_k \leq s$ ($s = 0, 1, \dots$) so that $f(x) = f_0(x)$ is of class C^* in Γ_s in terms of the $f_k(x)$ for each s . Let $\epsilon(x)$ be a continuous function, positive in $E - A'$ and zero in A' . Then there is a function $F(x)$ defined in $E - A_{-1}$ such that

- (1) $F(x)$ is of class C^* in $E - A_{s-1}$ ($s = 0, 1, \dots$),
- (2) $D_k F(x) = f_k(x)$ in $A - A_{s-1}$ ($\sigma_k \leq s, s = 0, 1, \dots$),
- (3) $|D_k F(x) - f_k(x)| < \epsilon(x)$ in $B - B_{s-1}$ ($\sigma_k \leq s, s = 0, 1, \dots$),
- (4) $F(x)$ is analytic in $E - A'$.

Any number of sets A_s, B_s may be void; any of the points a_{si} may not exist. Note that if $A_\infty = A - (A_{-1} + A_0 + \dots)$, then $F(x)$ is of class C^∞ at all points of A_∞ . Theorem I for m finite is obtained by letting B and A_{-1}, \dots, A_{m-1} be void, and setting $A = A_m$; and for m infinite, by letting B and every A_s be void. Theorem II is obtained similarly; we arrange the a_{si} in a sequence a_1, a_2, \dots , and set $m_s = s$ if a_s is in $A_s - A_{s-1}$. Lemma 6 is obtained by setting $A = A_{-1} = E - R$, $B_s = R_{s+1}$ ($s = 0, 1, \dots$), and taking $\epsilon(x)$ so that $\epsilon(x) \leq \epsilon_s$ in $R - R_s$.

We turn now to the proof. Take a subdivision of the open set $E - (A + B)$ as in §8, let $y^{\nu\sigma}$ ($\nu = 1, 2, \dots$) be the vertices of the cubes, and let $x^{\nu\sigma}$ be a point of Γ_0 whose distance from $y^{\nu\sigma}$ is not more than twice the distance from $y^{\nu\sigma}$ to Γ_0 . Define the functions $\phi_{\nu\sigma}(x)$ in $E - (A + B)$ as in §9, and define $g'_\sigma(x)$ by (11.1), using the functions $\phi_{\nu\sigma}(x)$ and $\psi_{0;\sigma}(x; x^{\nu\sigma}) = f(x^{\nu\sigma})$, and replacing $E - A$ and A by $E - (A + B)$ and Γ_0 respectively. Then $g'_\sigma(x)$ is defined throughout $E - A_{-1}$, and is easily seen to be a continuous extension of $f(x)$. Let $g_0(x)$ be a function $= g'_\sigma(x)$ in $A - A_{-1} + \Gamma_1$ and analytic in the open set $E - (A + \Gamma_1)$ so that

[†] A special case of this theorem has been proved by A. Besikowitch, *Über analytische Funktionen mit vorgeschriebenen Werten ihrer Ableitungen*, Mathematische Zeitschrift, vol. 21 (1924), pp. 111-118.

$$(24.1) \quad |g_0(x) - g'_0(x)| < \theta_1(x)/4 \text{ in } E - (A + \Gamma_1),$$

where $\theta_p(x) = \min [\epsilon(x), \text{distance from } x \text{ to } A + \Gamma_p]$ ($p=1, 2, \dots$). Then $g_0(x)$ is continuous in $E - A_{-1}$.

We shall now define in succession functions $g_1(x), g_2(x), \dots$, with the following properties:

(a) $g_p(x)$ is defined in $E - A_{-1}$, is of class C^s in $E - A_{s-1}$ ($s=0, \dots, p$), and is analytic in $E - (A + \Gamma_{p+1})$.

(b) $D_k g_p(x) = f_k(x)$ in $A - A_{s-1} + \Gamma_{p+1}$ ($\sigma_k \leq s, s=0, \dots, p$).

(c) $|D_k g_p(x) - D_k g_{p-1}(x)| < \epsilon(x)/2^{p+2}$ in $B_{p-1} - B_{s-1}$ ($\sigma_k \leq s, s=0, \dots, p-1$).

(d) $|D_k g_p(x) - f_k(x)| < \epsilon(x)/2^{p+2}$ in $B_p - B_{p-1}$ ($\sigma_k \leq p$).

Assuming $g_0(x), \dots, g_{p-1}(x)$ are defined, we shall define $g_p(x)$. Consider any point of Γ_p ; it is at a positive distance from the closed set A_{p-1} , and hence we can enclose it in an open set lying at a positive distance from A_{p-1} . We thus enclose Γ_p in an open set Γ'_p containing no points of A_{p-1} , and having no limit points in A_{p-1} than limit points of Γ_p . Take a subdivision of the open set $E - (A + \Gamma_p)$, let $y^{p\nu}$ ($\nu=1, 2, \dots$) be the vertices of the cubes, and let $x^{p\nu}$ be a point of Γ_p whose distance from $y^{p\nu}$ is not more than twice the distance from $y^{p\nu}$ to Γ_p ($\nu=1, 2, \dots$). Define the functions $\phi_{p\nu}(x)$ in $E - (A + \Gamma_p)$ as in §9 and define $\psi_{p;k}(x'; x)$ by (6.1) ($\sigma_k \leq p$), replacing m by p . Remembering that $f(x)$ is of class C^p in Γ_p , set

$$(24.2) \quad g'_p(x) = \sum_{\nu} \phi_{p\nu}(x) \psi_{p;0}(x; x^{p\nu}) \text{ in } \Gamma'_p - \Gamma_p,$$

and set $g_p(x) = g_{p-1}(x)$ in Γ_p . From the proof in §11 it is seen that $g'_p(x)$ is an extension of class C^p of the values of $f(x)$ in Γ_p .

Set $\zeta_p(x) = g'_p(x) - g_{p-1}(x)$ in Γ'_p ; then $\zeta_p(x)$ is of class C^{p-1} in Γ'_p , and

$$(24.3) \quad D_k \zeta_p(x) = 0 \text{ in } \Gamma_p \quad (\sigma_k \leq p-1).$$

Set $\eta_p = 1/\{2^{p+4}c[(p+1)!]^n(36n^{1/2})^p N^{(p)}\}$ ($p=0, 1, \dots$), where $N^{(p)} = \max N_k$ for $\sigma_k \leq p$. Let K_{p-1} be the set of points of B_{p-1} for which

$$|D_k \zeta_p(x)| \geq \eta_{p-1} \epsilon(x) \delta_x^{p-1-\sigma_k} \text{ for some } k \quad (\sigma_k \leq p-1),$$

where δ_x is the distance from x to Γ_p , or 1 if that is smaller. Each point of $A - A_{p-1}$ is at a positive distance from K_{p-1} , as B_{p-1} has no limit points in $A - A_{p-1}$, and each point of $B - B_{p-1}$ is at a positive distance from K_{p-1} on account of (24.3), as $\epsilon(x) > 0$ in B ; hence each point of Γ_p is at a positive distance from K_{p-1} , and we can enclose Γ_p in an open set Γ''_p which lies in Γ'_p and contains no points of K_{p-1} . Now

$$(24.4) \quad |D_k \zeta_p(x)| < \eta_{p-1} \epsilon(x) \delta_x^{p-1-\sigma_k} \text{ in } \Gamma_p'' \cdot B_{p-1} \quad (\sigma_k \leq p-1).$$

We can also take Γ_p'' so that if $\rho_p(x)$ is the distance from x to A_{p-1} , then

$$(24.5) \quad |D_k \zeta_p(x)| < \eta_{p-1} \rho_p(x) \delta_x^{p-1-\sigma_k} \text{ in } \Gamma_p'' \quad (\sigma_k \leq p-1).$$

Let I_{p_i} be those sets I_s of the subdivision of $E - (A + \Gamma_p)$ lying wholly in Γ_p'' ($i=1, 2, \dots$), and set

$$(24.6) \quad g_p''(x) = g_{p-1}(x) + \zeta_p(x) \sum_{i=1}^{\infty} \phi_{p_i}(x) \text{ in } E - (A + \Gamma_p),$$

and $g_p''(x) = g_{p-1}(x) = f(x)$ in $A - A_{-1} + \Gamma_p$. Then $g_p''(x)$ is of class C^p in $E - A_{p-1}$, and with the help of (24.4) we find

$$(24.7) \quad |D_k g_p''(x) - D_k g_{p-1}(x)| < \epsilon(x)/2^{p+3} \text{ in } B_{p-1} \quad (\sigma_k \leq p-1)$$

(see §13). Also (24.5) gives

$$(24.8) \quad |D_k g_p''(x) - D_k g_{p-1}(x)| < \rho_p(x)/2^{p+3} \text{ in } E - A \quad (\sigma_k \leq p-1),$$

and hence $g_p''(x)$ is of class C^s in $E - A_{s-1}$ ($s=0, \dots, p$), as the same is true of $g_{p-1}(x)$ ($s=0, \dots, p-1$).

Finally let $g_p(x)$ be an analytic function such that

$$(24.9) \quad |D_k g_p(x) - D_k g_p''(x)| < \theta_{p+1}(x)/2^{p+3} \text{ in } E - (A + \Gamma_{p+1}) \quad (\sigma_k \leq p);$$

set $g_p(x) = g_{p-1}(x) = f(x)$ in $A - A_{-1} + \Gamma_{p+1}$. Then $g_p(x)$ has all the required properties. (c) is a direct consequence of the above inequality and (24.7); (d) follows from (24.9) and the fact that $D_k g_p''(x) = f_k(x)$ in $\Gamma_p(\sigma_k \leq p)$; (a) and (b) follow with the aid of Lemma 1.

Set

$$(24.10) \quad g(x) = \lim_{p \rightarrow \infty} g_p(x) \text{ in } E - A_{-1}.$$

By (24.8) and (24.9), $g(x)$ exists and is of class C^∞ in $E - A$. Let x be any point of any $A_s - A_{s-1}$; by (a), $g_p(x)$ is of class C^s in a neighborhood of x for $p \geq s$, and by (24.8) and (24.9), the same is true of $g(x)$. The same argument, using (b), shows that

$$(24.11) \quad D_k g(x) = f_k(x) \text{ in } A - A_{s-1} \quad (\sigma_k \leq s, s = 0, 1, \dots).$$

Finally (c), (d), (24.1) and the definition of $g'_0(x)$ show that

$$(24.12) \quad |D_k g(x) - f_k(x)| < \epsilon(x)/2 \text{ in } B - B_{p-1} \quad (\sigma_k \leq s, s = 0, 1, \dots).$$

We have now found an extension $g(x)$ with all the properties but (4). It is replaced by an analytic extension $F(x)$ just as in §23; we must be careful merely to make

$$(24.13) \quad |D_k F(x) - D_k g(x)| < \epsilon(x)/2 \text{ in } B - B_{s-1} \quad (\sigma_k \leq s, s = 0, 1, \dots).$$

Let a_1, a_2, \dots be the a_s arranged in a sequence, and set $m_s = s$ if a_s is in $A_s - A_{s-1}$. Set $R = E - A'$. Let R_p consist of those points of the R_p of §19 whose distances from the closed set $A + B - B_{p-1}$ are $> 1/p$ ($p = 1, 2, \dots$). Every point of $E - A'$ lies in some R_p , as $B - (B_0 + B_1 + \dots)$ is void. Take the β_s (§19) small enough so that if $|\lambda_{p(rk)}| \leq 2\beta_s$ (see (22.8)) and $g^*(x) = \sum \lambda_i \omega_i'(x)$, then

$$(24.14) \quad |D_k g^*(x)| < \epsilon(x)/8 \text{ in } R_{p+1} - R_p \quad (\sigma_k \leq p).$$

Let ϵ'_s be one eighth the lower bound of $\epsilon(x)$ for x in R_{s+1} ($s = 1, 2, \dots$), or 1 if that is smaller. Replace the ϵ_s of (20.2) by $\min(\epsilon_s, \epsilon'_s)$. Then for any such $g^*(x)$, (16.1) gives

$$(24.15) \quad |D_k L g^*(x)| < \epsilon_p + |D_k g^*(x)| < \epsilon(x)/4 \text{ in } R_{p+1} - R_p \quad (\sigma_k \leq p).$$

Replace the $g'(x)$ of §23 by the present $g(x)$, and determine $G'(x)$ so that

$$(24.16) \quad |D_k G'(x) - D_k g(x)| < \min[\beta_p, \epsilon(x)/4] \text{ in } R - R_p \quad (\sigma_k \leq p),$$

and, in particular, in $B - B_{p-1}$. Now if $G(x)$ and $F(x)$ are determined as in §23, then (24.13) and hence property (3) hold; the other properties are easily verified, and the proof of the theorem is complete.

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THE ASYMPTOTIC SOLUTIONS OF CERTAIN LINEAR ORDINARY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER*

BY
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1. Introduction. The ordinary differential equation

$$(1) \quad \frac{d^2v}{ds^2} + \lambda p_1(s, \lambda) \frac{dv}{ds} + \lambda^2 p_2(s, \lambda) v = 0,$$

in which the coefficients $p_i(s, \lambda)$ are expansible in descending powers of λ when $|\lambda|$ is large, includes as special cases many differential equations of classical importance. It is accordingly the subject of an extensive literature. In particular, the problem of the asymptotic dependence of its solutions upon the complex parameter λ has been the subject of many investigations, in which for suitably restricted configurations a theory of considerable generality has been deduced.

A familiar change of variable gives to the equation the form

$$(2) \quad \frac{d^2u}{ds^2} - \{\lambda^2 q_0(s) + \lambda q_1(s) + q_2(s, \lambda)\} u = 0,$$

in which $q_2(s, \lambda)$ is bounded for large values of λ . The salient hypothesis, then, which has most generally been assumed and under which a developed theory is known, is that the variable s be real, and that on the basic interval considered, the characteristic equation

$$\theta^2 - q_0(s) = 0$$

have roots $\theta_i(s)$ which are everywhere distinct.† In recent papers the author has studied the asymptotic forms for a type of equation not subject to this hypothesis by virtue of the fact that the coefficient $q_0(s)$ becomes zero at some point of the domain of the variable considered. The form of the solutions was determined, moreover, for the entire complex plane.‡ With such a configuration of variables, the feature of prime interest was found to lie in

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† In this connection special mention is due to Birkhoff and Tamarkin. For references cf. Langer, R. E., *On the asymptotic solutions of differential equations*, etc., these Transactions, vol. 34 (1932), pp. 447-480.

‡ Cf. the previous reference.

the incidence of the Stokes' phenomenon, under which the analytic forms used for the asymptotic representation of any specific solution must be changed in a discontinuous manner as the point (s, λ) varies across certain specifiable boundaries in the complex (s, λ) space. This is attributable to the fact that the asymptotic representation utilizes multiple-valued expressions for the description of the single-valued solutions of the equation.

Referring to the equation in form (2) the investigation cited was made for the case in which the coefficient $q_0(s)$ vanishes at some point like any real positive power of the variable, but under the assumption that the term in λ does not occur, i.e., $q_1(s) \equiv 0$. In the present paper the case in which the term in λ is present while at some point the term in λ^2 vanishes to the second order is to be studied.* It is planned as a sequel to apply the results of this discussion to a study of the functional form of the solutions of the Mathieu equation over the complex plane, in a manner resembling that in which the earlier results were applied to a study of the Bessel's functions.

It may be of interest to note that the case to be considered bears a certain formal limiting relationship to a class of cases which by suitable transformations may be brought under the theory developed in the earlier papers cited. Thus, consider the equation

$$\frac{d^2 u}{ds^2} - \{\lambda^2 q_0(s) + \lambda^r q_1(s) + q_2(s, \lambda)\} u = 0,$$

with

$$q_0(s) \equiv s^r \sum_{j=0}^{\infty} a_j s^j, \quad a_0 \neq 0,$$

and let the variable be changed by the substitution $s = \lambda^{-\sigma} z$, with σ an undetermined positive constant. The equation takes the form

$$\frac{d^2 u}{dz^2} - \{\lambda^{2-2\sigma-r} a_0 z^r + \lambda^{2-2\sigma-r} a_1 z^{r+1} + \dots + \lambda^{r-2\sigma} q_1 + q_2\} u = 0,$$

and if $r < 4/2 + \nu$ it is always possible to choose σ so as to make the second highest power of λ which occurs the zeroth power. Then if $-\rho^2$ is written for the highest power of λ , the equation takes the form

$$\frac{d^2 u}{dz^2} + \{\rho^2 z^r a_0 + \chi(z, \rho)\} u = 0.$$

It is seen that when $\nu = 2$, as in the equation (2), the transformation is possi-

* In this connection cf. for the case of a real variable Goldstein, S., *A note on certain approximate solutions of linear differential equations of the second order* (2), Proceedings of the London Mathematical Society, (2), vol. 33 (1932), p. 246.

ble whenever $r < 1$. For this class of equations the equation (2), in which $r = 1$, may evidently be regarded as a limiting form.

2. The normal equation. In the differential equation (2) let the zero of the coefficient $q_0(s)$ be designated by s_0 , and with c as a tentatively undetermined constant let the change of variable

$$z = s - s_0 - \frac{c}{\lambda}$$

be made. With the use of the formal relations

$$\begin{aligned} q_0(s) &= q_0(z + s_0) + \frac{c}{\lambda} q'_0(z + s_0) + \frac{c^2}{\lambda^2} q''_0\left(z + s_0 + \frac{\theta_1 c}{\lambda}\right), \\ q_1(s) &= q_1(z + s_0) + \frac{c}{\lambda} q'_1\left(z + s_0 + \frac{\theta_2 c}{\lambda}\right), \end{aligned}$$

the differential equation may then be written

$$(3) \quad u''(z) - \{\lambda^2 \chi_0^2(z) + \lambda \chi_1(z) + \chi_2(z, \lambda)\} u = 0,$$

in which, specifically,

$$\begin{aligned} \chi_0^2(z) &= q_0(z + s_0), \\ \chi_1(z) &= q_1(z + s_0) + c q'_0(z + s_0). \end{aligned}$$

Whatever choice of the constant c is made, the zero of the coefficient $\chi_0^2(z)$ clearly occurs at the origin. It is readily found that c may always be so chosen that the relation

$$(4) \quad 3\chi'_0(0)\chi'_1(0) - 2\chi''_0(0)\chi_1(0) = 0$$

is satisfied, and this choice will be made inasmuch as it is convenient for subsequent purposes. The equation (3) will then (i.e., when (4) is satisfied) be designated as in normal form, and the preliminary normalization of the equation will be assumed in the discussion which is to ensue.

The more specific description of the equation for which a theory is to be deduced may be given, as follows, through the means of an enunciation of the hypotheses to be made. The variable z is to be complex, and is to vary over a simply connected (finite or infinite) fundamental region R_* , which includes the origin and in which the hypotheses, to be enumerated from (i) to (v) below, are simultaneously fulfilled. It may be considered as a blanket hypothesis upon the equation that some such region exists. The explicit assumptions are the following:

(i) Within the region R , the coefficient $\chi_0^2(z)$ is analytic and has a zero of the second order at the origin. Moreover, except in the immediate neighborhood of the origin the functions

$$\chi_0(z) \text{ and } \int_0^z \chi_0(z) dz$$

are bounded from zero.

(ii) Within the region R , the coefficient $\chi_1(z)$ is analytic, and is such that the functions

$$\frac{\chi_1(z)}{\chi_0^2(z)}, \text{ and } \frac{\int_a^z \{\chi_1(z)/\chi_0(z)\} dz}{\int_0^z \chi_0(z) dz}, \text{ with } a \neq 0,$$

are bounded except possibly in the immediate neighborhood of the origin.

(iii) The coefficient $\chi_2(z, \lambda)$ is analytic in R , and in any finite portion of R is bounded uniformly when $|\lambda|$ is sufficiently large.

The enunciation of hypotheses (iv) and (v), which are less transparent, will be deferred to §7.

For definiteness the function $\arg \chi_0(z)$ will be determined by the relation

$$\arg \left\{ \lim_{z \rightarrow 0} z^{-1} \chi_0(z) \right\} = 0.$$

This involves no loss of generality since the adjustment may always be achieved by the transfer of a suitable constant factor from the parameter λ^2 to the coefficient $\chi_0^2(z)$.

3. The related equation. Let the relations

$$(5a) \quad k = -\frac{\chi_1(0)}{4\chi_0'(0)}, \quad (5d) \quad \Phi(z) = \int_0^z \phi(z) dz,$$

$$(5b) \quad \eta(z) = \frac{\chi_1(z)}{\chi_0(z)} + \frac{2k\chi_0(z)}{\int_0^z \chi_0(z) dz}, \quad (5e) \quad \xi = \lambda\Phi(z),$$

$$(5c) \quad \phi(z) = 2\chi_0(z) + \frac{\eta(z)}{\lambda}, \quad (5f) \quad \Psi(z) = \frac{\Phi^{1/4}(z)}{\phi^{1/2}(z)}$$

serve respectively to define the symbols which occur upon the left. The function $\eta(z)$, which is analytic in R , because of the hypotheses (i) and (ii), is

found to vanish at the origin in virtue of the normalization (4). It follows from this and the hypotheses that outside of any neighborhood of the origin the functions $\phi(z)$ and $\Phi(z)$ are bounded from zero uniformly when $|\lambda|$ is sufficiently large, while at the origin they have zeros of the first and second orders respectively. The function $\Psi(z)$ is accordingly analytic (with proper definition at $z=0$) and uniformly bounded in any finite portion of R_s . Lastly, it may be shown that since the zero of $\phi(z)$ is a simple one, the expression

$$\omega(\phi) = \frac{-3}{16} \left(\frac{\phi}{\Phi} \right)^2 - \frac{\phi''}{2\phi} + \frac{3}{4} \left(\frac{\phi'}{\phi} \right)^2$$

is analytic (with proper definition at $z=0$).

Let $M_{\mu,\nu}(\xi)$ denote the confluent hypergeometric function customarily so designated,* and let the functions $y_1(z)$, $y_2(z)$ be defined by the formulas

$$(6) \quad y_j(z) = \Psi(z) \xi^{-1/4} M_{k, \pm 1/4}(\xi), \quad j = 1, 2. \dagger$$

The functions M satisfy the equation

$$M''_{\mu,\nu}(\xi) - \left\{ \frac{1}{4} - \frac{\nu}{\xi} - \frac{1-4\mu^2}{4\xi^2} \right\} M_{\mu,\nu}(\xi) = 0,$$

from which it may be found by direct substitution that the functions (6) are solutions of the differential equation

$$(7) \quad y''(z) - \{ \lambda^2 \chi_0^2(z) + \lambda \chi_1(z) + \Omega(z, \lambda) \} y(z) = 0,$$

in which

$$\Omega(z, \lambda) = \frac{\eta^2}{4} + \frac{2k\chi_0^2 \int_0^z \eta dz}{\Phi \int_0^z \chi_0 dz} - \frac{k(4\chi_0\eta + \lambda^{-1}\eta^2)}{\Phi} + \omega(\phi).$$

This equation, (7), will be referred to briefly as the related equation. The similarity of its structure to that of the given equation (3), which is evident in so far as the coefficients of λ and λ^2 are concerned, extends also to the remaining element. For with the analyticity of the expression $\omega(\phi)$ established, it is easily seen that the coefficient $\Omega(z, \lambda)$, as well as $\chi_2(z, \lambda)$, satisfies the hypothesis (iii).

4. The solutions of the related equation. The solutions (6) of the related equation take on and are determined by the initial values

* The formulas and notations of this and the following sections are taken from Whittaker and Watson, *A Course in Modern Analysis*, 3d edition, Cambridge, University Press, 1920, chapter XVI.

† It is to be consistently understood that when the index j is used in conjunction with double signs, then $j=1$ is to be associated with the upper signs and $j=2$ with the lower ones.

$$(8) \quad \begin{aligned} y_1(0) &= 0, & y_2(0) &= \Psi(0), \\ y_1'(0) &= \frac{\lambda^{1/2}}{2\Psi(0)}, & y_2'(0) &= \Psi'(0). \end{aligned}$$

For values of the variable ξ which are numerically small they are conveniently described by the formulas

$$y_1(z) = \Psi(z)\xi^{1/2}e^{-\xi/2} \left\{ 1 + \frac{3-4k}{2 \cdot 3}\xi + \dots \right\},$$

$$y_2(z) = \Psi(z)e^{-\xi/2} \left\{ 1 + \frac{1-4k}{1 \cdot 2}\xi + \dots \right\}.$$

For large values of ξ they are describable by asymptotic formulas. Due to the incidence of the Stokes' phenomenon, however, such description is dependent upon the location of the variable ξ .

The origin, $z=0$, is an ordinary point for both the given and the related equations, and the region R_s accordingly consists of a single sheet. The relation (5e), however, maps the z plane upon a Riemann surface having a simple branch point at $\xi=0$, and hence there corresponds to R_s a two-sheeted region R_t which is the domain of variation for the variable ξ . In this domain let the sectors $\Xi^{(h)}$ be defined by the relations

$$(9) \quad \Xi^{(h)}: \left(h - \frac{1}{2} \right) \pi + \epsilon \leq \arg \xi \leq \left(h + \frac{3\pi}{2} \right) \pi - \epsilon,$$

$$h = -2, -1, 0, 1,$$

ϵ denoting an arbitrarily small but positive and fixed constant. It is clear that these sectors overlap considerably and also that they completely cover the domain R_t . The corresponding sub-regions of R_s will be designated by the same symbols. It is to be noted that they depend upon the parameter λ .

When ξ lies in the sector $\Xi^{(0)}$ and $|\xi|$ is sufficiently large, it is known that

$$M_{k, \pm 1/4}(\xi) = \Gamma\left(1 \pm \frac{1}{2}\right) e^{-k\pi i} \left\{ \frac{1}{\Gamma(\frac{1}{2} \pm \frac{1}{4} - k)} W_{-k, 1/4}(\xi e^{-\pi i}) \right. \\ \left. + \frac{ie^{\pm \pi i/4}}{\Gamma(\frac{1}{2} \pm \frac{1}{4} + k)} W_{k, 1/4}(\xi) \right\},$$

with

$$W_{\mp k, 1/4}(-i\xi e^{\mp \pi i/2}) = (-i\xi e^{\mp \pi i/2})^{\mp k} e^{\pm \xi/2} \left[1 + \frac{E(\xi)}{\xi} \right]^*$$

It may be noted that the formula as written is formally correct even in the exceptional cases when the index k is a quarter of a real odd integer. The term in which the gamma function is infinite is then merely to be omitted.

When ξ does not lie in $\Xi^{(0)}$ the appropriate formulas may be deduced by the use of those above in conjunction with the relations

$$M_{k, \pm 1/4}(\xi) = ie^{\mp \pi i/4} M_{-k, \pm 1/4}(\xi e^{-\pi i}).$$

From these facts it may be computed that when $|\xi| > N$,† and ξ lies in the sector $\Xi^{(h)}$, then

$$(10) \quad y_j(z) = \Psi(z) \xi^{-1/4} \Gamma\left(1 \pm \frac{1}{2}\right) \left\{ C_{j,1}^{(h)} \xi^{-k} e^{\xi/2} \left[1 + \frac{E(\xi)}{\xi} \right] + C_{j,2}^{(h)} \xi^k e^{-\xi/2} \left[1 + \frac{E(\xi)}{\xi} \right] \right\},$$

with coefficients given by the formulas

$$(11) \quad \begin{aligned} \Gamma\left(\frac{1}{2} \pm \frac{1}{4} - k\right) C_{j,1}^{(h)} &= \begin{cases} \pm ie^{-2k\pi i}, & \text{for } h = -2, \\ 1, & \text{for } h = -1, \text{ or } 0, \\ \mp ie^{2k\pi i}, & \text{for } h = 1; \end{cases} \\ \Gamma\left(\frac{1}{2} \pm \frac{1}{4} + k\right) C_{j,2}^{(h)} &= \begin{cases} -ie^{(k\mp 1/4)\pi i}, & \text{for } h = -2, \text{ or } -1, \\ ie^{-(k\mp 1/4)\pi i}, & \text{for } h = 0, \text{ or } 1. \end{cases} \end{aligned}$$

It is to be understood that if k is such that one of the gamma functions is infinite, then the coefficient C which multiplies it in (11) is to be assigned the value zero. The formulas (10) differ for different values of h . It is readily observed, however, that in any region common to two of the sectors $\Xi^{(h)}$ their difference is asymptotically negligible. When it is a matter of choice as to which sector is to be considered as containing the point ξ , then the choice is always immaterial.

The formulas

$$(12a) \quad y_{h,j}(z) = \mp \left\{ \Gamma(-\frac{1}{2}) C_{j,3-j}^{(h)} y_1(z) + \Gamma(\frac{1}{2}) C_{1,3-j}^{(h)} y_2(z) \right\},$$

which are found to have the inverse form

* The symbol E will be used consistently to designate some function which is bounded. There is to be no implication that the symbol denotes the same function in different instances.

The formula given for $M_{\mu,\nu}$ by Whittaker and Watson, p. 346, appears to be in error.

† The symbolism $|\xi| > N$ is to be interpreted merely as an abbreviation of the statement when $|\xi|$ is sufficiently large.

$$(12b) \quad y_j(z) = \Gamma(1 \pm \frac{1}{2}) \{ C_{j,1}^{(h)} y_{h,1}(z) + C_{j,2}^{(h)} y_{h,2}(z) \},$$

define for each index h a set of solutions alternative to those described above. When ξ lies in the sector $\Xi^{(m)}$ they are found to be of the forms

$$(13) \quad y_{h,j}(z) = \Psi(z) \xi^{-1/4} \left\{ A_{j,1}^{(h,m)} \xi^{-k} e^{\xi/2} \left[1 + \frac{E(\xi)}{\xi} \right] + A_{j,2}^{(h,m)} \xi^k e^{-\xi/2} \left[1 + \frac{E(\xi)}{\xi} \right] \right\},$$

with

$$(14) \quad \begin{aligned} A_{j,1}^{(h,m)} &= \pm \pi \{ C_{2,2-j}^{(h)} C_{1,1}^{(m)} - C_{1,2-j}^{(h)} C_{2,1}^{(m)} \}, \\ A_{j,2}^{(h,m)} &= \pm \pi \{ C_{2,2-j}^{(h)} C_{1,2}^{(m)} - C_{1,2-j}^{(h)} C_{2,2}^{(m)} \}. \end{aligned}$$

These formulas reduce when $m=h$ to give

$$(13a) \quad \begin{aligned} y_{h,j}(z) &= \Psi(z) \xi^{\mp k-1/4} e^{\pm \xi/2} \left[1 + \frac{E(\xi)}{\xi} \right], \text{ for } \xi \text{ in } \Xi^{(h)}, \\ y'_{h,j}(z) &= \frac{\pm \lambda^{1/2}}{\Psi(z)} \xi^{\mp k+1/4} e^{\pm \xi/2} \left[1 + \frac{E(\xi)}{\xi} \right], \end{aligned}$$

and in the simplicity of these forms lies the advantage of the solutions in question.

From the formulas (13a), and the fact that the Wronskian of any pair of solutions is independent of z , it is found directly that

$$W(y_{h,1}, y_{h,2}) \equiv \lambda^{1/2}.$$

5. Formal developments. Let the function $\theta(z)$ be defined by the relation

$$\theta(z) = \chi_2(z, \lambda) - \Omega(z, \lambda).$$

Then the given equation may be written in the form

$$u''(z) - \{ \lambda^2 \chi_0^2(z) + \lambda \chi_1(z) + \Omega(z, \lambda) \} u(z) = \theta(z) u(z),$$

and is accordingly solved by any function $u(z)$ which satisfies the relation

$$(15) \quad u(z) = y(z) + \frac{1}{\lambda^{1/2}} \int_{*}^z \{ y_{h,1}(z) y_{h,2}(z_1) - y_{h,2}(z) y_{h,1}(z_1) \} \theta(z_1) u(z_1) dz_1.$$

In this $y(z)$ may be any solution of the related equation and the integration may be extended over any desired path which extends from any fixed point $*$ to the variable point z . To each choice of these elements there evidently cor-

responds a solution u . The correspondence between a solution u and the associated function y will be indicated by the use of the same subscripts upon the former as may be attached to the latter.

From the relations (5e) and (5f) it follows that

$$dz = \frac{\Psi^2(z)}{\lambda^{1/2} \xi^{1/2}} d\xi,$$

and hence if for any values of the arguments involved the functions Q are defined by the formulas

$$(16) \quad Q_i(\alpha, \beta, \delta) = \pm \theta(z_1) \Psi^2(z_1) \{ \alpha_{h,i}(z) \beta_{h,3-i}(z_1) - \alpha_{h,3-i}(z) \beta_{h,i}(z_1) \delta \},$$

the relation (15) may be written

$$(17) \quad u(z) = y(z) + \frac{1}{\lambda} \int_*^z Q_i(y, y, 1) u(z_1) \frac{d\xi_1}{\xi_1^{1/2}}.$$

It should be observed that differentiation of this formula leads simply to the associated one

$$(18) \quad u'(z) = y'(z) + \frac{1}{\lambda} \int_*^z Q_i(y', y, 1) u(z_1) \frac{d\xi_1}{\xi_1^{1/2}}.$$

When the form of $u(z)$ has been derived, this formula serves readily to yield the form of $u'(z)$.

For the case in which the formula (17) is to be applied to the particular solutions $u_{h,i}(z)$ associated with the related solutions $y_{h,i}(z)$ it is convenient to define the symbols U and Y by the formulas

$$(19) \quad U_{h,i}(z) = \frac{\xi^{\pm h+1/4}}{\Psi(z)} e^{\mp \xi/2} u_{h,i}(z), \quad Y_{h,i}^{(n)}(z) = \frac{\xi^{\pm h+1/4}}{\Psi(z)} e^{\mp \xi/2} y_{h,i}^{(n)}(z),$$

in which $y^{(0)}(z) \equiv y(z)$, and $y^{(n)}(z)$ for $n > 0$ is to be subsequently defined. Then formula (17) may be given in either one of the forms

$$(20) \quad \begin{aligned} u_{h,i}(z) &= y_{h,i}(z) + I(u_{h,i}, *, \xi), \\ U_{h,i}(z) &= Y_{h,i}^{(0)}(z) + J(U_{h,i}, *, \xi), \end{aligned}$$

in which the final terms may each again be given by alternative formulas thus:

$$(21) \quad I(u, *, \xi) \begin{cases} = \frac{1}{\lambda} \int_*^\xi Q_i(y, y, 1) u(z_1) \frac{d\xi_1}{\xi_1^{1/2}}, \\ = \frac{1}{\lambda} \int_*^\xi Q_i(y, Y^{(0)}, \xi_1^{\mp 2h} e^{\pm \xi_1}) \Psi^2(z_1) U(z_1) \frac{d\xi_1}{\xi_1}, \end{cases}$$

and

$$(22) \quad J(U, *, \xi) \begin{cases} = \frac{1}{\lambda} \int_*^\xi Q_j(Y^{(0)}, y, \xi^{\pm 2k} e^{\mp \xi}) u(z_1) \frac{d\xi_1}{\xi_1^{1/2}}, \\ = \frac{1}{\lambda} \int_*^\xi Q_j(Y^{(0)}, Y^{(0)}, \xi^{\pm 2k} \xi_1^{\mp 2k} e^{\mp (\xi - \xi_1)}) \Psi^2(z_1) U(z_1) \frac{d\xi_1}{\xi_1}. \end{cases}$$

The process of iteration, familiar from the theory of integral equations, may be applied to the equations (17) or (20). It leads formally to the relations

$$(23) \quad u(z) = y(z) + \sum_{n=1}^{\infty} y^{(n)}(z), \quad U(z) = Y^{(0)}(z) + \sum_{n=1}^{\infty} Y^{(n)}(z),$$

with terms given by the recurrence formulas

$$(24) \quad \begin{aligned} y^{(n)}(z) &= I(y^{(n-1)}, *, \xi), \\ Y^{(n)}(z) &= J(Y^{(n-1)}, *, \xi). \end{aligned}$$

It is to be shown in the following that the series in (23) which have been obtained formally are convergent under appropriate circumstances, so that actual solutions are represented by them.

6. The solutions $u_j(z)$ when $|\xi| \leq N$. With any choice of the constant N the region $|\xi| \leq N$ lies entirely within the domain R_ξ , provided $|\lambda|$ is sufficiently large. Hence the straight line from the origin to any point of the region lies within R_ξ , and may be chosen as the path of integration in the formulas (17) and (18). With this choice and with the rôle of $y(z)$ taken by $y_j(z)$, either one of the functions (6), the corresponding solution $u_j(z)$ takes at $\xi=0$, i.e., at $z=0$, the same values as y_j . Hence, from (8),

$$(25) \quad \begin{aligned} u_1(0) &= 0, & u_2(0) &= \Psi(0), \\ u_1'(0) &= \frac{\lambda^{1/2}}{2\Psi(0)}, & u_2'(0) &= \Psi'(0). \end{aligned}$$

With ξ bounded the variables z and ξ_1 are likewise bounded. The boundedness of the functions $y_{h,j}(z)$ and hence of the expression $Q_j(y, y, 1)$ follows directly, and accordingly formulas (24) and (21) yield the relation

$$|y_j^{(n)}(z)| \leq \frac{M}{|\lambda|} \int_0^\xi |y_j^{(n-1)}(z_1) \frac{d\xi_1}{\xi_1^{1/2}}|^\dagger.$$

Since $y_j^{(0)}(z)$ is bounded it follows directly by induction that for any index n

† It is to be understood that M is used merely to denote some sufficiently large constant.

$$|y_j^{(n)}(z)| \leq M \left\{ \frac{M}{|\lambda|} \right\}^n.$$

The series in the formula (23) accordingly converges for sufficiently large values of $|\lambda|$, and is in fact of the order $O(|\lambda|^{-1})$.

The conclusion thus deduced, together with that which immediately follows from (18) since $y_{h,j}'(z)\lambda^{-1/2}$, and hence $Q_j(y', y, 1)\lambda^{-1/2}$, are bounded, is the following, i.e., that

$$(26) \quad \begin{aligned} u_j(z) &= y_j(z) + \frac{E(z, \lambda)}{\lambda}, \\ u_j'(z) &= y_j'(z) + \frac{E(z, \lambda)}{\lambda^{1/2}}, \quad \text{when } |\xi| \leq N. \end{aligned}$$

7. Additional hypotheses. When ξ is not restricted to be numerically bounded the considerations are less simple than those of §6. In particular some stipulations restricting the configuration of the domain of the variables must be made, and will be framed in the following way.

A sub-region of R_ξ will be styled as a region of the type r if it is simply connected (finite or infinite), and fulfills the following specifications:

- (a) that it contains the origin and lies entirely within some one of the half planes bounded by the axis $\Re(\xi) = 0$;
- (b) that it contains no more than one segment of any line on which $\Re(\xi)$ is constant.

Concerning a region of this type, r , it will be observed that it invariably contains upon its boundary (possibly at infinity if the region is infinite) a specific point ξ_m , so located that there passes through each point ξ of the region an ordinary curve, Γ , which joins the origin with the point ξ_m , and upon which the abscissa, $\Re(\xi)$, varies monotonically with the arc length (in the sense of non-decreasing or non-increasing). The symbol Γ will be reserved for the designation of arcs of curves of the type described.

As a hypothesis upon the fundamental domain R_λ , and upon the range of the parameter λ , it will be assumed that

- (iv) The region R_λ is such that for each admissible value of λ every point of the corresponding region R_ξ lies in some sub-region of the type r .

If the domain R_λ is finite no additional assumptions respecting the given equation need be made. However, if the domain is infinite the discussion to follow necessitates the further and final hypothesis, i.e., that

(v) For all arcs upon which $|z| \geq A$ (A an arbitrarily large but fixed constant) and which for some admitted value of λ correspond to a curve of the type Γ , a relation

$$\int \left| \frac{\theta(z)}{\phi(z)} dz \right| < M$$

is uniformly satisfied.

8. The solutions $u_{h,j}(z)$. Let any region of the type r be considered and let the index h be determined by a (any) sector $\Xi^{(h)}$ which contains the region in question. With $y(z)$ in the formula (17) replaced by $y_{h,j}(z)$ the relation assumes the form (20) in which the limit of integration $*$ still remains to be specified. This limit will be chosen as either $* = \xi_m$ or $* = 0$ as dictated by the requirement that in proceeding along a Γ curve from the point $*$ to the point ξ the abscissa shall be algebraically non-decreasing in the case $j=1$, and non-increasing in the case $j=2$. Inasmuch as the reasoning is entirely similar the discussion will be given only for the case $j=1$. Then when the chosen region is one in which $\Re(\xi) > 0$, $* = 0$, while if $\Re(\xi) < 0$ in r , then $* = \xi_m$.

Case 1. $\Re(\xi) > 0$ in r . Since the integration extends from 0 to ξ , it is clear that the discussion of §6 applies without modification if $|\xi| \leq N$. Hence

$$(27) \quad |y_{h,1}^{(n)}(z)| \leq M \left\{ \frac{M}{|\lambda|} \right\}^n, \quad \text{when } |\xi| \leq N.$$

When $|\xi| > N$ the path of integration, which may be chosen as a Γ curve, contains a point ξ_0 for which $|\xi_0| = N$. Then let the relation (24) in its second form be written

$$(28) \quad Y_{h,1}^{(n)}(z) = J(Y_{h,1}^{(n-1)}, 0, \xi_0) + J(Y_{h,1}^{(n-1)}, \xi_0, \xi).$$

The formulas (13a) and (19) show directly that when $|\xi|$ is large the functions $Y_{h,j}^{(0)}(z)$ are bounded. Since in the assumed configuration the quantities $\xi^{2k}e^{-\xi}$ and $(\xi - \xi_1^{-1})^{2k}e^{-(\xi - \xi_1)}$ are likewise bounded, it is clear from the first and second of the formulas (22) respectively that

$$(29) \quad \begin{aligned} |J(Y_{h,1}^{(n-1)}, 0, \xi_0)| &< \frac{M}{|\lambda|} \int_0^{\xi_0} |y_{h,1}^{(n-1)}(z_1)| \frac{d\xi_1}{\xi_1^{1/2}}, \\ |J(Y_{h,1}^{(n-1)}, \xi_0, \xi)| &< \frac{M}{|\lambda|} \int_{\xi_0}^{\xi} |Y_{h,1}^{(n-1)}(z_1)| \cdot \left| \theta(z_1) \Psi^4(z_1) \frac{d\xi_1}{\xi_1} \right|. \end{aligned}$$

Since

$$\theta(z_1) \Psi^4(z_1) \frac{d\xi_1}{\xi_1} = \frac{\theta(z_1)}{\phi(z_1)} dz_1,$$

it follows, in virtue of hypothesis (v), that when $n=1$ the integral on the right of the first of the relations (29) is bounded while that in the second relation is of the order $O(\log |\lambda|)$. For this value of n , therefore, the result

$$(30) \quad |Y_{h,1}^{(n)}(z)| < M \left\{ \frac{M \log |\lambda|}{|\lambda|} \right\}^n$$

may be concluded, and in virtue of (27) the general validity of the relation (30) follows by induction.

Case 2. $\Re(\xi) < 0$ in r . In this case the integration extends from ξ_m to the point ξ . If $|\xi| > N$ the formula

$$Y_{h,1}^{(n)}(z) = J(Y_{h,1}^{(n-1)}, \xi_m, \xi)$$

may be used, and since the right member is of precisely the structure of the final term of the relation (28) the conclusion (30) may be reached as in the preceding case.

If $|\xi| \leq N$ the first of the formulas (24) may be written

$$y_{h,1}^{(n)}(z) = I(y_{h,1}^{(n-1)}, \xi_0, \xi) + I(y_{h,1}^{(n-1)}, \xi_m, \xi_0).$$

With the present configuration the quantity $\xi_1^{-2k} e^{\xi_1}$ is bounded, and the formulas (21) show respectively that

$$\begin{aligned} |I(y_{h,1}^{(n-1)}, \xi_0, \xi)| &< \frac{M}{|\lambda|} \int_{\xi_0}^{\xi} \left| y_{h,1}^{(n-1)}(z_1) \frac{d\xi_1}{\xi_1^{1/2}} \right|, \\ |I(y_{h,1}^{(n-1)}, \xi_m, \xi_0)| &< \frac{M}{|\lambda|} \int_{\xi_m}^{\xi_0} \left| Y_{h,1}^{(n-1)}(z_1) \cdot \left| \theta(z_1) \Psi^4(z_1) \frac{d\xi_1}{\xi_1} \right| \right|. \end{aligned}$$

In the manner of the preceding discussion, it follows readily that

$$(31) \quad |y_{h,1}^{(n)}(z)| < M \left\{ \frac{M \log |\lambda|}{|\lambda|} \right\}^n, \quad \text{when } |\xi| \leq N.$$

This relation obviously displaces (27) and is, therefore, valid for all regions r in which $|\xi| \leq N$.

The results (30) and (31) assure the convergence of the series which occur in the formulas (23), when $|\lambda|$ is large. Recalling the relations (19) the results (together with those for the derived functions and for the case $j=2$) may be enunciated as follows.

Corresponding to any region of the type r there exists a pair of solutions of the given differential equation which in that region are subject to the descriptions

$$(32) \quad \begin{aligned} u_{h,j}(z) &= y_{h,j}(z) + \frac{E(z, \lambda) \log \lambda}{\lambda}, \\ u'_{h,j}(z) &= y'_{h,j}(z) + \frac{E(z, \lambda) \log \lambda}{\lambda^{1/2}}, \quad \text{when } |\xi| \leq N, \end{aligned}$$

and

$$(33) \quad \begin{aligned} u_{h,j}(z) &= y_{h,j}(z) + \frac{\Psi(z) \xi^{\mp k-1/4} e^{\pm \xi/2} E(z, \lambda) \log \lambda}{\lambda}, \\ u'_{h,j}(z) &= y'_{h,j}(z) + \frac{\xi^{\mp k+1/4}}{\Psi(z)} e^{\pm \xi/2} \frac{E(z, \lambda) \log \lambda}{\lambda^{1/2}}, \quad \text{when } |\xi| > N. \end{aligned}$$

It may be noted in connection with these formulas that in any region r within which $|\Re(\xi)|$ is unbounded the solution of the sub-dominant form is unique except for a constant factor. This follows from the fact that every solution $u(z)$ must be expressible in the form

$$u(z) \equiv \delta_1 u_{h,1}(z) + \delta_2 u_{h,2}(z),$$

with coefficients which are free from z , and unless the coefficient of the dominant solution on the right is zero the solution is itself of the dominant form.

It should also be remarked that each set of solutions $u_{h,j}(z)$ of the description (33) has been deduced for a specific region r , and, although the notation has not been designed to indicate the fact, this set of solutions for any one region r_1 is not, in general, identical with that for another region r_2 . In a special but important case the existence of a set of solutions which retain the forms (32), (33) over two abutting regions r may be deduced as follows. Let r_1 and r_2 be two regions of this type which lie within one and the same sector $\Xi^{(h)}$ and which abut along the imaginary ξ axis, r_1 lying in the half-plane $\Re(\xi) \leq 0$, and r_2 in the half-plane $\Re(\xi) \geq 0$. Denote by $u_{h,1}(z)$ the solution which is sub-dominant in r_1 and by $u_{h,2}(z)$ the solution which is sub-dominant in r_2 . These solutions are linearly independent, as may be seen by comparing the respective formulas (33) along the imaginary axis. Hence $u_{h,1}(z)$ and $u_{h,2}(z)$ are of the dominant form respectively in the regions r_2 and r_1 , and are thus described by the formulas (32) and (33) over the two regions r_1 and r_2 simultaneously. Such a pair of regions may constitute a large portion or even the whole of the region $\Xi^{(h)}$.

9. The solutions for general values of ξ . Between any three of the various solutions $u(z)$ which have been defined there exists a linear relation with constant coefficients. Thus, in particular, for each index h

$$(34) \quad u_{h,j}(z) \equiv \alpha_{1,j}^{(h)} u_1(z) + \alpha_{2,j}^{(h)} u_2(z), \quad j = 1, 2,$$

with coefficients given by the familiar formulas

$$\alpha_{j,i}^{(h)} = \mp \frac{W(u_{3-j}, u_{h,i})}{W(u_1, u_2)}, \quad i, j = 1, 2.$$

Since the Wronskians are all independent of z , they may be evaluated at the origin, and in virtue of the relations (8), (25), and (32) it is found that

$$\alpha_{j,i}^{(h)} = \left\{ \mp \frac{W(y_{3-j}, y_{h,i})}{W(y_1, y_2)} + \frac{E(\lambda) \log \lambda}{\lambda} \right\}.$$

On the other hand, the relations (12) may be made to yield the equalities

$$\mp \frac{W(y_{3-j}, y_{h,i})}{W(y_1, y_2)} = (-1)^i \Gamma \left(\mp \frac{1}{2} \right) C_{3-j, 3-i}^{(h)},$$

whence it follows that

$$(35) \quad \alpha_{j,i}^{(h)} = (-1)^i \Gamma \left(\mp \frac{1}{2} \right) \left\{ C_{3-j, 3-i}^{(h)} + \frac{E(\lambda) \log \lambda}{\lambda} \right\},$$

the coefficients on the right being given by the formulas (11).

With the values (35) the relations inverse to (34) are readily computed to be

$$(36) \quad u_j(z) = \Gamma \left(1 \pm \frac{1}{2} \right) \left\{ u_{h,1}(z) \left\{ C_{j,1}^{(h)} + \frac{E(\lambda) \log \lambda}{\lambda} \right\} \right. \\ \left. + u_{h,2}(z) \left\{ C_{j,2}^{(h)} + \frac{E(\lambda) \log \lambda}{\lambda} \right\} \right\},$$

a result which is obtainable independently of the value of h . Let any point z be given then, and let h be determined as the index of the region $\Xi^{(h)}$ in which the point z lies. The formula (36) may then be resorted to, and since for the value of z given the formulas (33) and (13a) are valid, the asymptotic expression for $u_j(z)$ is obtained. Quantitatively the results may be summarized as follows.

The solutions of the given differential equation which are determined by the initial values (25) are given when $|\xi| \leq N$ by the formulas

$$(37) \quad u_i(z) = \frac{1}{\lambda^{1/4} \phi^{1/2}(z)} M_{k, \pm 1/4}(\xi) + \frac{E(z, \lambda)}{\lambda}, \\ u'_j(z) = \lambda^{3/4} \phi^{1/2}(z) \left\{ M'_{k, \pm 1/4}(\xi) - \frac{\phi'(z)}{2\lambda \phi^2(z)} M_{k, \pm 1/4}(\xi) \right\} + \frac{E(z, \lambda)}{\lambda^{1/2}}.$$

When $|\xi| > N$ and ξ lies in a sector $\Xi^{(h)}$, then

$$u_j(z) = \frac{\Gamma(1 \pm \frac{1}{2})}{\lambda^{1/4} \phi^{1/2}(z)} \{ [C_{j,1}^{(h)}] \xi^{-k} e^{\xi/2} + [C_{j,2}^{(h)}] \xi^k e^{-\xi/2} \}, \quad (38)$$

$$u'_j(z) = \lambda^{3/4} \phi^{1/2}(z) \Gamma(1 \pm \frac{1}{2}) \{ [C_{j,1}^{(h)}] \xi^{-k} e^{\xi/2} - [C_{j,2}^{(h)}] \xi^k e^{-\xi/2} \},$$

in which the coefficients are given by the formulas (11), and the symbol $[C]$ designates in each case an expression which differs from C by quantities of the order $O(|\xi|^{-1})$ and $O(\log |\lambda|/|\lambda|)$.

Lastly, on substituting the forms (38) into the relation (34) (the values of h in the two expressions being not necessarily the same) the forms of the solutions $u_{h,j}(z)$ for general values of z may be derived. These results are the following:

For any index $h = -2, -1, 0$, or 1 , there exists for the given differential equation a pair of solutions $u_{h,j}(z)$, which, when $|\xi| > N$ and z lies in a region r (or a pair of such regions which abut along $\Re(\xi) = 0$) in the domain $\Xi^{(h)}$, are of the form

$$u_{h,j}(z) = \frac{1}{\lambda^{1/4} \phi^{1/2}(z)} \xi^{\mp k} e^{\pm \xi/2} [1],$$

$$u'_{h,j}(z) = \pm \lambda^{3/4} \phi^{1/2}(z) \xi^{\mp k} e^{\pm \xi/2} [1]. \quad (39)$$

When $|\xi| > N$ but z is in the domain $\Xi^{(m)}$, then the respective forms are

$$u_{h,j}(z) = \frac{1}{\lambda^{1/4} \phi^{1/2}(z)} \{ [A_{j,1}^{(h,m)}] \xi^{-k} e^{\xi/2} + [A_{j,2}^{(h,m)}] \xi^k e^{-\xi/2} \},$$

$$u'_{h,j}(z) = \lambda^{3/4} \phi^{1/2}(z) \{ [A_{j,1}^{(h,m)}] \xi^{-k} e^{\xi/2} - [A_{j,2}^{(h,m)}] \xi^k e^{-\xi/2} \}, \quad (40)$$

with coefficients given by the formulas (14) and (11).

It is sometimes possible to justify replacing a coefficient $[0]$ by 0 itself on the ground that the forms given for two sectors $\Xi^{(h)}$ which have a region in common must be asymptotically equivalent within that region. Thus, by way of illustration, if every sector $\Xi^{(h)}$ consists of but two regions r , and there accordingly exist solutions having the forms (39) over the entire sectors $\Xi^{(h)}$, it is found that formulas (40) yield for ξ in the sector $\Xi^{(-1)}$ the value

$$u_{0,2}(z) = \frac{1}{\lambda^{1/4} \phi^{1/2}(z)} \{ [0] \xi^{-k} e^{\xi/2} + [1] \xi^k e^{-\xi/2} \}.$$

Now unless the coefficient indicated as [0] is actually 0, the term in which it occurs is dominant in the region common to $\Xi^{(-1)}$ and $\Xi^{(0)}$ and the formula conflicts with the appropriate formula (39). Hence in this case it must be concluded that the coefficient in question actually vanishes.

Lastly, when $|\xi| \leq N$ these solutions may be described by the formulas

$$\begin{aligned}
 (41) \quad u_{h,j}(z) &= \frac{\mp 1}{\lambda^{1/4} \phi^{1/2}(z)} \left\{ \Gamma\left(-\frac{1}{2}\right) C_{2,3-j}^{(h)} M_{k,1/4}(\xi) + \Gamma\left(\frac{1}{2}\right) C_{1,3-j}^{(h)} M_{k,-1/4}(\xi) \right\} \\
 &\quad + \frac{E(z, \lambda) \log \lambda}{\lambda}, \\
 u'_{h,j}(z) &= \mp \lambda^{3/4} \phi^{1/2}(z) \left\{ \Gamma\left(-\frac{1}{2}\right) C_{2,3-j}^{(h)} \left(M'_{k,1/4}(\xi) - \frac{\phi'(z)}{2\lambda\phi^2(z)} M_{k,1/4}(\xi) \right) \right. \\
 &\quad \left. + \Gamma\left(\frac{1}{2}\right) C_{1,3-j}^{(h)} \left(M'_{k,-1/4}(\xi) - \frac{\phi'(z)}{2\lambda\phi^2(z)} M_{k,-1/4}(\xi) \right) \right\} + \frac{E(z, \lambda) \log \lambda}{\lambda}.
 \end{aligned}$$

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THE INVERSION OF THE LAPLACE INTEGRAL AND THE RELATED MOMENT PROBLEM*

BY

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INTRODUCTION

Let $\alpha(t)$ be a complex function of the real variable t , of bounded variation in the interval $(0, R)$ for every positive R , and such that the integral

$$(1) \quad f(x) = \int_0^\infty e^{-xt} d\alpha(t)$$

converges for some complex value of x . It is then known that the integral will converge for all complex x of greater real part and will consequently define a function, which we have denoted by $f(x)$, in a half-plane. By the inversion of the integral (1) we mean the determination of the function $\alpha(t)$ in terms of the function $f(x)$. There is one special case of (1) of particular interest, that in which $\alpha(t)$ is a step-function with jumps only at the integral points. Then

$$(2) \quad f(x) = \sum_{n=0}^{\infty} a_n e^{-nx}.$$

In this case the problem of inversion reduces to the determination of the coefficients of the power series (2). If we set $z = e^{-x}$ in (2) we get

$$F(z) = f(\log(1/z)) = \sum_{n=0}^{\infty} a_n z^n,$$

and there are two familiar determinations of the coefficients. One is in terms of a contour integral

$$a_n = \frac{1}{2\pi i} \int_C \frac{F(z)}{z^{n+1}} dz$$

where the contour C may be taken as a circle with center at $z=0$ and with any sufficiently small radius, the integration being in the positive sense. The other is

$$a_n = F^{(n)}(0)/n!.$$

If we return to the function $f(x)$, the contour C becomes a vertical line in

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the x -plane. Likewise we should have

$$a_n = \lim_{z \rightarrow \infty} \frac{1}{n!} \frac{d^n}{dz^n} f\left(\log \frac{1}{z}\right)$$

as x becomes infinite along the positive real axis.*

Reasoning from this special case, we should expect that for the general case (1) there would be two determinations of $\alpha(t)$, one in terms of a contour integral along a vertical line, the other in terms of the derivatives of $f(x)$ for large real positive values of x . The first of these is in fact very well known. For the special case in which

$$(3) \quad \alpha(t) = \int_0^t \phi(u) du$$

and

$$(4) \quad f(x) = \int_0^\infty e^{-xt} \phi(t) dt$$

it was already known to Cauchy in the form

$$\phi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(x) e^{xt} dx,$$

where c is a sufficiently large real constant and the path of integration is a vertical line. The general case (1) seems first to have been treated by H. Hamburger.† It is remarkable that the second method, discovered the first in the special case (2), has received no attention until very recent times. It was apparently discovered first by W. P. Mason in his researches in electrical theory of 1929. No rigorous derivation of the inversion formula was published, however. In 1930 E. L. Post did obtain an inversion formula of the type in question for the special case in which the function $\alpha(t)$ has the form (3) with $\phi(u)$ a continuous function.‡ It is the purpose of the present paper to obtain an inversion formula for the general integral (1).

By way of introducing this inversion operator let us consider first functions $f(x)$ which are analytic at infinity and which vanish there. That is, $f(x)$

* We could of course take the approach along any parallel line or along any curve proceeding indefinitely to the right.

† H. Hamburger, *Über eine Riemannsche Formel aus der Theorie der Dirichletschen Reihen*, Mathematische Zeitschrift, vol. 6 (1920), p. 6. See also D. V. Widder, *A generalization of Dirichlet's series and of Laplace's integrals by means of a Stieltjes integral*, these Transactions, vol. 31 (1929), p. 708. We shall refer to this latter paper as I.

‡ E. L. Post, *Generalized differentiation*, these Transactions, vol. 32 (1930), p. 772.

can be represented for $|x|$ sufficiently large by a convergent power series in $1/x$,

$$(5) \quad f(x) = \sum_{n=0}^{\infty} \frac{b_n}{x^{n+1}}.$$

If we note that

$$\frac{1}{x^{n+1}} = \int_0^{\infty} e^{-xt} \frac{t^n}{n!} dt$$

for all x whose real part is positive, we see that

$$f(x) = \int_0^{\infty} e^{-xt} \phi(t) dt,$$

$$\phi(t) = \sum_{n=0}^{\infty} \frac{b_n t^n}{n!}.$$

Now introduce the operator

$$L_{k,t}[f(x)] = \frac{(-1)^k}{k!} f^{(k)}\left(\frac{k}{t}\right) \left(\frac{k}{t}\right)^{k+1}.$$

For the series (5),

$$L_{k,t}[f(x)] = \sum_{n=0}^{\infty} \frac{(n+k)!}{n!k!} b_n \left(\frac{t}{k}\right)^n.$$

As k becomes infinite,

$$\lim_{k \rightarrow \infty} \frac{(n+k)!}{k!k^n} = 1 \quad (n = 0, 1, \dots),$$

so that, if it were permissible to interchange summation and limit signs, we should have

$$\lim_{k \rightarrow \infty} L_{k,t}[f(x)] = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} = \phi(t).$$

This leads us to introduce an operator $L_t[f(x)]$ by the equation

$$L_t[f(x)] = \lim_{k \rightarrow \infty} L_{k,t}[f(x)].$$

We could easily justify the formal steps taken above and show rigorously that

$$L_t[f(x)] = \phi(t) \quad (0 < t < \infty).$$

However, our purpose for the moment is merely to provide a heuristic ap-

proach. Much more general cases, in which the method of series will not be applicable, will be treated later.

We may now see what sort of operator will apply to the Stieltjes integral (1) by performing an integration by parts,

$$f(x) = x \int_0^{\infty} e^{-xt} \alpha(t) dt.$$

If we apply the operator L_t to the function $f(x)/x$, we expect, guided by the above formal work, to obtain the function $\alpha(t)$. On the other hand we can show, as we do in §3, that

$$L_{k,t}[f(x)/x] = f(\infty) + (-1)^{k+1} \int_{k/t}^{\infty} \frac{u^k}{k!} f^{(k+1)}(u) du.$$

We are thus led to introduce a second operator S_t by the equations

$$\begin{aligned} S_{k,t}[f(x)] &= f(\infty) + (-1)^{k+1} \int_{k/t}^{\infty} \frac{u^k}{k!} f^{(k+1)}(u) du, \\ (6) \quad S_t[f(x)] &= \lim_{k \rightarrow \infty} S_{k,t}[f(x)]. \end{aligned}$$

We shall show in §3 that our conjecture is verified, that for every convergent integral (1)

$$S_t[f(x)] = \frac{\alpha(t+) + \alpha(t-)}{2} \quad (0 < t < \infty)$$

at least if a suitable constant is added to the function $\alpha(t)$ so as to make $\alpha(0) = 0$. (This change in $\alpha(t)$ of course produces no change in $f(x)$.)

In §4 we treat the most general integral (4) where the function $\phi(t)$ is integrable in the sense of Lebesgue, and we find that the operator L_t inverts the integral for almost all positive values of x . The remaining sections of Part I are devoted to further inversion formulas, it being always understood that the function $f(x)$ is known to have a representation (1). In Part II we drop this assumption and consider all functions $f(x)$ to which the operators L and S are applicable. Their very nature demands, of course, that the functions $f(x)$ must have derivatives of all orders and must have certain asymptotic properties as x becomes positively infinite. We are thus able to develop necessary and sufficient conditions that a function $f(x)$ should have a representation (1). Among other results we prove a theorem of S. Bernstein to the effect that if $f(x)$ is completely monotonic,

$$(7) \quad (-1)^k f^{(k)}(x) \geq 0 \quad (k = 0, 1, \dots),$$

it must have the representation (1) with $\alpha(t)$ a non-decreasing function. By use of our operator S this result becomes almost self-evident. For, if (7) holds, $S_{k,t}[f(x)]$ is clearly a non-decreasing function of t . The same must be true of $S_t[f(x)]$ if this function exists. The existence of this limit of course requires proof.

In Part III we take up several applications. First we discuss the zeros of a function $f(x)$ represented in the form (1). To make our results more descriptive let us consider here the case (4) where $\phi(t)$ is real and continuous. We are able to show that if $\phi(t)$ has just n changes of sign in $(0, \infty)$ then $f^{(k)}(x)$ will have exactly the same number in the interval of convergence of (4) for all k sufficiently large. We are even able to compute the coordinates of the zeros of $\phi(t)$ in terms of those of $f^{(k)}(x)$. Thus if the zeros of $\phi(t)$ are at the points t_1, t_2, \dots, t_n , where

$$0 < t_1 < t_2 < \dots < t_n,$$

and if those of $f^{(k)}(x)$ are at the points $x_{1,k}, x_{2,k}, \dots, x_{n,k}$, where

$$x_{1,k} > x_{2,k} > \dots > x_{n,k},$$

then

$$\lim_{k \rightarrow \infty} \frac{k}{x_{i,k}} = t_i \quad (i = 1, 2, \dots).$$

In an article in the Proceedings of the National Academy of Sciences* we announced this result with the restriction that $\phi(t)$ should approach a limit as t becomes infinite, observing that the condition was probably redundant. J. Karamata† in a subsequent note of the same Proceedings removed this restriction but imposed another. In the present paper we remove all conditions of the type, demanding only that the integral (1) should converge, a condition imposed by the nature of the problem. We do not even demand that $\alpha(t)$ or $\phi(t)$ should be continuous.

In Part IV we treat the complex case. The integral is taken in the form (4), and the function $\phi(t)$ is supposed analytic in the half-plane for which the real part of t is positive and of such a nature that the integral (4) converges when the path of integration is the positive real axis. We are then able to show that

$$L_t[f(x)] = \phi(t)$$

for all complex t whose real part is positive. By use of this result we are able to treat also the complex zeros of $\phi(t)$ in terms of the complex zeros of $f^{(k)}(x)$.

* D. V. Widder, *On the changes of sign of the derivatives of a function defined by a Laplace integral*, Proceedings of the National Academy of Sciences, vol. 18 (1932), p. 112.

† J. Karamata, *Remarks on a theorem of D. V. Widder*, Proceedings of the National Academy of Sciences, vol. 18 (1932), p. 406.

In Part V we take up a moment problem intimately related to the Laplace integral. If in (1) we allow x to take on only positive integral values we have by an obvious change of variable

$$(8) \quad f(n) = \mu_n = \int_0^\infty e^{-nt} d\alpha(t) = \int_0^1 t^n d\beta(t) \quad (n = 0, 1, 2, \dots).$$

The determination of $\beta(t)$ in terms of the sequence $\{\mu_n\}$ is the moment problem of Hausdorff. Since the variable n now runs through a discrete set of values we expect to get an operator applicable to the sequence $\{\mu_n\}$ as the operator L was to the integral (4) by replacing the derivative of order k by a difference of order k and by replacing a $(k+1)$ th power of x by a product $n(n+1) \cdots (n+k)$. Proceeding in this way we are led to define the operator $L_t\{\mu_n\}$ by the equations

$$L_{k,t}\{\mu_n\} = \frac{(n+k+1)!}{n!k!} (-1)^k \Delta^k \mu_n, \quad n = \left[\frac{kt}{1-t} \right],$$

$$L_t\{\mu_n\} = \lim_{k \rightarrow \infty} L_{k,t}\{\mu_n\}.$$

Here $[u]$ means the greatest integer contained in u . We find in fact that this operator does invert the moment sequence (8) if $\beta(t)$ has the form (3) with $\phi(t)$ integrable in $(0, 1)$. That is,

$$L_t\{\mu_n\} = \phi(t)$$

almost everywhere in $(0, 1)$. In defining an operator S which will be applicable to the general sequence (8) we again proceed by analogy replacing the integral sign by the summation sign and the derivatives by differences in (6). In this way we arrive at the operators

$$S_{k,t}\{\mu_n\} = -\mu_\infty - \sum_{i=n+1}^\infty \frac{(i+k)!}{i!k!} (-1)^{k+1} \Delta^{k+1} \mu_i,$$

$$S_t\{\mu_n\} = \lim_{k \rightarrow \infty} S_{k,t}\{\mu_n\}, \quad n = \left[\frac{kt}{1-t} \right].$$

We then prove that

$$S_t\{\mu_n\} = \frac{\beta(t+) + \beta(t-)}{2} \quad (0 < t < 1).$$

We then turn to sequences $\{\mu_n\}$ which are not known to be moment sequences and discuss the effect of the operators L and S on them. We are able

to obtain necessary and sufficient conditions that a given sequence should be a moment sequence. In particular Hausdorff's theorem to the effect that every completely monotonic sequence,

$$(-1)^k \Delta^k \mu_n \geq 0 \quad (k = 0, 1, \dots; n = 0, 1, \dots),$$

has the form (8) with $\beta(t)$ non-decreasing is obtained by use of the moment operator $S_t\{\mu_n\}$ with the same ease as Bernstein's theorem was obtained by use of the integral operator $S_t[f(x)]$.

As a further application of the operators L and S a study is made of the changes of sign of a sequence $\{\mu_n\}$ as affected by the changes of trend in $\beta(t)$ or by the changes of sign in $\phi(t)$. Results analogous to these already mentioned for the Laplace integral are obtained, results which serve to generalize certain theorems of M. Fekete.† In conclusion the complex case is treated. A slight modification of the operator L is necessary to give it meaning for complex t . In the foregoing definition we merely replace $[kt/(1-t)]$ by $kt/(1-t)$ thus defining an operator which we denote by $L_t^*\{\mu_n\}$. We find that if $\phi(t)$ is analytic in a circle of unit diameter with center at $t = \frac{1}{2}$, then throughout that circle

$$L_t^*\{\mu_n\} = \phi(t).$$

The method found to be most serviceable in the major part of this paper is the Laplace method of determining an asymptotic expression for an integral of the form

$$\int [f(t)]^k \phi(t) dt$$

when k becomes infinite. In the last section of the paper we use a modification of this method due to Perron for the case in which the integrand is complex and the path of integration is in the complex plane.

PART I

INVERSION FORMULAS

1. **The problem.** In Part I we shall discuss functions $f(x)$ of the real variable x which are known to be expressible by means of a Laplace-Stieltjes integral

$$(1.1) \quad f(x) = \int_0^\infty e^{-xt} d\alpha(t)$$

where the function $\alpha(t)$ is a real function of bounded variation in the interval

† M. Fekete, *Sur les changements de signe d'une fonction continue dans un intervalle*, Paris Comptes Rendus, vol. 190 (1930), p. 1366. References to Fekete's earlier work on the subject are given in this article.

$0 \leq t \leq R$ for every positive R , where $\alpha(0) = 0$, and where the integral converges for some value of x . We shall obtain a formula for the determination of $\alpha(t)$ in terms of the values of $f(x)$ and its derivatives. It will appear that a knowledge of these values in a neighborhood of $x = +\infty$ will be sufficient. In particular if $\alpha(t)$ is the integral of a function $\phi(t)$ so that (1.1) becomes

$$(1.2) \quad f(x) = \int_0^\infty e^{-xt} \phi(t) dt,$$

we shall obtain a similar formula for $\phi(t)$.

2. A preliminary limit. In order to develop the inversion formula for (1.1) we find it useful to know the value of the following limit:

$$\lim_{k \rightarrow \infty} e^{-k/t} \left[1 + \frac{k}{t} + \frac{1}{2!} \left(\frac{k}{t} \right)^2 + \cdots + \frac{1}{k!} \left(\frac{k}{t} \right)^k \right]$$

for all positive t . To show the existence of this limit and to obtain its value we first express it as a definite integral by use of Taylor's formula with exact remainder,

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots + f^{(k)}(0)\frac{x^k}{k!} + \int_0^x \frac{(x-u)^k}{k!} f^{(k+1)}(u) du.$$

For our purposes take $f(x) = e^x$ and replace x by k/t . Then

$$e^{k/t} = 1 + \frac{k}{t} + \frac{1}{2!} \left(\frac{k}{t} \right)^2 + \cdots + \frac{1}{k!} \left(\frac{k}{t} \right)^k + \frac{1}{k!} \int_0^{k/t} \left(\frac{k}{t} - u \right)^k e^u du,$$

$$\begin{aligned} e^{-k/t} \left[1 + \frac{k}{t} + \frac{1}{2!} \left(\frac{k}{t} \right)^2 + \cdots + \frac{1}{k!} \left(\frac{k}{t} \right)^k \right] \\ = 1 - \frac{1}{k!} \int_0^{k/t} \left(\frac{k}{t} - u \right)^k e^{u-(k/t)} du = 1 - \frac{1}{k!} \int_0^{k/t} u^k e^{-u} du. \end{aligned}$$

If we set $u = ky$ this becomes

$$H_k(t) = 1 - \frac{k^{k+1}}{k!} \int_0^{1/t} [ye^{-y}]^k dy.$$

We can show at once that $H_k(t)$ approaches 1 as k becomes infinite for $t > 1$. For, the function ye^{-y} has a single maximum at $y=1$, and is consequently increasing for $y < 1$, decreasing for $y > 1$. For $t > 1$ this maximum is outside the interval of integration. Hence

$$|H_k(t) - 1| < \frac{k^{k+1}}{k!} \frac{e^{-k/t}}{t^{k+1}}.$$

The right-hand side of this inequality approaches zero as k becomes infinite, as one sees by use of Stirling's formula.

Next suppose that $t < 1$. Since

$$k! = \int_0^{\infty} e^{-u} u^k du$$

we may clearly obtain the following expression for $H_k(t)$:

$$H_k(t) = \frac{1}{k!} \int_{k/t}^{\infty} u^k e^{-u} du = \frac{k^{k+1}}{k!} \int_{1/t}^{\infty} [ye^{-y}]^k dy.$$

If $t < 1$, the function ye^{-y} is a decreasing function for $y > 1/t$, so that

$$|H_k(t)| < \left[\frac{1}{t} e^{-1/t} \right]^{k-1} \frac{k^{k+1}}{k!} \int_{1/t}^{\infty} ye^{-y} dy.$$

The right-hand side of this inequality approaches zero as k becomes infinite, so that $H_k(t)$ must do so also.

It remains to treat the case $t = 1$. We have

$$H_k(1) = 1 - \frac{1}{k!} \int_0^k u^k e^{-u} du,$$

and by a direct application of Laplace's method* we see that

$$\lim_{k \rightarrow \infty} H_k(1) = \frac{1}{2}.$$

We have thus proved

THEOREM 1. *On setting*

$$H_k(t) = e^{-k/t} \left[1 + \frac{k}{t} + \frac{1}{2!} \left(\frac{k}{t} \right)^2 + \cdots + \frac{1}{k!} \left(\frac{k}{t} \right)^k \right],$$

we have

$$H_k(t) = 1 - \int_0^{k/t} \frac{u^k}{k!} e^{-u} du = \int_{k/t}^{\infty} \frac{u^k}{k!} e^{-u} du,$$

and

$$\lim_{k \rightarrow \infty} H_k(t) = \begin{cases} 0 & \text{if } 0 < t < 1 \\ \frac{1}{2} & \text{if } t = 1 \\ 1 & \text{if } 1 < t < \infty. \end{cases}$$

If we set $g(t) = \lim_{k \rightarrow \infty} H_k(t)$, then

* G. Pólya and G. Szegő, *Aufgaben und Lehrsätze aus der Analysis*, vol. I, chapter 2, p. 80, problem 210. The result was known to Jacobi; cf. *Gesammelte Werke*, vol. 7, p. 213.

$$e^{-x} = \int_0^{\infty} e^{-xt} dg(t),$$

so that Theorem 1 gives us an inversion formula for the integral (1.1) in this special case.

3. **The inversion of the Laplace-Stieltjes integral.** From the result of Theorem 1 we can now conjecture the inversion operator for the general integral (1.1). We introduce the following

DEFINITION. An operator $S_t[f(x)]$ is defined by the equations

$$(3.1) \quad S_{k,t}[f(x)] = f(\infty) + (-1)^{k+1} \int_{k/t}^{\infty} \frac{u^k}{k!} f^{(k+1)}(u) du \quad (k = 0, 1, 2, \dots),$$

$$(3.2) \quad S_t[f(x)] = \lim_{k \rightarrow \infty} S_{k,t}[f(x)].$$

For this operator to be applicable to a given function $f(x)$ it is necessary that each of the operations involved in the definition should be well defined for the function. Thus $f(x)$ must have continuous derivatives of all orders, must approach a finite limit as x becomes infinite, the improper integrals (3.1) must converge, and the limit (3.2) must exist. The operator is clearly distributive. We shall show in this section that it is well defined if $f(x)$ has the representation (1.1) and that it serves to invert that integral. The result to be established is the following:

THEOREM 2. If the integral

$$(3.3) \quad f(x) = \int_0^{\infty} e^{-xt} d\alpha(t) \quad (\alpha(0) = 0)$$

converges for $x > c$, then

$$S_t[f(x)] = \frac{\alpha(t+) + \alpha(t-)}{2} \quad (t > 0).$$

Since the given integral converges for $x > c$ we know that there exists a constant M such that

$$(3.4) \quad |\alpha(t)| < Me^{gt} \quad (0 \leq t < \infty),$$

where g is a positive constant greater than c .^{*} Hence, on integrating by parts, we obtain

^{*} D. V. Widder, I, p. 703, Lemma 2.

$$(3.5) \quad \begin{aligned} f(x) &= \lim_{t \rightarrow \infty} [e^{-xt}\alpha(t)] + x \int_0^{\infty} e^{-xt}\alpha(t)dt \quad (x > g), \\ f(x) &= x \int_0^{\infty} e^{-xt}\alpha(t)dt. \end{aligned}$$

Set $f(x)/x = F(x)$ and introduce a new operator S by the

DEFINITION. An operator $L_t[f(x)]$ is defined by the equations

$$\begin{aligned} L_{k,t}[f(x)] &= (-1)^k f^{(k)}(k/t)(k/t)^{k+1}/k! \quad (k = 0, 1, 2, \dots), \\ L_t[f(x)] &= \lim_{k \rightarrow \infty} L_{k,t}[f(x)]. \end{aligned}$$

We shall show that this operation is well defined when applied to $F(x)$ and that the result of the operation is $[\alpha(t+) + \alpha(t-)]/2$. If $x > g$ equation (3.5) gives us*

$$F^{(k)}(x) = (-1)^k \int_0^{\infty} e^{-xt} t^k \alpha(t) dt.$$

Hence

$$L_{k,t}[F(x)] = \frac{1}{k!} \left(\frac{k}{t}\right)^{k+1} \int_0^{\infty} e^{-ky/t} y^k \alpha(y) dy.$$

Let t_0 be an arbitrary positive value of t and make the transformation $u = y/t_0$. Then

$$L_{k,t_0}[F(x)] = \frac{k^{k+1}}{k!} \int_0^{\infty} e^{-ku} u^k \alpha(t_0 u) du.$$

By use of the function $g(t)$ of §2 we now define the function

$$\psi(u) = [\alpha(t_0+) - \alpha(t_0-)]g(u) + \alpha(t_0-).$$

This function has the properties

$$\begin{aligned} \psi(1+) &= \alpha(t_0+), \\ \psi(1-) &= \alpha(t_0-), \\ \psi(1) &= \frac{\alpha(t_0+) + \alpha(t_0-)}{2}, \end{aligned}$$

so that the function

$$\phi(u) = \alpha(t_0 u) - \psi(u)$$

has the properties

* D. V. Widder, I, p. 702, Corollary 2.

$$(3.6) \quad \phi(1+) = \phi(1-) = 0.$$

We wish to show that the difference

$$L_{k,t_0}[F(x)] - \frac{\alpha(t_0+) + \alpha(t_0-)}{2}$$

approaches zero with $1/k$. But

$$\begin{aligned} \frac{k^{k+1}}{k!} \int_0^\infty e^{-ku} u^k \psi(u) du &= [\alpha(t_0+) - \alpha(t_0-)] \frac{k^{k+1}}{k!} \int_0^\infty e^{-ku} u^k g(u) du + \alpha(t_0-) \\ &= [\alpha(t_0+) - \alpha(t_0-)] \frac{k^{k+1}}{k} \int_1^\infty e^{-ku} u^k du + \alpha(t_0-) \\ &= [\alpha(t_0+) - \alpha(t_0-)] H_k(1) + \alpha(t_0-) \rightarrow \frac{1}{2} [\alpha(t_0+) + \alpha(t_0-)] \quad (k \rightarrow \infty). \end{aligned}$$

Hence

$$(3.7) \quad L_{t_0}[F(x)] - \frac{\alpha(t_0+) + \alpha(t_0-)}{2} = \lim_{k \rightarrow \infty} \frac{k^{k+1}}{k!} \int_0^\infty e^{-ku} u^k [\alpha(t_0 u) - \psi(u)] du.$$

By (3.6) we see that to an arbitrary positive quantity ϵ there corresponds a number η such that

$$(3.8) \quad |\phi(u)| < \epsilon/3 \quad (0 < |1-u| < \eta < 1).$$

Now divide the interval of integration in the right-hand member of (3.7) into $(0, 1-\eta)$, $(1-\eta, 1+\eta)$, $(1+\eta, \infty)$, the corresponding contributions being defined as I_1 , I_2 , and I_3 respectively. Then

$$|I_2| < \frac{\epsilon}{3} \frac{k^{k+1}}{k!} \int_{1-\eta}^{1+\eta} e^{-ku} u^k du < \frac{\epsilon}{3} \frac{k^{k+1}}{k!} \int_0^\infty e^{-ku} u^k du = \frac{\epsilon}{3}.$$

If we denote by K an upper bound of $|\phi(u)|$ in the interval $(0, 1)$, we have as in §2

$$|I_1| \leq \frac{k^{k+1}}{k!} K e^{-k(1-\eta)} (1-\eta)^k.$$

Since the right-hand side of this inequality approaches zero with $1/k$ we can determine k_0 so large that

$$|I_1| < \epsilon/3 \quad (k > k_0).$$

Finally, since

$$\begin{aligned} |\phi(u)| &\leq |\alpha(t_0 u)| + |\psi(u)|, \\ |\phi(u)| &\leq M e^{\sigma t_0 u} + 3 M e^{\sigma t_0} < N e^{\gamma u} \quad (\gamma > 0), \end{aligned}$$

we have

$$(3.9) \quad |I_3| < \frac{k^{k+1}}{k!} e^{-(k-\lambda)(1+\eta)} (1+\eta)^{k-\lambda} \int_{1+\eta}^{\infty} e^{-\lambda u} u^{\lambda} N e^{\gamma u} du.$$

Here λ is any fixed number greater than γ , so that the integral (3.9) converges. The right-hand side of (3.9) tends to zero with $1/k$. Hence we can find k_1 greater than k_0 such that

$$|I_3| < \epsilon/3 \quad (k > k_1).$$

Then for $k > k_1$

$$\left| L_{k,t_0}[F(x)] - \frac{\alpha(t_0+) + \alpha(t_0-)}{2} \right| < \epsilon$$

and

$$L_{t_0}[F(x)] = \frac{\alpha(t_0+) + \alpha(t_0-)}{2}.$$

It will now be established that

$$L_{k,t}[F(x)] = S_{k,t}[f(x)],$$

and this will complete the proof of the theorem. It is not evident that the operator S has a meaning as applied to $f(x)$. To show that it has we prove first that

$$(3.10) \quad (-1)^n f^{(n)}(x) = \int_0^{\infty} e^{-xt} t^n d\alpha(t) = o(x^{-n}) \quad (x \rightarrow \infty; n = 1, 2, \dots).$$

Indeed, if we set

$$(3.11) \quad \beta(0) = 0, \quad \beta(t) = \int_0^t (-1)^n u^n d\omega(u) \quad (t > 0),$$

where

$$\omega(0) = 0, \quad \omega(u) = \alpha(u) - \alpha(0+) \quad (u > 0),$$

we have

$$\begin{aligned} f^{(n)}(x) &= \int_0^{\infty} e^{-xt} d\beta(t) \\ &= x \int_0^{\infty} e^{-xt} \beta(t) dt, \end{aligned}$$

and

$$\beta(t) = o(t^n) \quad (t \rightarrow 0).$$

This follows since $\omega(u)$ is continuous at $u=0$. The total variation of $\omega(u)$ in the interval $(0, t)$, which we denote by $V(t)$, approaches zero with t and

$$|\beta(t)| < t^n V(t).$$

If ϵ is an arbitrary positive number we can determine a number δ such that

$$|\beta(t)| < \frac{\epsilon}{2} \frac{t^n}{n!} \quad (0 \leq t \leq \delta).$$

Hence

$$(3.12) \quad \left| x \int_0^\delta e^{-xt} \beta(t) dt \right| < \frac{\epsilon}{2} \frac{x}{n!} \int_0^\delta e^{-xt} t^n dt < \frac{\epsilon}{2} \frac{x}{n!} \int_0^\infty e^{-xt} t^n dt = \frac{\epsilon}{2x^n}.$$

By integrating (3.11) by parts one sees at once that $\beta(t)$ satisfies an inequality

$$|\beta(t)| < M' e^{g't} \quad (0 \leq t < \infty),$$

where M' and g' are suitable positive constants. Consequently

$$(3.13) \quad \left| x \int_\delta^\infty e^{-xt} \beta(t) dt \right| < M' x \int_\delta^\infty e^{-t(x-g')} dt = M' x e^{-\delta(x-g')}/(x-g') \\ = o(x^{-n}) \quad (x \rightarrow \infty).$$

Combining inequalities (3.12) and (3.13) we have

$$(3.14) \quad f^{(n)}(x) = o(x^{-n}) \quad (x \rightarrow \infty, n > 0).$$

For $n=0$

$$f(x) - \alpha(0+) = \int_0^\infty e^{-xt} d\omega(t).$$

In this case

$$\beta(t) = \omega(t) = o(1) \quad (t \rightarrow 0),$$

and the above proof shows that

$$f(x) - \alpha(0+) = o(1) \quad (x \rightarrow \infty),$$

whence

$$f(\infty) = \alpha(0+).$$

To show that the improper integral

$$(3.15) \quad \int_{k/t}^\infty \frac{u^k}{k!} f^{(k+1)}(u) du \quad \left(\frac{k}{t} > c \right)$$

converges, we proceed by induction. The integral

$$\int_{k/t}^{\infty} f'(u) du = \lim_{R \rightarrow \infty} f(R) - f(k/t)$$

clearly converges to the value $\alpha(0+) - f(k/t)$. Suppose that we have shown that the integral (3.15) converges for $k=m-1$. The equation

$$\int_{k/t}^R f^{(m+1)}(u) \frac{u^m}{m!} du = \frac{f^{(m)}(R) R^m}{m!} - \frac{f^{(m)}(k/t) \left(\frac{k}{t}\right)^m}{m!} - \int_{k/t}^R f^{(m)}(u) \frac{u^{m-1}}{(m-1)!} du$$

together with the relation (3.14) shows that (3.15) also converges for $k=m$, and thus we see that $S_{k,t}[f(x)]$ exists for every positive t .

Successive integration by parts shows that

$$\begin{aligned} S_{k,t}[f(x)] &= f\left(\frac{k}{t}\right) - f'\left(\frac{k}{t}\right) \frac{k}{t} \\ &\quad + \frac{1}{2!} f''\left(\frac{k}{t}\right) \left(\frac{k}{t}\right)^2 - \cdots + (-1)^k \frac{1}{k!} f^{(k)}\left(\frac{k}{t}\right) \left(\frac{k}{t}\right)^k. \end{aligned}$$

On the other hand we have

$$\begin{aligned} F^{(k)}(x) &= \frac{d^k}{dx^k} \left[f(x) \frac{1}{x} \right] = \sum_{n=0}^k (-1)^n \binom{k}{n} f^{(k-n)}(x) \frac{n!}{x^{n+1}}, \\ L_{k,t}[F(x)] &= \frac{1}{k!} \left(\frac{k}{t}\right)^{k+1} \sum_{n=0}^k (-1)^{k-n} \binom{k}{n} f^{(k-n)}\left(\frac{k}{t}\right) n! \left(\frac{t}{k}\right)^{n+1} = S_{k,t}[f(x)]. \end{aligned}$$

Hence

$$L_t[F(x)] = S_t[f(x)].$$

This completes the proof of the theorem. In the course of the proof we have established a result of interest in itself and which we state as

THEOREM 3. *If the function $\phi(t)$ is of bounded variation in the interval $(0, R)$ for every finite R , if for some constant c*

$$(3.16) \quad \phi(t) = O(e^{ct}) \quad (t \rightarrow \infty),$$

and if

$$F(x) = \int_0^{\infty} e^{-xt} \phi(t) dt,$$

then

$$L_t[F(x)] = \frac{\phi(t+) + \phi(t-)}{2}.$$

Later we shall prove a much more general result of this same character.

It is of interest to illustrate the foregoing theory by the following examples:

$$F(x) = \Gamma(\gamma + 1)x^{-\gamma-1}, \quad \phi(t) = t^\gamma \quad (\gamma \geq 1);$$

$$f(x) = \Gamma(\gamma + 1)x^{-\gamma-1}, \quad \alpha(t) = t^{\gamma+1}/(\gamma + 1) \quad (0 < \gamma < 1);$$

$$f(x) = (x + c)^{-1}, \quad \alpha(t) = (1 - e^{-ct})/c \quad (c > 0);$$

where the limits involved in the definition of the operators L and S may be directly computed. Theorem 3 is not applicable to the function $\Gamma(\gamma+1)x^{-\gamma-1}$ for $\gamma < 1$ since the function $\phi(t) = t^\gamma$ is not of bounded variation in any interval including the origin.

4. **The inversion of the Laplace-Lebesgue integral.** In this section we develop an inversion formula for the case in which $\alpha(t)$ in (1.1) is the integral of a function $\phi(t)$,

$$\alpha(t) = \int_0^t \phi(u) du,$$

where $\phi(u)$ is merely integrable in the sense of Lebesgue. Then (1.1) takes the form (1.2). We shall be able to show that if (1.2) converges for some value of x (and hence for every greater value), then

$$L_t[f(x)] = \phi(t)$$

for almost all positive values of t . It is important to note that no restriction of the type (3.16) is imposed on $\phi(t)$, so that the result is the best possible one for integrals of the type (1.2). We state our result in

THEOREM 4. *If the function $\phi(t)$ is integrable in the interval $(0, R)$ for every positive R and if the integral*

$$(4.1) \quad f(x) = \int_0^\infty e^{-xt} \phi(t) dt$$

converges for $x > c$, then

$$L_t[f(x)] = \phi(t)$$

for almost all positive values of t .

Since $\phi(t)$ is integrable we have*

$$(4.2) \quad \int_{t_0}^t |\phi(u) - \phi(t_0)| du = o(|t - t_0|) \quad (t \rightarrow t_0)$$

for almost all positive values of t_0 .

* See, for example, L. Tonelli, *Serie Trigonometriche*, p. 174.

Let us fix attention on such a value t_0 . Set

$$\beta(t, t_0) = \int_{t_0}^t [\phi(u) - \phi(t_0)] du.$$

We wish to show that

$$L_{t_0}[f(x)] = \phi(t_0).$$

In view of

$$L_{k, t_0} \left[\frac{\phi(t_0)}{x} \right] = L_{t_0} \left[\frac{\phi(t_0)}{x} \right] = \phi(t_0),$$

we have only to show that the integral

$$L_{k, t_0} \left[f(x) - \frac{\phi(t_0)}{x} \right] = \int_0^\infty e^{-ku/t_0} \left(\frac{k}{t_0} \right)^{k+1} \frac{u^k}{k!} [\phi(u) - \phi(t_0)] du$$

approaches zero as k becomes infinite. By introducing the function $\beta(t, t_0)$ this becomes

$$(4.3) \quad L_{k, t_0} \left[f(x) - \frac{\phi(t_0)}{x} \right] = \int_0^\infty e^{-ku/t_0} \left(\frac{k}{t_0} \right)^{k+1} \frac{u^k}{k!} d\beta(u, t_0).$$

Set

$$\gamma(t) = \int_0^t \phi(u) du.$$

Since the integral (4.1) converges for $x > c$, there exist positive constants M and γ such that

$$(4.4) \quad |\gamma(t)| < M e^{\gamma t} \quad (0 \leq t < \infty).$$

On account of the relation

$$\beta(t, t_0) = \gamma(t) - \gamma(t_0) - \phi(t_0)(t - t_0)$$

it is clear that $\beta(t, t_0)$ also satisfies an inequality of the type (4.4). Hence, on integrating (4.3) by parts, we see that the integrated term vanishes if k is sufficiently large ($k > g$, say). Thus

$$L_{k, t_0} \left[f(x) - \frac{\phi(t_0)}{x} \right] = I(k),$$

where

$$I(k) = \left(\frac{k}{t_0} \right)^{k+1} \frac{1}{k!} \int_0^\infty e^{-ku/t_0} \beta(u, t_0) \left[\frac{ku^k}{t_0} - ku^{k-1} \right] du.$$

In the integral $I(k)$ make the change of variable $u = t_0 y$. We thus obtain

$$I(k) = \frac{1}{t_0} \frac{k^{k+1}}{(k-1)!} \int_0^\infty e^{-k u} \beta(t_0 y, t_0) y^{k-1} (y-1) dy.$$

Corresponding to an arbitrary positive ϵ there is a number η so small that

$$(4.5) \quad |\beta(y t_0, t_0)| < \epsilon |y-1|/3 \quad (|y-1| < \eta)$$

by virtue of (4.2). Divide the interval of integration into the three parts $(0, 1-\eta)$, $(1-\eta, 1+\eta)$, $(1+\eta, \infty)$, denoting the corresponding contributions by $I_1(k)$, $I_2(k)$, $I_3(k)$ respectively. Then

$$(4.6) \quad |I_1(k)| \leq \frac{1}{t_0} \frac{k^{k+1}}{(k-1)!} e^{-k(1-\eta)} (1-\eta)^{k-1} \int_0^{1-\eta} |\beta(t_0 y, t_0)| (1-y) dy.$$

The right-hand side of this inequality, and hence also the left, approaches zero with $1/k$.

Next consider the integral $I_3(k)$. It converges absolutely for numbers k greater than g . Let k_1 be such a number. Then the integral

$$A = \int_1^\infty e^{-k_1 u} |\beta(t_0 y, t_0)| y^{k_1-1} (y-1) dy$$

converges and

$$(4.7) \quad |I_3(k)| \leq \frac{1}{t_0} \frac{k^{k+1}}{(k-1)!} e^{-(k-k_1)(1+\eta)} (1+\eta)^{k-k_1} A.$$

The constant A is independent of k and one sees easily that the right-hand side of (4.7) approaches zero with $1/k$. Hence we may determine k_0 so large that for $k > k_0$

$$|I_1(k)| < \epsilon/3, \quad |I_3(k)| < \epsilon/3.$$

As for $I_2(k)$, we have by virtue of (4.5)

$$\begin{aligned} |I_2(k)| &< \frac{\epsilon}{3} \frac{1}{t_0} \frac{k^{k+1}}{(k-1)!} \int_{1-\eta}^{1+\eta} e^{-k u} (y-1)^2 y^{k-1} dy \\ &< \frac{\epsilon}{3} \frac{1}{t_0} \frac{k^{k+1}}{(k-1)!} \int_0^\infty e^{-k u} (y-1)^2 y^{k-1} dy. \end{aligned}$$

By use of the gamma function we see that the right-hand side of the latter inequality reduces to $\epsilon/3$ so that

$$|I(k)| < \epsilon \quad (k > k_0).$$

Hence

$$L_{t_0}[f(x)] = \phi(t_0)$$

and the proof is complete. We add the following

COROLLARY. *If the function $\phi(t)$ is integrable in the interval $(0, R)$ for every positive R , is continuous at $t=t_0$, and if the integral*

$$\int_0^{\infty} e^{-xt} \phi(t) dt$$

converges for $x > c$, then

$$L_{t_0}[f(x)] = \phi(t_0) \quad (t_0 > 0).$$

By use of this result we are now able to generalize Theorem 3, removing the condition (3.16). Without loss of generality we may suppose that

$$(4.8) \quad \phi(t) = \frac{\phi(t+) + \phi(t-)}{2}.$$

From the function $g(t)$ of §2 we construct the function

$$\omega(u) = [\phi(t_0+) - \phi(t_0-)]g(u/t_0) + \phi(t_0-).$$

This function is seen to satisfy the conditions

$$\omega(t_0+) = \phi(t_0+),$$

$$\omega(t_0-) = \phi(t_0-),$$

$$\omega(t_0) = \phi(t_0).$$

Consequently the function $\phi(u) - \omega(u)$ has the value zero at $u = t_0$ and is continuous there. Set

$$G(x) = \int_0^{\infty} e^{-xu} \omega(u) du.$$

The corollary of Theorem 4 gives us

$$L_{t_0}[F(x) - G(x)] = \phi(t_0) - \omega(t_0) = 0,$$

$$L_{t_0}[F(x)] = L_{t_0}[G(x)].$$

Since $\omega(u)$ is bounded, and thus satisfies condition (3.16), it follows that we may apply Theorem 3 to $G(x)$. Thus

$$L_{t_0}[G(x)] = \omega(t_0) = \phi(t_0)$$

and

$$L_{t_0}[F(x)] = \phi(t_0).$$

We have thus proved

THEOREM 5. *If the function $\phi(t)$ is of bounded variation in the interval $(0, R)$ for every positive R , and if*

$$F(x) = \int_0^{\infty} e^{-xt} \phi(t) dt,$$

the integral converging for some value of x , then

$$L_t[F(x)] = \frac{\phi(t+) + \phi(t-)}{2}.$$

5. Uniform convergence. We have seen that if $\phi(t)$ is continuous at $t = t_0$ then $L_{t_0}[f(x)] = \phi(t_0)$. If $\phi(t)$ is continuous in an interval $a \leq t \leq b$, then the equation holds for each t of the interval. We now show further that as k becomes infinite the sequence of functions $L_{k,t}[f(x)]$ tends to $\phi(t)$ uniformly in any closed sub-interval of (a, b) not including the end points. The precise result is stated in

THEOREM 6. *If the function $\phi(t)$ is integrable in the interval $(0, R)$ for every positive R , is continuous in the interval $0 \leq a \leq t \leq b$, and if the integral*

$$(5.1) \quad f(x) = \int_0^{\infty} e^{-xt} \phi(t) dt$$

converges for some value of x , then

$$\lim_{k \rightarrow \infty} L_{k,t}[f(x)] = \phi(t)$$

uniformly in the interval $a' \leq t \leq b'$, where $a < a' < b' < b$.

Defining the function $\beta(t, t_0)$ as in the proof of Theorem 4, we obtain

$$(5.2) \quad L_{k,t}[f(x)] - \phi(t) = \int_0^{\infty} e^{-ku/t} \left(\frac{k}{t}\right)^{k+1} \frac{u^k}{k!} d\beta(u, t).$$

By virtue of (4.4) we have

$$|\beta(u, t)| \leq Me^{\gamma u} + Me^{\gamma t} + |\phi(t)| |u - t|.$$

Denote by N the maximum of $\phi(t)$ in the interval $a \leq t \leq b$. Then

$$|\beta(u, t)| \leq Me^{\gamma u} + Me^{\gamma b} + N(u + b) \leq M'e^{\gamma u} \quad (0 \leq u < \infty)$$

if M' is suitably chosen. Hence, on integrating (5.2) by parts the integrated

term vanishes for all t in the interval (a, b) provided $k > b\gamma$. Then

$$I(k) = L_{k,t}[f(x)] - \phi(t) = \frac{k^{k+1}}{(k-1)!} \frac{1}{t^{k+2}} \int_0^\infty e^{-ku/t} \beta(u, t) u^{k-1} (u-t) du.$$

Make the change of variable $v = u/t$:

$$I(k) = \frac{k^{k+1}}{(k-1)!} \frac{1}{t} \int_0^\infty e^{-kv} \beta(tv, t) v^{k-1} (v-1) dv.$$

Since $\phi(t)$ is continuous in the closed interval (a, b) , to an arbitrary positive ϵ there corresponds a number δ such that for any two points t' and t'' of (a, b)

$$(5.3) \quad |\phi(t') - \phi(t'')| < \epsilon/3$$

provided only that $|t' - t''| < \delta$. Now choose a number η satisfying the inequalities

$$(5.4) \quad 0 < \eta < 1, \eta \leq \delta/b', \eta \leq (b-b')/b', \eta \leq (a'-a)/a'.$$

With this number η define the integrals $I_1(k)$, $I_2(k)$, $I_3(k)$ as in §4. From (4.6) we now obtain

$$|I_1(k)| \leq \frac{1}{a'} \frac{k^{k+1}}{(k-1)!} e^{-k(1-\eta)} (1-\eta)^{k-1} M' \int_0^1 e^{\gamma b \eta} (1-y) dy.$$

The right-hand side of this inequality is independent of t and tends to zero with $1/k$.

We consider next the integral $I_3(k)$. Determine $k_1 > \gamma b$. Then from (4.7) we obtain

$$|I_3(k)| \leq \frac{1}{a'} \frac{k^{k+1}}{(k-1)!} e^{-(k-k_1)(1+\eta)} (1+\eta)^{k-k_1} B,$$

where

$$B = \int_1^\infty e^{-k_1 v} e^{\gamma b \eta} y^{k_1-1} (y-1) dy.$$

Again the right-hand side is independent of t and approaches zero with $1/k$.

In order to discuss $I_2(k)$ we note first that

$$|\beta(tv, t)| \leq \epsilon t |v-1|/3$$

if $1-\eta \leq v \leq 1+\eta$ and if $a' \leq t \leq b'$. For, these inequalities imply that

$$|tv - t| = t |v - 1| \leq b' \eta \leq \delta.$$

Moreover tv and t lie in the closed interval (a, b) since the inequalities

$$0 \leq t \leq b', 0 \leq v \leq 1 + \eta$$

imply

$$tv \leq (1 + \eta)b',$$

or, by virtue of (5.4),

$$tv \leq b.$$

In a similar way the inequalities $t \geq a'$ and $v \geq 1 - \eta$ imply $tv \geq a$. It follows that

$$|\beta(tv, t)| \leq \left| \int_t^{tv} |\phi(u) - \phi(t)| du \right| \leq t|v - 1|\epsilon/3.$$

Hence

$$\begin{aligned} |I_2(k)| &\leq \frac{k^{k+1}}{(k-1)!} \frac{1}{t} \frac{\epsilon}{3} \int_{1-\eta}^{1+\eta} e^{-kv} v^{k-1} |v - 1|^2 t dv \\ &< \frac{k^{k+1}}{(k-1)!} \frac{\epsilon}{3} \int_0^\infty e^{-kv} v^{k-1} (v - 1)^2 dv = \frac{\epsilon}{3}. \end{aligned}$$

Consequently we may determine a number k_2 independent of t such that for $k > k_2$

$$|I(k)| < \epsilon,$$

and the proof is complete.

From this result follows immediately

THEOREM 7. *If the function $\alpha(t)$ is of bounded variation in the interval $(0, R)$ for every positive R , is continuous in the interval $0 \leq a \leq t \leq b$, if $\alpha(0) = 0$, and if the integral*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t)$$

converges for some x , then

$$\lim_{k \rightarrow \infty} S_{k,t}[f(x)] = \alpha(t)$$

uniformly in the interval $a' \leq t \leq b'$, where $a < a' < b' < b$.

For if x is sufficiently large

$$f(x)/x = \int_0^\infty e^{-xt} \alpha(t) dt,$$

and we have already seen that

$$L_{k,t}[f(x)/x] = S_{k,t}[f(x)].$$

By Theorem 6 the left-hand member of this equation, and hence also the right, approaches $\alpha(t)$ uniformly in the interval (a', b') .

The interval of uniform convergence may, under certain conditions, be extended to infinity. For example, we have

THEOREM 8. *If the function $\phi(t)$ is continuous in the interval $0 \leq t < \infty$, and if $\phi(t)$ tends to a finite limit as t becomes infinite, then*

$$\lim_{k \rightarrow \infty} L_{k,t}[f(x)] = \phi(t)$$

uniformly in that interval provided $L_{k,0}[f(x)]$ is defined as $\phi(0)$.

Under the present hypotheses the integral (5.1) converges absolutely for $x > 0$ and the method of proof is greatly simplified on that account. It will be seen that we must show that

$$\lim_{k \rightarrow \infty} \frac{k^{k+1}}{k!} \int_0^{\infty} e^{-ky} y^k [\phi(ty) - \phi(t)] dy$$

approaches zero uniformly in the interval $0 \leq t < \infty$. The details of the proof are left to the reader.

This theorem is a result which the author stated without proof in an earlier note.* Another result which we enunciated in that note we record here as a

COROLLARY. *The function*

$$(1+x)^{-k} - e^{-kx}$$

tends uniformly to zero in the interval $0 \leq x < \infty$ as k becomes infinite.

To prove this take the function $f(x)$ of Theorem 8 as $(1+x)^{-1}$ and $\phi(t) = e^{-t}$. Then

$$\frac{1}{1+x} = \int_0^{\infty} e^{-xt} e^{-t} dt \quad (x > -1).$$

Since e^{-t} approaches zero as t becomes infinite and is continuous in the interval $0 \leq t < \infty$, Theorem 8 is applicable, so that

$$\lim_{k \rightarrow \infty} L_{k,t} \left[\frac{1}{x+1} \right] = \lim_{k \rightarrow \infty} \left(1 + \frac{t}{k} \right)^{-k-1} = e^{-t}$$

* This was the Proceedings article mentioned in the Introduction. See Theorems 1 and 3 of that note. It will be found that the statement of Theorem 3 is somewhat different from the statement of Theorem 8 above, but the equivalence of the two results may be seen by making the change of variable $k/x = t$.

uniformly for $0 \leq t < \infty$. If we set $t = kx$ we have

$$\lim_{k \rightarrow \infty} [(1+x)^{-k-1} - e^{-kx}] = 0$$

uniformly for $0 \leq x < \infty$. But

$$(1+x)^{-k} - e^{-kx} = [(1+x)^{-k-1} - e^{-kx}] + x(1+x)^{-k-1}.$$

The minimum of the function $x(1+x)^{-k-1}$ in the interval $0 \leq x < \infty$ is $k^k(k+1)^{-k-1}$. This approaches zero with $1/k$, so that the corollary is proved.

6. Further inversion formulas. We now prove

THEOREM 9. *If the integral*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t) \quad (\alpha(0) = 0)$$

converges for some value of x , then

$$\begin{aligned} \lim_{k \rightarrow \infty} \left[f(\infty) + (-1)^{k+1} \int_{k/t}^\infty \frac{u^k}{k!} e^{ck/u} f^{(k+1)}(u+c) du \right] \\ = \frac{\alpha(t+) + \alpha(t-)}{2} \quad (t > 0) \end{aligned}$$

for any constant c .

This theorem is a generalization of Theorem 2 and reduces to that result when c is put equal to zero. To prove it we note as before that for sufficiently large positive values of x we have

$$F(x) = f(x)/x = \int_0^\infty e^{-xt} \alpha(t) dt.$$

Since

$$F(x+c) = \int_0^\infty e^{-xt} e^{-ct} \alpha(t) dt,$$

we have by an application of Theorem 3

$$\lim_{k \rightarrow \infty} \frac{(-1)^k e^{ct}}{k!} F^{(k)}\left(\frac{k}{t} + c\right) \left(\frac{k}{t}\right)^{k+1} = \frac{\alpha(t+) + \alpha(t-)}{2}.$$

Set

$$(6.1) \quad I_k = f(\infty) + (-1)^{k+1} \int_{k/t}^\infty \frac{u^k}{k!} e^{ck/u} f^{(k+1)}(u+c) du.$$

That this integral converges for any c and for sufficiently large values of k will become apparent when we replace $f(x)$ by $xF(x)$:

$$I_k = f(\infty) + (-1)^{k+1} \int_{k/t}^{\infty} \frac{u^k}{k!} e^{c k/u} [F^{(k+1)}(u+c)(u+c) + (k+1)F^{(k)}(u+c)] du.$$

By integration by parts we obtain

$$\begin{aligned} I_k = f(\infty) &+ \frac{(-1)^k}{k!} \left(\frac{k}{t} + c \right) \left(\frac{k}{t} \right)^k e^{c t F^{(k)}} \left(\frac{k}{t} + c \right) \\ (6.2) \quad &+ \lim_{u \rightarrow \infty} (-1)^{k+1} (u+c) \frac{u^k}{k!} e^{c k/u} F^{(k)}(u+c) \\ &+ (-1)^{k+1} \int_{k/t}^{\infty} \frac{e^{c k/u}}{k!} c^2 k u^{k-2} F^{(k)}(u+c) du. \end{aligned}$$

To show that the integral (6.1) exists it will be sufficient to show that the limit involved in the third term of (6.2) exists and that the integral of (6.2) converges. But we saw in §3 that

$$(-1)^k F^{(k)}(x) x^{k+1}/k! = f(x) - f'(x)x + f''(x)\frac{x^2}{2!} - \cdots + (-1)^k f^{(k)}(x)\frac{x^k}{k!}$$

and that

$$\lim_{x \rightarrow \infty} f^{(i)}(x) x^i = 0 \quad (i = 1, 2, \dots).$$

Hence

$$\lim_{x \rightarrow \infty} (-1)^k F^{(k)}(x) x^{k+1}/k! = f(\infty) \quad (k = 0, 1, 2, \dots),$$

or

$$\lim_{u \rightarrow \infty} (-1)^{k+1} \frac{u^k}{k!} (u+c) e^{c k/u} F^{(k)}(u+c) = -f(\infty),$$

so that the first and third terms of (6.2) may be omitted. To show that the integral (6.2) converges we appeal to the inequality (3.4), from which it follows immediately that

$$|F^{(k)}(x)| < M k! / (x-g)^{k+1} \quad (x > g).$$

By application of this inequality we see easily that the integral in question is $O(1/k)$ as k becomes infinite. Since the second term of (6.2) approaches $[\alpha(t+) + \alpha(t-)]/2$, we see that

$$\lim_{k \rightarrow \infty} I_k = \frac{\alpha(t+) + \alpha(t-)}{2}.$$

This completes the proof of the theorem.

7. Relation between the operators L and S . We have already seen that if

$$(7.1) \quad f(x) = \int_0^{\infty} e^{-xt} \phi(t) dt,$$

then

$$S_t[f(x)] = \int_0^t \phi(u) du, \quad L_t[f(x)] = \phi(t).$$

That is, we are able to determine $\phi(t)$ and its integral in terms of $f(x)$. We are thus led to seek to determine the successive integrals and the successive derivatives (provided the latter exist) of $\phi(t)$ in terms of $f(x)$. It will appear in this section that the successive integrals of $S_{k,t}[f(x)]$ approach the corresponding integrals of $\phi(t)$ and that the successive derivatives of $L_{k,t}[f(x)]$ approach the corresponding derivatives of $\phi(t)$. We begin by proving

THEOREM 10. If $f(x)$ is any function such that $S_{k,t}[f(x)]$ exists for every positive t , then

$$\frac{d}{dt} S_{k,t}[f(x)] = -\frac{1}{t} L_{k,t}[f'(x)]$$

almost everywhere in the interval $(0, \infty)$.

By hypothesis $f(x)$ must have derivatives of the first $k+1$ orders and the integral

$$(-1)^{k+1} \int_{k/t}^{\infty} \frac{u^k}{k!} f^{(k+1)}(u) du$$

must converge for $t > 0$. But this integral has a derivative with respect to t almost everywhere in the interval $(0, \infty)$ equal to

$$\frac{(-1)^{k+1}}{k!t} \left(\frac{k}{t}\right)^{k+1} f^{(k+1)}\left(\frac{k}{t}\right),$$

so that the result is proved.

If $f(x)$ is defined by (7.1) where $\phi(t)$ is integrable in $(0, R)$ for every positive R , Theorem 2 shows that

$$\lim_{k \rightarrow \infty} S_{k,t}[f(x)] = \int_0^t \phi(u) du.$$

Theorem 10 shows further that the first derivative of $S_{k,t}[f(x)]$ with respect to t approaches $\phi(t)$ almost everywhere. For

$$f'(x) = - \int_0^{\infty} e^{-xt} t \phi(t) dt,$$

and by Theorem 4

$$\lim_{k \rightarrow \infty} -\frac{1}{t} L_{k,t}[f'(x)] = \phi(t).$$

We next establish

THEOREM 11. *If the function $\phi(t)$ is integrable in $(0, R)$ for every positive R , and if the integral*

$$f(x) = \int_0^\infty e^{-xt} \phi(t) dt$$

converges for some value of x , then

$$(7.2) \quad \int_0^t du_m \int_0^{u_m} du_{m-1} \int_0^{u_{m-1}} \cdots \int_0^{u_1} \phi(u_0) du_0 \\ = \lim_{k \rightarrow \infty} \frac{(-1)^{k+1}}{k!m!} \int_{k/t}^\infty u^k f^{(k+1)}(u) \left(t - \frac{k}{u}\right)^m du.$$

It is a familiar fact that the iterated integral (7.2) can be expressed as the single integral

$$\int_0^t \frac{(t-u)^m}{m!} \phi(u) du.$$

The integral on the right-hand side of (7.2) can be expressed as follows:

$$(7.3) \quad \frac{(-1)^{k+1}}{k!m!} \int_{k/t}^\infty u^k f^{(k+1)}(u) \left(t - \frac{k}{u}\right)^m du \\ = \frac{(-1)^{k+1}}{k!m!} \sum_{i=0}^m (-1)^i \binom{m}{i} t^{m-i} k^i \int_{k/t}^\infty u^{k-i} f^{(k+1)}(u) du \quad (k > m).$$

That the integrals on the right-hand side of this equality converge one sees at once by use of (3.10) for $i=1, 2, \dots, m$. For $i=0$ the convergence of the integral was already established in §3. Now let k become infinite in (7.3). We shall be able to show that

$$(7.4) \quad \lim_{k \rightarrow \infty} \frac{(-1)^{k+1}}{k!} k^i \int_{k/t}^\infty f^{(k+1)}(u) u^{k-i} du = \int_0^t u^i \phi(u) du$$

and thus that

$$\lim_{k \rightarrow \infty} \frac{(-1)^{k+1}}{k!m!} \sum_{i=0}^m (-1)^i \binom{m}{i} t^{m-i} k^i \int_{k/t}^\infty u^{k-i} f^{(k+1)}(u) du \\ = \sum_{i=0}^m (-1)^i \binom{m}{i} \frac{t^{m-i}}{m!} \int_0^t u^i \phi(u) du = \frac{1}{m!} \int_0^t (t-u)^m \phi(u) du.$$

This will clearly establish the theorem. To prove (7.4) we note first that

$$\begin{aligned}\int_{k/t}^{\infty} u^{k-i} f^{(k+1)}(u) du &= \int_{(k'+i)/t}^{\infty} u^{k'} f^{(k'+i+1)}(u) du \\ &= \int_{k'/t}^{\infty} u^{k'} f^{(k'+i+1)}(u) du - \int_{k'/t}^{(k'+i)/t} u^{k'} f^{(k'+i+1)}(u) du,\end{aligned}$$

where $k' = k - i$. But

$$\lim_{k' \rightarrow \infty} (-1)^{k'+i+1} \frac{(k' + i + 1)^i}{(k' + i)!} \int_{k'/t}^{\infty} u^{k'} f^{(k'+i+1)}(u) du = \int_0^t u^i \phi(u) du,$$

as one sees by applying Theorem 2 to the function

$$f^{(i)}(x) = (-1)^i \int_0^{\infty} e^{-xt} t^i \phi(t) dt$$

and by noting that

$$\frac{1}{k'!} \sim \frac{(k' + i + 1)^i}{(k' + i)!} \quad (k' \rightarrow \infty).$$

That $f^{(i)}(\infty) = 0$ follows from (3.10).

It remains only to show that

$$\lim_{k \rightarrow \infty} I_k = 0,$$

where

$$I_k = \frac{(k + i + 1)^i}{(k + i)!} \int_{k/t}^{(k+i)/t} u^k f^{(k+i+1)}(u) du = 0.$$

Set

$$\alpha(t) = \int_0^t \phi(y) dy.$$

Then

$$F(x) = f(x)/x = \int_0^{\infty} e^{-xt} \alpha(t) dt,$$

and $\alpha(t)$ satisfies the inequality (3.4). Introducing the function $F(x)$ we have

$$I_k = \frac{(k + i + 1)^i}{(k + i)!} \int_{k/t}^{(k+i)/t} [F^{(k+i+1)}(u) u^{k+1} + (k + i + 1) u^k F^{(k+i)}(u)] du.$$

By integration by parts this becomes

$$\begin{aligned}(7.5) \quad I_k &= \frac{(k + i + 1)^i}{(k + i)!} \left[F^{(k+i)} \left(\frac{k+i}{t} \right) \left(\frac{k+i}{t} \right)^{k+1} \right. \\ &\quad \left. - F^{(k+i)} \left(\frac{k}{t} \right) \left(\frac{k}{t} \right)^{k+1} + i \int_{k/t}^{(k+i)/t} F^{(k+i)}(u) u^k du \right].\end{aligned}$$

If we apply Theorem 3 to the function

$$(-1)^i F^{(i)}(x) = \int_0^\infty e^{-xt} t^i \alpha(t) dt$$

and to the function $F(x)$ itself, we see that the sum of the first two terms on the right-hand side of (7.5) approaches zero with $1/k$. Finally by virtue of (2.3) we have

$$(7.6) \quad |F^{(k+i)}(u)| < M(k+i)!/(u-g)^{k+i+1} \quad (u > g),$$

whence the third term of (7.5) becomes $O(1/k)$ as k becomes infinite. The proof of the theorem is thus complete.

We turn next to the problem of determining the successive derivatives of $\phi(t)$ in terms of $f(x)$. We first establish the following

LEMMA. *If the function $f(x)$ is of class* C^n in the interval $0 \leq x < \infty$, then*

$$(7.7) \quad L_{k,t} \left[\frac{d^n}{dx^n} \{x^n f(x)\} \right] = (-1)^n t^n \frac{d^n}{dt^n} \{L_{k,t}[f(x)]\} \\ (t > 0; n, k = 0, 1, 2, \dots).$$

We prove this result by induction. For $n=1$ we have

$$\frac{d}{dx} \{xf(x)\} = xf'(x) + f(x),$$

$$L_{k,t}[xf(x)] = (-1)^k [f^{(k+1)}(k/t)(k/t)^{k+2} + (k+1)f^{(k)}(k/t)(k/t)^{k+1}]/k!.$$

On the other hand

$$\frac{d}{dt} L_{k,t}[f(x)] = \frac{(-1)^{k+1}}{k!} \left[f^{(k+1)} \left(\frac{k}{t} \right) \left(\frac{k}{t} \right)^{k+1} \frac{k}{t^2} + (k+1)f^{(k)} \left(\frac{k}{t} \right) \frac{k}{t^2} \left(\frac{k}{t} \right)^k \right],$$

so that (7.7) is established for $n=1$. Suppose it is true for $0, 1, \dots, n-1$. Rewrite the left-hand member of (7.7) as follows:

$$(7.8) \quad L_{k,t} \left[\frac{d^{n-1}}{dx^{n-1}} \left\{ x^{n-1} \frac{d}{dx} xf(x) + (n-1)x^{n-2}xf(x) \right\} \right].$$

If we replace n by $n-1$ and $f(x)$ by

$$(n-1)f(x) + \frac{d}{dx}[xf(x)]$$

in (7.7) we see that (7.8) becomes

* That is, continuous with its first n derivatives.

$$(-1)^{n-1} t^{n-1} \frac{d^{n-1}}{dt^{n-1}} L_{k,t} \left[\frac{d}{dx} \{ x f(x) \} + (n-1) f(x) \right].$$

Applying (7.7) again, now with $n=1$, (7.8) becomes

$$\begin{aligned} & (-1)^{n-1} t^{n-1} \frac{d^{n-1}}{dt^{n-1}} \left\{ (n-1) L_{k,t} [f(x)] - t \frac{d}{dt} L_{k,t} [f(x)] \right\} \\ & = (-1)^n t^n \frac{d^n}{dt^n} L_{k,t} [f(x)]. \end{aligned}$$

The induction is thus complete.

By use of this Lemma we establish

THEOREM 12. *If the function $\phi(t)$ is of class C^n in the interval $0 \leq t < \infty$, and if the integral*

$$(7.9) \quad \int_0^\infty e^{-xt} \phi^{(n)}(t) dt$$

converges for some value of x , then the integral

$$(7.10) \quad f(x) = \int_0^\infty e^{-xt} \phi(t) dt$$

also converges for large values of x and

$$\phi^{(n)}(t) = \lim_{k \rightarrow \infty} \frac{d^n}{dt^n} L_{k,t} [f(x)] \quad (t > 0).$$

Since the integral (7.9) converges, an application of inequality (3.4) gives

$$(7.11) \quad |\phi^{(n-1)}(t)| < M e^{at} \quad (0 \leq t < \infty).$$

Integrating (7.9) by parts and using (7.11) we obtain

$$\int_0^\infty e^{-xt} \phi^{(n)}(t) dt = -\phi^{(n-1)}(0) + x \int_0^\infty e^{-xt} \phi^{(n-1)}(t) dt$$

for all values of x sufficiently large. Successive applications of this result will show that (7.10) converges and that

$$f(x)x^n = \phi^{(n-1)}(0) + x\phi^{(n-2)}(0) + \cdots + x^{n-1}\phi(0) + \int_0^\infty e^{-xt} \phi^{(n)}(t) dt.$$

Differentiate both sides of this equation n times with respect to x ,

$$\frac{d^n}{dx^n} (x^n f(x)) = (-1)^n \int_0^\infty e^{-xt} t^n \phi^{(n)}(t) dt,$$

and apply the operator $L_{k,t}$ to both sides of the resulting equation. This gives, when the result of the Lemma is taken into account,

$$t^n \frac{d^n}{dt^n} L_{k,t}[f(x)] = L_{k,t} \left[\int_0^\infty e^{-xt} t^n \phi^{(n)}(t) dt \right].$$

Take the limit of both sides of this equation as k becomes infinite, applying Theorem 4, and obtain

$$t^n \phi^{(n)}(t) = \lim_{k \rightarrow \infty} t^n \frac{d^n}{dt^n} L_{k,t}[f(x)],$$

from which the result of the theorem follows immediately.

8. The operator L applied to the Laplace-Stieltjes integral. We conclude Part I with a discussion of the effect of the operator L on functions $f(x)$ which are defined by Laplace-Stieltjes integrals (1.1). We have already seen that

$$S_t[f(x)] = \frac{\alpha(t+) + \alpha(t-)}{2}.$$

We now show that $L_t[f(x)]$ also exists for certain values of t . We prove

THEOREM 13. *Let the function $\alpha(t)$ be of bounded variation in the interval $(0, R)$ for every positive R , and let it possess a derivative on the right $\alpha_+'(t_0)$ and a derivative on the left $\alpha_-'(t_0)$ at a point $t_0 > 0$. Then if the integral*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t)$$

converges for some value of x ,

$$L_{t_0}[f(x)] = \frac{\alpha_+'(t_0) + \alpha_-'(t_0)}{2}.$$

We note first that it will be no essential restriction to suppose that $t_0 = 1$. For, set

$$g(x) = f\left(\frac{x}{t_0}\right) = \int_0^\infty e^{-xt/t_0} d\alpha(t) = \int_0^\infty e^{-xu} d\alpha(t_0 u)$$

where $t = t_0 u$. Simple computation shows that

$$L_1[g(x)] = t_0 L_{t_0}[f(x)]$$

and that the derivatives on the right and left of $\alpha(t_0 u)$ at $u = 1$ are $t_0 \alpha_+'(t_0)$ and $t_0 \alpha_-'(t_0)$ respectively. Hence if we have proved the theorem for $t_0 = 1$ we have

$$L_{t_0}[f(x)] = \frac{L_1[g(x)]}{t_0} = \frac{\alpha'_+(t_0) + \alpha'_-(t_0)}{2}.$$

Form the expression

$$L_{k,1}[f(x)] = \frac{k^{k+1}}{k!} \int_0^\infty e^{-kt} t^k d\alpha(t).$$

The improper integral will converge if k is sufficiently large. It will be sufficient to show that the two integrals

$$I_k = \frac{k^{k+1}}{k!} \int_0^1 e^{-kt} t^k d[\alpha(t) - \alpha'_-(1)t],$$

$$J_k = \frac{k^{k+1}}{k!} \int_1^\infty e^{-kt} t^k d[\alpha(t) - \alpha'_+(1)t]$$

approach zero with $1/k$, since we saw in §2 that

$$\lim_{k \rightarrow \infty} \frac{k^{k+1}}{k!} \int_0^1 e^{-kt} t^k \alpha'_-(1) dt = \frac{\alpha'_-(1)}{2},$$

$$\lim_{k \rightarrow \infty} \frac{k^{k+1}}{k!} \int_1^\infty e^{-kt} t^k \alpha'_+(1) dt = \frac{\alpha'_+(1)}{2}.$$

If we introduce the functions

$$\beta(t) = \alpha(1) - \alpha(t) - \alpha'_-(1)(1-t),$$

$$\gamma(t) = \alpha(t) - \alpha(1) - \alpha'_+(1)(t-1),$$

we have

$$I_k = \frac{k^{k+1}}{k!} \int_0^1 e^{-kt} t^k d\beta(t),$$

$$J_k = \frac{k^{k+1}}{k!} \int_1^\infty e^{-kt} t^k d\gamma(t).$$

By noting that

$$\beta(t) = o(1-t) \quad (t \rightarrow 1-),$$

$$\gamma(t) = o(t-1) \quad (t \rightarrow 1+),$$

and by use of the methods of §4, we see that I_k and J_k approach zero with $1/k$. We omit the details.

The proof may easily be extended to include the case in which $\alpha(t)$ has right-hand or left-hand derivatives that are infinite at t_0 .

PART II

THE REPRESENTATION OF FUNCTIONS AS LAPLACE INTEGRALS

9. A preliminary formula. In Part I we dealt with functions $f(x)$ which were known to be expressible as Laplace integrals. Here we shall abandon this assumption and assume only that the functions $f(x)$ are such as to make the operators L and S have meaning. This will lead us to certain uniqueness theorems regarding the representation of functions as Laplace integrals and to necessary and sufficient conditions for such representation. We begin with

THEOREM 14. *If $f(x)$ is of class C^n in the interval $c \leq x < \infty$, and if the integrals*

$$(9.1) \quad \int_c^\infty u^k f^{(k+1)}(u) du \quad (k = 0, 1, 2, \dots, n-1)$$

converge, then

$$(9.2) \quad \frac{(-1)^k}{k!} F^{(k)}(x) x^{k+1} = f(\infty) + \frac{1}{k!} \int_x^\infty u^k (-1)^{k+1} f^{(k+1)}(u) du$$

($x \geq c$; $k = 0, 1, 2, \dots, n-1$)

where $F(x) = f(x)/x$.

By integration by parts we obtain

$$\begin{aligned} (-1)^{k+1} \int_x^\infty \frac{u^k}{k!} f^{(k+1)}(u) du &= \left[(-1)^{k+1} f^{(k)}(u) \frac{u^k}{k!} \right]_x^\infty \\ &\quad + (-1)^k \int_x^\infty \frac{u^{k-1}}{(k-1)!} f^{(k)}(u) du. \end{aligned}$$

Since both integrals converge by hypothesis it follows that the limits

$$\lim_{x \rightarrow \infty} x^k f^{(k)}(x)$$

exist for $k=0, 1, \dots, n-1$. The existence of all of these limits implies that they are zero except perhaps for $k=0$. We prove this by induction. Suppose that

$$\lim_{x \rightarrow \infty} x f'(x) = B > 0.$$

Then there exists a positive number x_0 such that

$$x f'(x) > B/2 \quad (x \geq x_0 > 0).$$

Hence

$$\int_{x_0}^x f'(u) du > \frac{B}{2} \int_{x_0}^x \frac{du}{u},$$

$$f(x) > \frac{B}{2} \log \frac{x}{x_0} + f(x_0).$$

As x becomes positively infinite the right-hand side of this inequality does also, so that $f(x)$ can not approach a limit as the hypothesis demands. If B is negative we need only replace $f(x)$ by $-f(x)$ in the foregoing proof.

Now suppose we have proved that

$$\lim_{x \rightarrow \infty} x^k f^{(k)}(x) = 0.$$

Then

$$x \frac{d}{dx} (x^k f^{(k)}(x)) = x^{k+1} f^{(k+1)}(x) + k x^k f^{(k)}(x)$$

approaches a limit by hypothesis. By the previous work this limit must be zero, whence

$$\lim_{x \rightarrow \infty} x^{k+1} f^{(k+1)}(x) = 0.$$

This completes the induction and gives us the equation

$$(-1)^{k+1} \int_x^\infty \frac{u^k}{k!} f^{(k+1)}(u) du = (-1)^k \frac{f^{(k)}(x) x^k}{k!} + (-1)^k \int_x^\infty \frac{u^{k-1}}{(k-1)!} f^{(k)}(u) du.$$

Successive application of this result gives (9.2).

COROLLARY 1. *Under the conditions of the theorem,*

$$L_{k,t}[F(x)] = S_{k,t}[f(x)] \quad (0 < t \leq k/c; k = 0, 1, \dots, n-1)$$

if $c > 0$.

COROLLARY 2. *If condition (9.1) is replaced by the condition*

$$(9.3) \quad \lim_{x \rightarrow \infty} x^k f^{(k)}(x) \text{ exists} \quad (k = 0, 1, 2, \dots, n-1),$$

the result of the theorem remains true.

To prove this we have only to show that (9.3) implies (9.1).

10. Uniqueness theorems. We turn next to the proof of

THEOREM 15. *If the function $F(x)$ is of class C^∞ in the interval $0 < x < \infty$ and satisfies the inequalities*

$$(10.1) \quad |F^{(k)}(x)| < \frac{M k!}{x^{k+1}} \quad (x > 0; k = 0, 1, \dots),$$

then

$$(10.2) \quad \lim_{k \rightarrow \infty} \int_0^{\infty} e^{-zt} L_{k,t}[F(x)] dt = F(x) \quad (x > 0).$$

That the integral (10.2) exists for $k=0, 1, 2, \dots$ follows at once from the inequalities

$$\begin{aligned} |L_{k,t}[F(x)]| &< M \quad (t > 0; k = 0, 1, 2, \dots), \\ \left| \int_0^{\infty} e^{-zt} L_{k,t}[F(x)] dt \right| &< \int_0^{\infty} e^{-zt} M dt = \frac{M}{x} \quad (x > 0). \end{aligned}$$

If we set

$$\begin{aligned} H_k(x) &= \int_0^{k/z} e^{-zt} L_{k,t}[F(x)] dt, \\ I_k(x) &= \int_{k/z}^{\infty} e^{-zt} L_{k,t}[F(x)] dt, \end{aligned}$$

we have

$$\int_0^{\infty} e^{-zt} L_{k,t}[F(x)] dt = H_k(x) + I_k(x),$$

and it will be sufficient to show that

$$\begin{aligned} \lim_{k \rightarrow \infty} H_k(x) &= F(x) & (x > 0), \\ \lim_{k \rightarrow \infty} I_k(x) &= 0 & (x > 0). \end{aligned}$$

By the change of variable $u = k/t$ we obtain

$$H_k(x) = (-1)^k \int_x^{\infty} \frac{u^{k-1}}{(k-1)!} e^{-kx/u} F^{(k)}(u) du.$$

On the other hand we have

$$\begin{aligned} (-1)^k \int_x^{\infty} \frac{(u-x)^{k-1}}{(k-1)!} F^{(k)}(u) du \\ = \frac{(-1)^k (u-x)^{k-1}}{(k-1)!} F^{(k-1)}(u) \Big|_x^{\infty} + (-1)^{k-1} \int_x^{\infty} \frac{(u-x)^{k-2}}{(k-2)!} F^{(k-1)}(u) du. \end{aligned}$$

The first term on the right-hand side of this equation is zero since

$$\lim_{u \rightarrow \infty} F^{(k-1)}(u) u^{k-1} = 0,$$

as we see from (10.1). By successive integration by parts we see that

$$F(x) = (-1)^k \int_x^\infty \frac{(u-x)^{k-1}}{(k-1)!} F^{(k)}(u) du.$$

Hence

$$H_k(x) - F(x) = \int_x^\infty \frac{u^{k-1}}{(k-1)!} (-1)^k F^{(k)}(u) \left[e^{-kx/u} - \left(1 - \frac{x}{u}\right)^{k-1} \right] du.$$

Again using (10.1),

$$|H_k(x) - F(x)| < Mk \int_x^\infty \frac{1}{u^2} \left| e^{-kx/u} - \left(1 - \frac{x}{u}\right)^{k-1} \right| du,$$

or, by the change of variable $v = x/u$,

$$|H_k(x) - F(x)| < \frac{Mk}{x} \int_0^1 |e^{-kv} - (1-v)^{k-1}| dv.$$

We write

$$e^{-kv} - (1-v)^{k-1} = [e^{-kv} - (1-v)^k] - v(1-v)^{k-1},$$

and note that

$$e^{-kv} - (1-v)^k > 0 \quad (0 < v < 1),$$

as one sees by virtue of the familiar inequality

$$e^{-x} > 1 - x \quad (x \neq 0).$$

Hence

$$\begin{aligned} |H_k(x) - F(x)| &< \frac{Mk}{x} \left[\int_0^1 [e^{-kv} - (1-v)^k] dv + \int_0^1 v(1-v)^{k-1} dv \right] \\ &= \frac{Mk}{x} \left[\frac{1 - e^{-k}}{k} - \frac{1}{k+1} \right] + \frac{Mk}{x} \frac{\Gamma(k)\Gamma(2)}{\Gamma(k+2)} \\ &= \frac{M}{x} \left[1 - e^{-k} - \frac{k}{k+1} + \frac{1}{k+1} \right]. \end{aligned}$$

From this inequality it follows that

$$\lim_{k \rightarrow \infty} H_k(x) = F(x)$$

for all positive values of x .

For $I_k(x)$ we clearly have the inequality

$$|I_k(x)| < \int_{k/x}^\infty M e^{-xt} dt = M e^{-k}/x \quad (x > 0).$$

This shows that

$$\lim_{k \rightarrow \infty} I_k(x) = 0$$

for all positive values of x , and the proof of the theorem is complete.

We now prove a companion theorem for the operator S .

THEOREM 16. *If the function $f(x)$ is of class C^∞ in the interval $0 < x < \infty$, and if*

$$(10.3) \quad \left| \int_x^\infty \frac{u^k}{k!} f^{(k+1)}(u) du \right| < M \quad (x > 0; k = 0, 1, \dots),$$

then

$$\lim_{k \rightarrow \infty} \int_0^\infty e^{-xt} dS_{k,t}[f(x)] = f(x) - f(\infty) \quad (x > 0)$$

where $S_{k,0}[f(x)]$ is defined as $f(\infty)$.

The existence of the integral (10.3) for all k enables us to employ Theorem 14 for an arbitrary value of k . Thus

$$\begin{aligned} & (-1)^k \frac{F^{(k)}(x) x^{k+1}}{k!} \\ &= f(\infty) + (-1)^{k+1} \int_x^\infty \frac{u^k}{k!} f^{(k+1)}(u) du \quad (x > 0; k = 0, 1, \dots), \end{aligned}$$

where $F(x) = f(x)/x$. By (10.3) we have

$$(10.4) \quad |F^{(k)}(x) x^{k+1}/k!| \leq |f(\infty)| + M = N \quad (x > 0; k = 0, 1, \dots).$$

Hence we may apply Theorem 15 to $F(x)$ and obtain

$$\lim_{k \rightarrow \infty} \int_0^\infty e^{-xt} L_{k,t}[F(x)] dt = F(x).$$

By Corollary 1 of Theorem 14,

$$(10.5) \quad L_{k,t}[F(x)] = S_{k,t}[f(x)] \quad (t > 0; k = 0, 1, 2, \dots).$$

The explicit expression for $S_{k,t}[f(x)]$ shows that

$$(10.6) \quad \lim_{t \rightarrow 0} S_{k,t}[f(x)] = f(\infty).$$

An integration by parts, using (10.4), (10.5), and (10.6), gives

$$(10.7) \quad \int_0^\infty e^{-xt} L_{k,t}[F(x)] dt = \frac{f(\infty)}{x} + \frac{1}{x} \int_0^\infty e^{-xt} dS_{k,t}[f(x)] \quad (x > 0).$$

It is to be noted that we have not proved that $S_{k,t}[f(x)]$ is of bounded variation in $0 \leq t \leq R$. This is not necessary for the existence of the above Stieltjes integral. We have only to note that e^{-xt} is a function of bounded variation in the interval $0 \leq t \leq R$ for every positive x and R , and that $S_{k,t}[f(x)]$ is continuous in $0 \leq t \leq R$ (if defined as $f(\infty)$ at $t=0$). Allowing k to become infinite in (10.7) we have

$$\lim_{k \rightarrow \infty} \frac{1}{x} \int_0^{\infty} e^{-xt} dS_{k,t}[f(x)] = \frac{f(x)}{x} - \frac{f(\infty)}{x},$$

from which the result of the theorem follows immediately.

Theorems 15 and 16 may be regarded as uniqueness theorems. The former shows at once that if $F_1(x)$ and $F_2(x)$ are two functions satisfying the inequalities (10.1) and such that

$$(10.8) \quad L_t[F_1(x)] = L_t[F_2(x)],$$

then $F_1(x) = F_2(x)$ for all positive values of x . For, the function

$$\Phi(x) = F_1(x) - F_2(x)$$

also satisfies inequalities (10.1). By Theorem 15

$$(10.9) \quad \lim_{k \rightarrow \infty} \int_0^{\infty} e^{-xt} L_{k,t}[\Phi(x)] dt = \Phi(x) \quad (x > 0).$$

But by (10.8)

$$\lim_{k \rightarrow \infty} L_{k,t}[\Phi(x)] = 0 \quad (t > 0).$$

Since

$$|e^{-xt} L_{k,t}[\Phi(x)]| < e^{-xt} M,$$

and since the function $M e^{-xt}$ is integrable with respect to t on the infinite interval $(0, \infty)$ for every positive x , we may take the limit under the integral sign in (10.7) and obtain

$$\Phi(x) \equiv 0.$$

In a similar way we can show that if $\Phi(x)$ is a function satisfying (10.3) and such that

$$S_t[\Phi(x)] \equiv 0 \quad (t > 0),$$

then $\Phi(x)$ is identically zero for all positive values of x . For, by Theorem 16,

$$\lim_{k \rightarrow \infty} \int_0^{\infty} e^{-xt} dS_{k,t}[\Phi(x)] = \Phi(x) - \Phi(\infty).$$

But

$$\int_0^{\infty} e^{-xt} dS_{k,t}[\Phi(x)] = -\Phi(\infty) + x \int_0^{\infty} e^{-xt} S_{k,t}[\Phi(x)] dt,$$

and since

$$|S_{k,t}\Phi(x)| < N,$$

we have, on allowing k to become infinite,

$$\Phi(x) - \Phi(\infty) = -\Phi(\infty).$$

The result is thus established.

11. Bernstein's theorem. By use of Theorem 16 we can now give a much simplified proof of a theorem of S. Bernstein.* As a preliminary result we prove

THEOREM 17. *If the function $f(x)$ is completely monotonic in the interval $-\eta < x < \infty$ ($\eta > 0$), then*

$$f(x) = \int_0^{\infty} e^{-xt} d\alpha(t)$$

where the function $\alpha(t)$ is a non-decreasing function and the integral converges for $x > 0$.

We recall that a function is completely monotonic in an interval $-\eta < x < \infty$ if it possesses derivatives of all orders there which satisfy the inequalities

$$(-1)^k f^{(k)}(x) \geq 0 \quad (-\eta < x < \infty).$$

We prove first that the limits

$$(11.1) \quad \lim_{x \rightarrow \infty} x^k f^{(k)}(x) \quad (k = 0, 1, 2, \dots)$$

exist. The result is obvious for $k=0$ since a completely monotonic function is non-negative and non-decreasing. Now form the function

$$f(x) - xf'(x).$$

It is clearly non-negative for $x > 0$ and has a non-positive derivative

$$\frac{d}{dx}(f(x) - xf'(x)) = -xf''(x).$$

* S. Bernstein, *Sur les fonctions absolument monotones*, Acta Mathematica, vol. 52 (1929), p. 1. See also F. Hausdorff, *Summationsmethoden und Momentfolgen*, Mathematische Zeitschrift, vol. 9 (1921), pp. 280-299.

It must therefore approach a limit as x becomes infinite, and hence the function $xf'(x)$ does also. We now proceed by induction. Suppose that the limit (11.1) exists for $k=0, 1, \dots, n$. We can prove that it also exists for $k=n+1$ by considering the function

$$(11.2) \quad f(x) - xf'(x) + \frac{x^2 f''(x)}{2!} - \dots + \frac{(-1)^{n+1}}{(n+1)!} x^{n+1} f^{(n+1)}(x)$$

whose derivative is

$$(-1)^{n+1} x^{n+1} f^{(n+2)}(x)/n!.$$

The function (11.2) being non-negative non-increasing approaches a limit as x becomes infinite. All terms except the last approach a limit by assumption, so that this last must also. This completes the induction.

By Corollary 2 of Theorem 14 we see that the integrals

$$\int_x^\infty u^k f^{(k+1)}(u) du \quad (k = 0, 1, 2, \dots)$$

converge for positive x . Hence the function

$$(11.3) \quad S_{k,t}[f(x)] = f(\infty) + \int_{k/t}^\infty \frac{u^k}{k!} (-1)^{k+1} f^{(k+1)}(u) du$$

is well defined for all positive values of t and for all positive integers k . We define the function as $f(\infty)$ for $t=0$. Since $f(x)$ is completely monotonic for $x>0$, it is clear that the integrand of the integral (11.3) is non-negative. Hence $S_{k,t}[f(x)]$ is a non-negative non-decreasing function of t . Since the function $f^{(k+1)}(u)$ is continuous in the neighborhood of the origin, we have

$$0 \leq S_{k,t}[f(x)] \leq f(\infty) + \int_0^\infty \frac{u^k}{k!} (-1)^{k+1} f^{(k+1)}(u) du \quad (0 \leq t < \infty).$$

But this integral is independent of k . In fact

$$(-1)^{k+1} \int_0^\infty \frac{u^k}{k!} f^{(k+1)}(u) du = f(0) - f(\infty),$$

as one sees by successive integrations by parts. Hence

$$0 \leq S_{k,t}[f(x)] \leq f(0) \quad (t \geq 0; k = 0, 1, 2, \dots).$$

We are thus in a position to apply Theorem 16, for the boundedness of $S_{k,t}[f(x)]$ implies the condition (10.3). Hence

$$f(x) = f(\infty) + \lim_{k \rightarrow \infty} \int_0^\infty e^{-xt} dS_{k,t}[f(x)].$$

Now by a theorem of E. Helly* it is possible to pick from the bounded sequence of functions $S_{k,t}[f(x)]$ ($k=0, 1, 2, \dots$) a sub-sequence $S_{j,t}[f(x)]$ which approaches a non-decreasing function $\beta(t)$ as j becomes infinite. By the Helly-Bray theorem† it is permissible to take the limit under the integral sign so that we have

$$f(x) = f(\infty) + \lim_{j \rightarrow \infty} \int_0^{\infty} e^{-xt} dS_{j,t}[f(x)] = f(\infty) + \int_0^{\infty} e^{-xt} d\beta(t).$$

Clearly

$$\beta(0) = f(\infty) \geq 0.$$

Hence if we define

$$\begin{aligned} \alpha(t) &= \beta(t) & (t > 0), \\ \alpha(0) &= 0, \end{aligned}$$

it is evident that $\alpha(t)$ remains a non-decreasing function and that

$$f(x) = \int_0^{\infty} e^{-xt} d\alpha(t) \quad (x > 0).$$

The theorem is completely established. We now prove the theorem of Bernstein:

THEOREM 18. *A necessary and sufficient condition that $f(x)$ should be completely monotonic for $x > c$ is that*

$$f(x) = \int_0^{\infty} e^{-xt} d\alpha(t),$$

where the function $\alpha(t)$ is non-decreasing and the integral converges for $x > c$.

The sufficiency of the condition is established simply by noting that the function

$$(-1)^k f^{(k)}(x) = \int_0^{\infty} e^{-xt} t^k d\alpha(t)$$

must be non-negative for those values of x which make the integral converge if $\alpha(t)$ is non-decreasing.

The necessity of the condition is easily established by use of Theorem 17. The important distinction between our present hypothesis and that of Theorem 17 is that here we do not know the function $f(x)$ to be completely

* Helly, *Über lineare Funktionaloperationen*, Wiener Sitzungsberichte, vol. 121 (1921), p. 265.

† See, for example, G. C. Evans, *The Logarithmic Potential, Discontinuous Dirichlet and Neumann Problems*, Colloquium Publications, vol. 6, of the American Mathematical Society, 1927, p. 15.

monotonic *outside* the interval in which the Laplace integral is to converge. We note first that the function $f(x+c+\eta)$ is completely monotonic for $x > -\eta$. Applying Theorem 17 we have

$$f(x+c+\eta) = \int_0^{\infty} e^{-xt} d\beta_{\eta}(t),$$

where $\beta_{\eta}(t)$ is non-decreasing and the integral converges for $x > 0$. That is,

$$(11.4) \quad f(x) = \int_0^{\infty} e^{-xt} e^{(c+\eta)t} d\beta_{\eta}(t) = \int_0^{\infty} e^{-xt} d\alpha_{\eta}(t),$$

where

$$\alpha_{\eta}(t) = \int_0^t e^{(c+\eta)u} d\beta_{\eta}(u).$$

The integral (11.4) converges for $x > c + \eta$. This argument holds for each positive value of η and appears to give various integral expressions for $f(x)$ corresponding to the various values of η used. But since a function can have but a single Laplace integral representation,* we see that $\alpha_{\eta}(t)$ is independent of η and may be denoted by $\alpha(t)$. This is a non-decreasing function. Hence

$$f(x) = \int_0^{\infty} e^{-xt} d\alpha(t),$$

where the integral converges for $x > c + \eta$. Since η was arbitrary, the integral converges for $x > c$ and the proof is complete.

12. Representation by absolutely convergent Laplace-Stieltjes integrals. We are now able to give by the present methods a much simplified proof of a theorem of the author.† We state it first in the slightly less general form:

THEOREM 19. *A necessary and sufficient condition that the function $f(x)$ can be expressed as*

$$(12.1) \quad f(x) = \int_0^{\infty} e^{-xt} d\alpha(t)$$

where $\alpha(t)$ is of bounded variation in the infinite interval $0 \leq x < \infty$ is that $f(x)$ should be of class C^{∞} in that interval and that

* D. V. Widder, loc. cit., p. 705. We are assuming of course that the functions $\alpha_{\eta}(t)$, $\beta_{\eta}(t)$ etc. are all normalized. That is,

$$\alpha_{\eta}(0) = 0, \quad [\alpha_{\eta}(t+) + \alpha_{\eta}(t-)]/2 = \alpha_{\eta}(t) \quad (t > 0).$$

† D. V. Widder, *Necessary and sufficient conditions for the representation of a function as a Laplace integral*, these Transactions, vol. 33 (1931), p. 851.

$$(12.2) \quad \int_x^\infty \frac{u^k}{k!} |f^{(k+1)}(u)| du \leq M \quad (x > 0; k = 0, 1, 2, \dots).$$

Our present methods enable us to improve the proof of both the necessity and the sufficiency of the condition. We begin with the necessity. Suppose the function $f(x)$ to have the representation (12.1), the total variation of $\alpha(t)$ in the interval $(0, \infty)$ being equal to M . Then the integral (12.1) must converge absolutely for $x \geq 0$. Denote the total variation of $\alpha(u)$ in the interval $0 \leq u \leq t$ by $V(t)$. Then $0 \leq V(t) \leq M$ for $0 \leq t < \infty$. Set

$$g(x) = \int_0^\infty e^{-xt} dV(t).$$

Then

$$|f^{(k)}(x)| = \left| \int_0^\infty e^{-xt} t^k d\alpha(t) \right| \leq \int_0^\infty e^{-xt} t^k dV(t) = (-1)^k g^{(k)}(x),$$

and

$$(12.3) \quad \int_x^\infty \frac{u^k}{k!} |f^{(k+1)}(u)| du \leq (-1)^{k+1} \int_x^\infty \frac{u^k}{k!} g^{(k+1)}(u) du$$

$$(x > 0; k = 0, 1, 2, \dots).$$

Since $V(t)$ is a non-decreasing function, $g(x)$ is completely monotonic for $x > 0$. Hence the integrals (12.3) surely converge for $x > 0$ as the argument used in the proof of Theorem 17 shows. Integrating the right-hand member of (12.3) by parts, we obtain

$$\begin{aligned} \int_x^\infty \frac{u^k}{k!} |f^{(k+1)}(u)| du &\leq g(x) - xg'(x) + \frac{x^2}{2!} g''(x) - \dots + (-1)^k \frac{x^k}{k!} g^{(k)}(x) \quad (x > 0) \\ &= \int_0^\infty e^{-xt} \left[1 + xt + \frac{x^2 t^2}{2!} + \dots + \frac{x^k t^k}{k!} \right] dV(t). \end{aligned}$$

Again apply integration by parts to this last integral:

$$\int_x^\infty \frac{u^k}{k!} |f^{(k+1)}(u)| du \leq \int_0^\infty e^{-xt} V(t) \frac{x^{k+1} t^k}{k!} dt \quad (x > 0).$$

Finally, since $V(t) \leq M$ we have

$$\int_x^\infty \frac{u^k}{k!} |f^{(k+1)}(u)| du \leq M \quad (x > 0).$$

This completes the proof of the necessity of the condition.

We turn next to the proof of the sufficiency. The condition (12.2) on

$f(x)$ enables us to show that $S_{k,i}[f(x)]$ is of bounded variation in the infinite interval $(0, \infty)$ and that the total variation of $f(x)$ in that interval has an upper bound independent of k . For, let R be an arbitrary positive constant. Divide the interval $(0, R)$ into sub-intervals by points t_i such that

$$0 = t_0 < t_1 < \cdots < t_n = R.$$

Then we have

$$\begin{aligned} & \sum_{i=0}^{n-1} |S_{k,t_{i+1}}[f(x)] - S_{k,t_i}[f(x)]| \\ &= \sum_{i=0}^{n-1} \left| \int_{k/t_{i+1}}^{\infty} \frac{u^k}{k!} (-1)^{k+1} f^{(k+1)}(u) du - \int_{k/t_i}^{\infty} \frac{u^k}{k!} (-1)^{k+1} f^{(k+1)}(u) du \right| \\ &\leq \sum_{i=0}^{n-1} \int_{k/t_{i+1}}^{k/t_i} \frac{u^k}{k!} |f^{(k+1)}(u)| du = \int_{k/R}^{\infty} \frac{u^k}{k!} |f^{(k+1)}(u)| du \leq M. \end{aligned}$$

Since M is independent of the manner of sub-division of the interval $(0, R)$ the function $S_{k,i}[f(x)]$ is of bounded variation in the interval and its total variation in that interval is at most M , a number independent of k and of R . Hence the total variation of $S_{k,i}[f(x)]$ in $(0, \infty)$ is also at most equal to M for all positive integers k .

Now by the theorem of Helly already employed in the proof of Theorem 17 we can pick from the sequence $S_{k,i}[f(x)]$ a sub-sequence $S_{j,i}[f(x)]$ which approaches a limit $\alpha(t)$ defined for $0 < t < \infty$, whose total variation in that interval is at most M .

The condition (12.2) clearly implies the condition (10.3), so that we may apply Theorem 16 here to show that

$$(12.4) \quad f(x) - f(\infty) = \lim_{j \rightarrow \infty} \int_0^{\infty} e^{-xt} dS_{j,i}[f(x)].$$

Since

$$|S_{j,i}[f(x)]| \leq |f(\infty)| + M$$

we may use precisely the same argument as that used in the proof of Theorem 17 to show that we may take the limit under the sign of integration in (12.4). Thus

$$f(x) = f(\infty) + \int_0^{\infty} e^{-xt} d\alpha(t) \quad (\alpha(0) = f(\infty)),$$

or if $\alpha(0)$ is defined as zero,

$$f(x) = \int_0^{\infty} e^{-xt} d\alpha(t).$$

In either case $\alpha(t)$ is of bounded variation in $(0, \infty)$ and the proof is complete.

We now prove the more general result by use of Theorem 19.

THEOREM 20. *A necessary and sufficient condition that $f(x)$ can be expressed in the form*

$$(12.5) \quad f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is of bounded variation in the interval $(0, R)$ for every positive R , the integral converging absolutely for $x > c$, is that $f(x)$ should be of class C^∞ in the interval $c < x < \infty$ and that for every positive constant δ there should exist a constant M_δ such that

$$(12.6) \quad \int_x^\infty \frac{u^k}{k!} |f^{(k+1)}(u + c + \delta)| du \leq M_\delta \quad (x > 0; k = 0, 1, 2, \dots).$$

We prove first the necessity of the condition. Let $f(x)$ have the form (12.5). Then

$$(12.7) \quad f(x + c + \delta) = \int_0^\infty e^{-xt} d\beta(t)$$

where

$$\beta(t) = \int_0^t e^{-(c+\delta)u} d\alpha(u).$$

The integral (12.7) converges absolutely for $x > -\delta$. The total variation of $\beta(u)$ in the interval $0 \leq u \leq t$ is

$$\int_0^t e^{-(c+\delta)u} dV(u),$$

where $V(t)$ is the total variation of $\alpha(u)$ in that interval. Hence the total variation of $\beta(u)$ in the interval $0 \leq u < \infty$ is

$$M_\delta = \int_0^\infty e^{-(c+\delta)u} dV(u).$$

This integral converges since (12.5) converges absolutely for $x = c + \delta$. Now applying Theorem 19 to the function $f(x + c + \delta)$ we get (12.6). This establishes the necessity of the condition.

Now assume that (12.6) holds. By Theorem 19 we have

$$f(x + c + \delta) = \int_0^\infty e^{-xt} d\beta(t)$$

where $\beta(t)$ is of bounded variation in the interval $0 \leq t < \infty$. This integral consequently converges absolutely for $x \geq 0$. Hence the integral

$$f(x) = \int_0^{\infty} e^{-xt} d\alpha(t),$$

where

$$\alpha(t) = \int_0^t e^{(c+\delta)u} d\beta(u),$$

converges absolutely for $x \geq c + \delta$. This argument holds for each positive δ . The function $\alpha(t)$ appears to depend on δ . This is not the case, however, as one sees by again appealing to the uniqueness theorem for the representation of a function by a Laplace integral.* Since δ is arbitrary it follows that (12.5) converges absolutely for $x > c$ and the proof is complete.

It is natural to inquire what sort of condition is imposed on the function $f(x)$ by (12.6) if the absolute value signs are removed from the integrand and are applied instead to the integral itself. In this connection we prove

THEOREM 21. *A necessary and sufficient condition that the function $f(x)$ can be expressed in the form*

$$f(x) = x \int_0^{\infty} e^{-xt} \phi(t) dt$$

where $\phi(t)$ is integrable in $(0, R)$ for every positive R , is uniformly bounded in $(0, \infty)$, and is such that the limit

$$(12.8) \quad \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t \phi(u) du$$

exists, is that a constant M should exist for which

$$(12.9) \quad \left| \int_x^{\infty} \frac{u^k}{k!} f^{(k+1)}(u) du \right| < M \quad (x > 0; k = 0, 1, \dots).$$

We first establish the necessity of the condition. Set $F(x) = f(x)/x$. If $|\phi(t)| < N$ for $0 \leq t < \infty$, then

$$(12.10) \quad \left| \frac{F^{(k)}(x)}{k!} x^{k+1} \right| < N \quad (x > 0; k = 0, 1, 2, \dots).$$

The hypothesis (12.8) implies that

$$(12.11) \quad \int_0^t \phi(u) du \sim at \quad (t \rightarrow 0)$$

* The function $\beta(u)$ of course depends on δ . It is precisely this fact that makes it possible for $\alpha(t)$ to be independent of δ .

for a suitable constant a . From this it follows that

$$\int_0^t u^k \phi(u) du \sim at^{k+1}/(k+1) \quad (t \rightarrow 0).$$

For, by virtue of (12.11), we have

$$\int_0^t [\phi(u) - a] du = o(t) \quad (t \rightarrow 0).$$

If we set

$$\beta(t) = \int_0^t u^k [\phi(u) - a] du,$$

we must show that

$$(12.12) \quad \beta(t) = o(t^{k+1}) \quad (t \rightarrow 0).$$

By an integration by parts (12.12) is easily established.

This result enables us to show that

$$(12.13) \quad (-1)^k F^{(k)}(x) \sim ak!/x^{k+1} \quad (x \rightarrow \infty),$$

or that

$$I_k = \int_0^\infty e^{-xt} t^k [\phi(t) - a] dt = o(x^{-k-1}) \quad (x \rightarrow \infty).$$

Integration by parts gives

$$\begin{aligned} I_k &= x \int_0^\infty e^{-xt} \beta(t) dt \\ &= x \int_0^{\delta} e^{-xt} \beta(t) dt + x \int_{\delta}^\infty e^{-xt} \beta(t) dt. \end{aligned}$$

Using the relation (12.12) on the first integral on the right-hand side of this equation we obtain

$$|I_k| \leq \frac{(k+1)!}{x^{k+1}} \epsilon + x e^{-(x-x_0)\delta} \int_{\delta}^\infty e^{-x_0 t} (N+a) t^{k+1} dt \quad (x > x_0 > 0).$$

But

$$x e^{-(x-x_0)\delta} = o(x^{-k-1}) \quad (x \rightarrow \infty).$$

Hence it is evident that (12.13) is true. From this fact it follows at once that

$$\lim_{x \rightarrow \infty} f(x) = a, \quad \lim_{x \rightarrow \infty} x^k f^{(k)}(x) = 0 \quad (k = 1, 2, \dots).$$

By Corollary 2 of Theorem 14 we have

$$\frac{(-1)^k}{k!} F^{(k)}(x) x^{k+1} = f(\infty) + (-1)^{k+1} \int_x^\infty \frac{u^k}{k!} f^{(k+1)}(u) du.$$

That is,

$$\left| \int_x^\infty \frac{u^k}{k!} f^{(k+1)}(u) du \right| \leq N + |f(\infty)| = M.$$

This completes the proof of the necessity of the condition.

Conversely, suppose that (12.9) holds. Applying Theorem 14 we have

$$|F^{(k)}(x) x^{k+1}/k!| < |f(\infty)| + M \quad (x > 0; k = 0, 1, 2, \dots).$$

Then applying a theorem of the author* we see that

$$F(x) = \frac{f(x)}{x} = \int_0^\infty e^{-xt} \phi(t) dt,$$

where $\phi(t)$ is uniformly bounded in $(0, \infty)$. It remains only to show that (12.8) holds. We know that $f(\infty)$ exists since the integral (12.9) converges for $k=0$ by hypothesis. That is,

$$F(x) \sim f(\infty)/x.$$

Since $\phi(t)$ is bounded we may apply a familiar Tauberian theorem† and conclude that

$$\int_0^t \phi(u) du \sim f(\infty)t \quad (t \rightarrow \infty).$$

This completes the proof of the theorem.

PART III

APPLICATIONS

13. Zeros of Laplace integrals. In this section we shall discuss the relation between the zeros of the integral

$$(13.1) \quad f(x) = \int_0^\infty e^{-xt} d\alpha(t)$$

and the changes of trend of the function $\alpha(t)$. In case $\alpha(t)$ has a continu-

* D. V. Widder, *Necessary and sufficient conditions for the representation of a function as a Laplace integral*, these Transactions, vol. 33 (1931), p. 873, Theorem 13.

† G. H. Hardy and J. E. Littlewood, *On Tauberian theorems*, Proceedings of the London Mathematical Society, vol. 30 (1930), p. 23.

ous derivative $\alpha'(t)$, E. Laguerre* proved that the number of zeros of $f(x)$ in the interval of convergence of (13.1) can not exceed the number of changes of sign of $\alpha'(t)$. He obtained a similar result for the case in which (13.1) reduces to a Dirichlet series. We here extend the result to the general integral (13.1).

We first make a precise definition of the notion of change of trend. In this definition we use the term interval to include an interval of zero length. A function $\alpha(t)$ is increasing (or decreasing) in the zero interval (a, a) if $\alpha(a+) > \alpha(a-)$ (or $\alpha(a+) < \alpha(a-)$).

DEFINITION. Let $\alpha(t)$ be a normalized function of bounded variation in the interval $0 \leq t \leq R$. Then $\alpha(t)$ has n changes of trend in that interval if there exist points

$$0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{n+1} = R,$$

with at most two consecutive t_i equal, such that

$$\begin{aligned} \alpha(t_{i+1}-) &\neq \alpha(t_i+) & \text{if } t_{i+1} \neq 0, t_i \neq R, \\ \alpha(0) &\neq \alpha(0+) & \text{if } t_0 = t_1 = 0, \\ \alpha(R) &\neq \alpha(R-) & \text{if } t_{n+1} = t_n = R; \end{aligned}$$

(B) $\alpha(t)$ is alternately increasing and decreasing in the intervals

$$(t_0, t_1), (t_1, t_2), \cdots, (t_n, t_{n+1}).$$

If a function has n changes of trend in $(0, R)$ for every positive R sufficiently large, we say that it has n changes of trend in the infinite interval $(0, \infty)$. In particular if $\alpha(t)$ has a continuous derivative $\alpha'(t)$ then n is the number of changes in sign of $\alpha'(t)$. On the other hand if $\alpha(t)$ is a step-function, so that (13.1) reduces to a Dirichlet series, n is the number of changes of sign in the sequence of the coefficients. We now establish

THEOREM 22. If the function $\alpha(t)$ is of bounded variation in the interval $(0, R)$ for every positive R , and if it has n changes of trend in the interval $(0, \infty)$, then the function

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t)$$

has at most n zeros in the interval of convergence of the integral.

By the definition of change of trend it is clear that the function

$$\beta(x) = \int_0^x (t - t_1)(t - t_2) \cdots (t - t_n) d\alpha(t)$$

* E. Laguerre, Oeuvres, vol. 1, p. 29.

is monotonic in the interval $(0, \infty)$. Now form the function

$$(13.2) \quad f^*(x) = \int_0^\infty e^{-xt} d\beta(t) = \int_0^\infty e^{-xt}(t-t_1)(t-t_2)\cdots(t-t_n)d\alpha(t).$$

Since $\beta(t)$ is monotonic it follows that $f^*(x)$ has no zeros in the interval of convergence of the integral (13.2). Set

$$(t-t_1)(t-t_2)\cdots(t-t_n) = a_n(-t)^n + a_{n-1}(-t)^{n-1} + \cdots + a_0.$$

Then

$$f^*(x) = a_n f^{(n)}(x) + a_{n-1} f^{(n-1)}(x) + \cdots + a_0 f(x).$$

If $f(x)$ had more than n zeros, this linear differential expression would have at least one zero, as one sees by a generalized form of Rolle's theorem.[†] Since $f^*(x)$ has no zeros, our result is proved.

By use of our inversion formula we can now get a more exact relation between the number of zeros of $f^{(k)}(x)$ and the number of changes of trend in $\alpha(t)$. We prove

THEOREM 23. *If the function $\alpha(t)$ is a normalized function of bounded variation with n changes of trend in the interval $(0, R)$ for every sufficiently large positive R , and if $\alpha(0+) = \alpha(0) = 0$, then*

$$f^{(k)}(x) = (-1)^k \int_0^\infty e^{-xt} t^k d\alpha(t)$$

has exactly n changes of sign in the interval of convergence of the integral for all k sufficiently large.

Before proving the theorem we point out that the restriction $\alpha(0+) = \alpha(0)$ is a necessary one. If it were omitted, $\alpha(t)$ could be defined as

$$\begin{aligned} \alpha(0) &= 0, & \alpha(t) &= 1 & (0 < t < 1), \\ \alpha(1) &= \frac{1}{2}, & \alpha(t) &= 0 & (1 < t < \infty), \end{aligned}$$

a function with one change of trend. Yet the derivatives of

$$1 - e^{-x} = \int_0^\infty e^{-xt} d\alpha(t)$$

have no change of sign, no matter how high the order.

To prove the theorem consider the points t_0, t_1, \dots, t_n whose existence is

[†] G. Pólya, *On the mean-value theorem corresponding to a given linear homogeneous differential equation*, these Transactions, vol. 24 (1922), pp. 312-324.

D. V. Widder, *A general mean-value theorem*, these Transactions, vol. 26 (1924), pp. 385-394. Since the coefficients a_i are constants, the property W is satisfied in any interval.

guaranteed by the foregoing definition. Consider two adjoining intervals (t_{i-1}, t_i) and (t_i, t_{i+1}) . Suppose that $\alpha(t)$ is increasing in the first and decreasing in the second. We show first that points ξ, η, ζ exist such that

$$(13.3) \quad \alpha(\xi) < \alpha(\eta) > \alpha(\zeta), \quad t_{i-1} \leq \xi < \eta < \zeta \leq t_{i+1}.$$

We consider several cases.

CASE I. $t_{i-1} \neq t_i, t_i \neq t_{i+1}$. In this case $\alpha(t)$ has at least one point of increase in $t_{i-1} \leq t \leq t_i$. Hence we can find ξ and η' such that

$$\alpha(\xi) < \alpha(\eta') \quad (t_{i-1} \leq \xi < \eta' \leq t_i).$$

Since $\alpha(t)$ has at least one point of decrease in $t_i \leq t \leq t_{i+1}$ we can determine η'' and ζ such that

$$\alpha(\eta'') > \alpha(\zeta) \quad (t_i \leq \eta'' < \zeta \leq t_{i+1}).$$

Choose η equal to η' or η'' so that

$$\begin{aligned} \alpha(\eta) &\geq \alpha(\eta'), \\ \alpha(\eta) &\geq \alpha(\eta''). \end{aligned}$$

Then clearly (13.3) is satisfied.

CASE II. $t_{i-1} \neq t_i, t_i = t_{i+1}$. Choose ξ and η' as in Case I. By B of the definition we see that $\alpha(t_{i+1}+) - \alpha(t_i-) < 0$. Hence we can determine η'' such that

$$\alpha(\eta'') > \alpha(t_{i+1}) = \frac{\alpha(t_{i+1}+) + \alpha(t_i-)}{2}.$$

Choose η as in Case I and $\zeta = t_{i+1}$. Then (13.3) is satisfied.

CASE III. $t_{i-1} = t_i, t_i \neq t_{i+1}$. The treatment of this case is similar to that of Case II and is omitted.

It is to be noted that $t_1 \neq t_0$ since $\alpha(0+) = \alpha(0)$. Hence $\xi > 0$ if $i = 1$.

Now choose a positive number ϵ so small that

$$\alpha(\xi) + \epsilon < \alpha(\eta) - \epsilon < \alpha(\zeta) + \epsilon.$$

By Theorem 2 we can determine an integer k_0 so large that

$$|S_{k,i}[f(x)] - \alpha(t)| < \epsilon \quad (t = \xi, \eta, \zeta; k > k_0).$$

Hence

$$S_{k,\xi}[f(x)] < S_{k,\eta}[f(x)] > S_{k,\zeta}[f(x)].$$

Since $S_{k,i}[f(x)]$ is a function of class C' at least, it follows that it has at least one maximum in the interval $t_{i-1} < t < t_{i+1}$, where its derivative vanishes. A similar proof applies if $\alpha(t)$ is decreasing in (t_{i-1}, t_i) and increasing in (t_i, t_{i+1}) .

Since there are but a finite number of intervals (t_i, t_{i+1}) we can determine k_0 so large that for $k > k_0$ the function $S_{k,t}$ will have at least one

$$\begin{array}{l} \text{maximum (minimum) in } (t_0, t_2), \\ \text{minimum (maximum) in } (t_1, t_3), \\ \text{maximum (minimum) in } (t_2, t_4), \\ \dots \quad \dots \quad \dots \quad \dots \\ \dots \quad \dots \quad \dots \quad \dots \end{array}$$

It thus becomes clear that the derivative of $S_{k,t}[f(x)]$ with respect to t will change sign at least n times in $(0, \infty)$. That is, the function

$$\frac{k^k}{(k-1)!} \frac{1}{t^{k+2}} f^{(k+1)}\left(\frac{k}{t}\right),$$

and hence also $f^{(k+1)}(x)$, must change sign at least n times in $(0, \infty)$. By Theorem 22 we see that $f^{(k+1)}(x)$ must change sign exactly n times in that interval and the theorem is completely established.

COROLLARY 1. *If $\alpha(t)$ has a maximum (minimum)* at a point t_0 , then for k sufficiently large $f^{(k)}(x)$ will have a change of sign at a point x_k such that*

$$\lim_{k \rightarrow \infty} \frac{k}{x_k} = t_0.$$

Let ϵ be an arbitrary positive number. We must show that there corresponds a number k_0 such that $f^{(k)}(x)$ will have a zero x_k for which

$$\left| \frac{k}{x_k} - t_0 \right| < \epsilon \quad (k > k_0),$$

or that $f^{(k)}(k/t)$ will have a zero in the interval $t_0 - \epsilon < t < t_0 + \epsilon$. This follows from the inequalities

$$S_{k,t_0-\epsilon}[f(x)] < S_{k,t_0}[f(x)] > S_{k,t_0+\epsilon}[f(x)]$$

precisely as in the proof of the theorem.

COROLLARY 2. *If*

$$f(x) = a_1 e^{-\lambda_1 x} + a_2 e^{-\lambda_2 x} + a_3 e^{-\lambda_3 x} + \dots,$$

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots, \quad \lim_{m \rightarrow \infty} \lambda_m = \infty,$$

and if the sequence a_1, a_2, a_3, \dots has n changes of sign, $f^{(k)}(x)$ will have exactly n

* We do not require that $\alpha(t)$ should be continuous at t_0 . We must, however, have a maximum (minimum) in the strict sense.

changes of sign in the interval of convergence of the series for all k sufficiently large.

COROLLARY 3. If $\alpha(t)$ has n_R changes of trend in $(0, R)$ and if

$$\lim_{R \rightarrow \infty} n_R = \infty,$$

then the number of changes of sign of $f^{(k)}(x)$ in the interval of convergence of the integral becomes infinite with k .

Here we introduce the further

DEFINITION. Let $\phi(t)$ be integrable in $(0, R)$. It has n changes of sign in that interval if the function

$$\alpha(t) = \int_0^t \phi(y) dy$$

has n changes of trend there.

For example, if $(-1)^i \phi(t) \geq 0$ almost everywhere in (t_i, t_{i+1}) , the sign $>$ holding for a set of positive measure ($i=0, 1, 2, \dots, n$), then the conditions of the definition are satisfied.

COROLLARY 4. If $\phi(t)$ is integrable in $(0, R)$ and has n changes of sign there for every positive R , then the function

$$f^{(k)}(x) = (-1)^k \int_0^\infty e^{-xt} t^k \phi(t) dt$$

has n changes of sign in the interval of convergence of the integral for all k sufficiently large.

This result includes as a special case a result which the author stated earlier without proof.* The increased generality of the present result is noteworthy.

In case the function $\phi(t)$ of Corollary 4 has its changes of sign at points in the neighborhood of which it is different from zero, our inversion formula enables us to locate the positions of the changes of sign of $\phi(t)$ if we know the positions of the changes of sign of $f^{(k)}(x)$. We first make exact the notion of change of sign at a point.

DEFINITION. Let $\phi(t)$ be defined in the interval $(0, R)$. Then it has a change of sign at a point $t=t_i$ ($0 < t_i < R$) if for all positive ϵ sufficiently small

$$\phi(t_i - \epsilon) > 0, \phi(t_i + \epsilon) < 0,$$

or

$$\phi(t_i - \epsilon) < 0, \phi(t_i + \epsilon) > 0.$$

* See the author's Proceedings article cited in the Introduction, Theorem 5.

We now establish

THEOREM 24. *If $\phi(t)$ is integrable in $(0, R)$ for every positive R , the integral*

$$(13.4) \quad f(x) = \int_0^\infty e^{-xt} \phi(t) dt$$

converging for some value of x , and if $\phi(t)$ has a change of sign at a point t_0 , then for k sufficiently large $f^{(k)}(x)$ will have a change of sign at a point x_k such that

$$\lim_{k \rightarrow \infty} \frac{k}{x_k} = t_0.$$

Given an arbitrary positive ϵ we wish to show that we can find an integer k_0 such that for $k > k_0$ the function $f^{(k)}(x)$ will have a change of sign x_k such that

$$\left| \frac{k}{x_k} - t_0 \right| < \epsilon$$

or that the function $f^{(k)}(k/t)$ will have a change of sign between $t_0 - \epsilon$ and $t_0 + \epsilon$ for all $k > k_0$. By Theorem 4 the function $L_{k,t}[f(x)]$ approaches $f(x)$ almost everywhere in $(0, \infty)$. Choose a point η in the interval $t_0 < t < t_0 + \epsilon$ and a point ξ in the interval $t_0 - \epsilon < t < t_0$ such that

$$\lim_{k \rightarrow \infty} L_{k,\xi}[f(x)] = \phi(\xi),$$

$$\lim_{k \rightarrow \infty} L_{k,\eta}[f(x)] = \phi(\eta).$$

But

$$\phi(\xi) \neq 0, \phi(\eta) \neq 0, \phi(\xi)\phi(\eta) < 0.$$

Hence we can determine k_0 so large that for $k > k_0$

$$L_{k,\xi}[f(x)] > \frac{\phi(\xi)}{2} > 0, \quad L_{k,\eta}[f(x)] < \frac{\phi(\eta)}{2} < 0,$$

or else

$$L_{k,\xi}[f(x)] < \frac{\phi(\xi)}{2} < 0, \quad L_{k,\eta}[f(x)] > \frac{\phi(\eta)}{2} > 0.$$

In either case the continuous function $L_{k,t}[f(x)]$ must vanish between ξ and η . But this function vanishes only when $f^{(k)}(k/t)$ vanishes, so that the theorem is established. We point out that the theorem could also be derived by use of Corollary 1 to Theorem 22.

COROLLARY. If $\phi(t)$ is continuous in $(0, \infty)$ and has changes of sign at the points t_i ,

$$0 < t_1 < t_2 < \cdots < t_n,$$

and at no others, then $f^{(k)}(x)$ has exactly n changes of sign for k sufficiently large at points

$$x_{1,k} > x_{2,k} > \cdots > x_{n,k},$$

and

$$\lim_{k \rightarrow \infty} \frac{x_{i,k}}{k} = \frac{1}{t_i} \quad (i = 1, 2, \cdots, n).$$

Included in this Corollary is a result stated earlier by the author.*

We now illustrate the theory by an example. Take

$$\phi(t) = (t-a)(t-b) \quad (0 < a < b).$$

Then

$$f(x) = \frac{2}{x^3} - \frac{(a+b)}{x^2} + \frac{ab}{x},$$

$$f^{(k)}(x) = \frac{(k+2)!}{x^{k+3}} - \frac{(a+b)(k+1)!}{x^{k+2}} + \frac{abk!}{x^{k+1}}.$$

Simple computations show that

$$x_{1,k} = \{(a+b)(k+1) + [(a+b)^2(k+1)^2 - 4ab(k+1)(k+2)]^{1/2}\} / (2ab),$$

$$x_{2,k} = \{(a+b)(k+1) - [(a+b)^2(k+1)^2 - 4ab(k+1)(k+2)]^{1/2}\} / (2ab).$$

These roots will be real if k is sufficiently large. It must be so large that

$$\frac{k+1}{k+2} > \frac{4ab}{(a+b)^2}.$$

The right-hand side of this inequality is less than unity if $a \neq b$, whereas the left-hand side approaches unity as k becomes infinite. It is clear that

$$\lim_{k \rightarrow \infty} \frac{x_{1,k}}{k} = \frac{1}{a}, \quad \lim_{k \rightarrow \infty} \frac{x_{2,k}}{k} = \frac{1}{b}.$$

This example shows that $f^{(k)}(x)$ may have a smaller number of changes of sign than $\phi(t)$ for small values of k . For example, if $a=9$, $b=10$, the function $f^{(k)}(x)$ has no zeros for $k < 359$, has two zeros for $k > 359$.

As a further application of Theorem 23 we prove

* Theorem 6 of the Proceedings article cited above.

THEOREM 25. *Let the series*

$$f(z) = \sum_{m=1}^{\infty} a_m z^m$$

have radius of convergence R , and let the sequence of real coefficients a_1, a_2, \dots have n changes of sign. Then the function

$$f_k(z) = \sum_{m=1}^{\infty} a_m(m)^k z^m$$

will have exactly n changes of sign in the interval $0 < z < R$ for all k sufficiently large.

Set $z = e^{-x}$ in the given power series. We thus obtain a Dirichlet series convergent for $x > \log(1/R)$, to which we apply Corollary 2 of Theorem 23. Its coefficients have exactly n changes of sign. Hence its k th derivative will have exactly n changes of sign in the interval $\log(1/R) < x < \infty$ for all k sufficiently large. Replacing x by $\log(1/z)$ we have the result stated.*

We may also obtain a similar result concerning factorial series.

THEOREM 26. *Let the sequence a_1, a_2, \dots have n changes of sign, and let the series*

$$(13.5) \quad f_l(x) = \sum_{m=1}^{\infty} \frac{a_m m! m^l}{x(x+1) \cdots (x+m)}$$

converge for $x > 0$. Then it is possible to determine a number l_0 such that for any fixed $l > l_0$ the function $f_l^{(k)}(x)$ will have n changes of sign in the interval $0 < x < \infty$ for all k sufficiently large.

Since the series (13.5) converges we have†

$$(13.6) \quad f_l(x) = \int_0^{\infty} e^{-xt} \phi_l(t) dt,$$

where

$$(13.7) \quad \phi_l(t) = \sum_{m=1}^{\infty} a_m m^l (1 - e^{-t})^{m-1}.$$

The series (13.6) converges for $0 < t < \infty$, or the series

$$\psi_l(z) = \sum_{m=1}^{\infty} a_m m^l z^{m-1}$$

* Cf. Pólya and Szegő, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2 (1925), p. 44, No. 44.

† D. V. Widder, I, p. 739.

converges for $0 < z < 1$. By Theorem 25 the function $\psi_l(z)z$ will have exactly n changes of sign in $0 < z < 1$ if l is greater than some number l_0 . The same will be true of $\psi_l(z)$ and hence of $\phi_l(t)$ in $0 < t < \infty$. We have now only to apply Corollary 3 of Theorem 23 to (13.7) to obtain the result stated.*

14. **Special inversion formulas.** We conclude Part III with several specific inversion formulas which, although they are immediate consequences of the general theory, are nevertheless worthy of separate statement. The first has to do with functions of the form

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t},$$

considered by Stieltjes. He gave a complex inversion formula when $\alpha(t)$ is an increasing function.† Our present methods enable us to give the following simple real solution of the Stieltjes problem.

THEOREM 27. *If the function $\alpha(t)$ is of bounded variation in the infinite interval $(0, \infty)$, and if*

$$(14.1) \quad f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t},$$

then

$$(14.2) \quad \frac{\alpha(t+) + \alpha(t-)}{2} = S_i[L_\nu[f(x)]] \quad (t > 0).$$

To prove this result set

$$\phi(y) = \int_0^\infty e^{-y^2} d\alpha(t).$$

Since $\alpha(t)$ is of bounded total variation in $(0, \infty)$, this integral converges absolutely for $0 \leq y < \infty$. It follows that the integral (14.1) converges for $x > 0$, that $f(x)$ is analytic in the whole complex x -plane with the negative real axis removed, and that

$$f(x) = \int_0^\infty e^{-xy} \phi(y) dy.$$

This integral converges for $x > 0$. But

$$\frac{\alpha(t+) + \alpha(t-)}{2} = S_i[\phi(y)] \quad (t > 0),$$

* Compare Pólya and Szegő, loc. cit., vol. 2, p. 51, No. 84.

† See, for example, O. Perron, *Die Lehre von den Kettenbrüchen*, 1929, p. 372.

and

$$\phi(y) = L_y[f(x)] \quad (y > 0)$$

by Theorems 2 and 3 respectively. Combining these two results we have (14.2).

Another application of Theorem 2 to Dirichlet series is contained in

THEOREM 28. *If*

$$f(x) = a_1 e^{-\lambda_1 x} + a_2 e^{-\lambda_2 x} + \dots, \\ 0 < \lambda_1 < \lambda_2 < \dots, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty,$$

the series converging for $x > c$, then

$$a_n = \lim_{k \rightarrow \infty} (-1)^{k+1} \int_{2k/(\lambda_n + \lambda_{n+1})}^{2k/(\lambda_{n-1} + \lambda_n)} \frac{u^k}{k!} f^{(k+1)}(u) du \quad (n = 1, 2, \dots),$$

where λ_0 is defined as zero.

This follows since

$$a_n = S_{(\lambda_n + \lambda_{n+1})/2}[f(x)] - S_{(\lambda_{n-1} + \lambda_n)/2}[f(x)].$$

In a similar way we may obtain the coefficients of a series in powers of $1/x$ in terms of the function it represents.

THEOREM 29. *If*

$$f(x) = \frac{a_0}{x} + \frac{a_1}{x^2} + \frac{a_2}{x^3} + \dots,$$

the series converging for $x > c$, then

$$a_n = \left\{ \lim_{k \rightarrow \infty} \frac{d^n}{dt^n} L_{k,t}[f(x)] \right\}_{t=0}.$$

This follows since $f(x)$ can be represented as a Laplace integral*

$$f(x) = \int_0^\infty e^{-xt} \phi(t) dt,$$

where

$$\phi(t) = a_0 + a_1 t + \frac{a_2 t^2}{2!} + \frac{a_3 t^3}{3!} + \dots$$

An application of Theorem 12 now gives the result.

Finally, the coefficients of a factorial series can also be determined in

* See for example, D. V. Widder, I, p. 728.

terms of the function which it represents.

THEOREM 30. *If*

$$f(x) = \frac{a_0}{x} + \frac{a_1}{x(x+1)} + \frac{a_2}{x(x+1)(x+2)} + \cdots,$$

the series converging for $x > c$, then

$$a_n = \left\{ \lim_{k \rightarrow \infty} \frac{d^n}{dt^n} L_{k, \log(1/(1-t))} [f(x)] \right\}_{t=0}.$$

The proof is similar to that of Theorem 29 and is omitted.

PART IV

COMPLEX VARIABLE

15. **Generating function analytic at infinity.** In previous work we have regarded the generating function f and the determining function ϕ as real functions of the real variable. Let us now suppose that both are functions of the complex variable. Set

$$z = x + iy, \quad s = \sigma + i\tau,$$

and write

$$(15.1) \quad f(s) = \int_0^\infty e^{-sz} \phi(z) dz.$$

We are still supposing that the path of integration is along the positive real axis. If the integral converges for some value of s we easily see by breaking $\phi(x)$ into its real and imaginary parts that

$$L_z[f(s)] = \phi(x)$$

for all real positive values of x . It is natural to inquire if this formula still holds when x is replaced by the complex variable z . In other words, will our inversion formula hold off the real axis? We shall be able to show that it holds in the half-plane $x > 0$ if the function $\phi(z)$ is analytic there. In the present section we shall assume in addition that $f(s)$ is analytic at infinity and vanishes there. We recall that such a function can be expressed in the form (15.1). If

$$(15.2) \quad f(s) = \sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}},$$

then

$$(15.3) \quad \phi(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!},$$

and $\phi(z)$ is entire.*

We now establish

THEOREM 31. *If $f(s)$ is analytic at infinity and vanishes there, then $L_s[f(s)]$ exists for all complex z and defines an entire function $\phi(z)$ such that*

$$f(s) = \int_0^{\infty} e^{-sz} \phi(z) dz,$$

the integral converging in some half-plane $\sigma > \sigma_c$.

Since $f(s)$ has the representation (15.2), the series converging in some neighborhood of infinity, there exist numbers M and ρ such that

$$|a_n| < M\rho^n \quad (n = 0, 1, 2, \dots).$$

Simple computation gives

$$(15.4) \quad L_{k,s}[f(s)] = \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} \frac{a_n z^n}{k^n k!} \quad (z < 1/\rho).$$

It will now be shown that the series (15.4), whose terms are to be regarded as functions of the two variables k and z , converges uniformly in the region $|z| \leq l$, $k \geq k_0$, where l is an arbitrary positive constant and k_0 is a suitably chosen positive integer. We note that

$$\frac{(n+k)!}{k^n k!} = \left(1 + \frac{1}{k}\right) \left(1 + \frac{2}{k}\right) \cdots \left(1 + \frac{n}{k}\right)$$

is a decreasing function of k for any positive integer n . Having chosen l arbitrarily we choose the integer k_0 greater than $l\rho$. We thus have

$$\sum_{n=0}^{\infty} \frac{(n+k)!}{k^n k!} \frac{a_n z^n}{n!} \ll \sum_{n=0}^{\infty} \frac{(n+k_0)!}{k_0^n k_0!} \frac{M\rho^n l^n}{n!}.$$

Since the convergent dominant series is independent of k and of z , the uniform convergence of (15.4) is established. Consequently we may take the limit of the series (15.4) term by term as k becomes infinite for any fixed z whose modulus is $\leq l$ provided that the limit of the general term exists. But

$$\lim_{k \rightarrow \infty} \frac{(n+k)!}{n!} \frac{a_n z^n}{k^n k!} = \frac{a_n z^n}{n!}.$$

It follows that

* See, for example, D. V. Widder, I, p. 728.

$$L_s[f(s)] = \lim_{k \rightarrow \infty} L_{k,s}[f(s)] = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!} = \phi(z)$$

for $|z| \leq l$. Since l was arbitrary, the theorem is completely established.

COROLLARY. *Under the conditions of the theorem,*

$$\lim_{k \rightarrow \infty} L_{k,z}[f(s)] = \phi(z)$$

uniformly in any closed region of the z -plane.

This follows at once from the fact that the series (15.4) converged uniformly in z as well as in k .

16. **Determining function analytic in a half-plane.** We now turn to the case in which the determining function is analytic in the half-plane. We shall show that in this case our inversion formula is valid throughout the half-plane. The result to be proved is

THEOREM 32. *Let the function $\phi(z)$ be analytic in the half-plane $x > 0$ and let the integral*

$$f(s) = \int_0^{\infty} e^{-su} \phi(u) du$$

converge for some value of s . Then

$$\lim_{k \rightarrow \infty} L_{k,z}[f(s)] = \phi(z)$$

uniformly in any closed region in the half-plane $x > 0$.

It is to be noted that we are not assuming that $\phi(z)$ is analytic at the origin. Thus our proof will apply to such functions as $z^{-1/2}$. This degree of generality is reflected in a corresponding complication in the proof.

We obtain at once the following integral representation of $L_{k,z}[f(s)]$,

$$L_{k,z}[f(s)] = \frac{1}{k!} \left(\frac{k}{z} \right)^{k+1} \int_0^{\infty} e^{-ku/z} u^k \phi(u) du.$$

Here u is for the present a real variable. Later we shall alter the path of integration and u will be a complex variable. Set $z = \rho e^{i\psi}$ and $u = r e^{i\theta}$. Let D be an arbitrary closed region in the half-plane $x > 0$. We can determine positive constants ρ_0 , ρ_1 and $\psi_1 < \pi/2$ such that for all points z of D we have $\rho_0 \leq \rho \leq \rho_1$, $|\psi| < \psi_1$. Now consider the function

$$|e^{-u/z} u/z| = (r/\rho) \exp [(-r/\rho) \cos(\theta - \psi)].$$

As z varies in D and u varies along the positive real axis we have

$$(r/\rho) \exp [(-r/\rho) \cos \psi] < (r/\rho_0) \exp [(-r/\rho_1) \cos \psi_1].$$

The right-hand side of this inequality is independent of the value of z in D and tends to zero as r becomes infinite or as r approaches zero. Consequently we can determine a positive number $\delta < \rho_0$ and a positive number $\Delta > \rho_1$ both independent of z in D such that

$$(16.1) \quad |e^{-u/z} u/z| < e^{-1}$$

for $r \leq \delta$ and for $r \geq \Delta$.

We can now show that

$$(16.2) \quad \lim_{k \rightarrow \infty} \frac{1}{k!} \left(\frac{k}{z}\right)^{k+1} \int_{\Delta}^{\infty} e^{-ku/z} u^k \phi(u) du = 0,$$

$$(16.3) \quad \lim_{k \rightarrow \infty} \frac{1}{k!} \left(\frac{k}{z}\right)^{k+1} \int_0^{\delta} e^{-ku/z} u^k \phi(u) du = 0$$

uniformly in D . Set

$$\alpha(u) = \int_0^u \phi(r) dr$$

for positive real values of u . Then constants M and g exist such that

$$(16.4) \quad |\alpha(u)| < M e^{gu} \quad (0 \leq u < \infty).$$

Hence

$$(16.5) \quad \begin{aligned} \frac{1}{k!} \left(\frac{k}{z}\right)^{k+1} \int_{\Delta}^{\infty} e^{-ku/z} u^k \phi(u) du &= -\alpha(\Delta) \frac{1}{k!} \left(\frac{k}{z}\right)^{k+1} e^{-k\Delta/z} \\ &\quad - \frac{1}{(k-1)!} \left(\frac{k}{z}\right)^{k+1} \int_{\Delta}^{\infty} \alpha(u) e^{-ku/z} u^{k-1} \left[1 - \frac{u}{z}\right] du \end{aligned}$$

for k sufficiently large. The first term on the right-hand side of this equation satisfies the inequality

$$|\alpha(\Delta)| \left| e^{-\Delta/z} \frac{\Delta}{z} \right|^k \frac{k^{k+1}}{k!} \rho^{-1} \leq |\alpha(\Delta)| \frac{k^{k+1}}{k!} l^k \rho_0^{-1},$$

where $l < e^{-1}$, as one sees by (16.1). The dominant function is independent of z in D and approaches zero with $1/k$.

The second term on the right-hand side of (16.5) is in modulus at most equal to

$$\frac{k^{k+1}}{(k-1)!} \frac{1}{\rho_0^2} l^{k-k_0} \int_{\Delta}^{\infty} |\alpha(u)| e^{-[k_0 u \cos \psi_1]/\rho_1} \left| 1 + \frac{u}{\rho_0} \right| u^{k_0-1} du,$$

where k_0 is chosen so large that the integral converges. This is possible by

(16.4). The dominant function is again independent of z in D and approaches zero with $1/k$, so that (16.2) is completely established. In a precisely similar way (16.3) is established, an integration by parts being necessary since $|\phi(u)|$ need not be bounded in $(0, \delta)$. However, it is unnecessary to use (16.4) in the present case since $\alpha(u)$ is continuous and hence bounded in $0 \leq u \leq \delta$. We omit the details of the proof.

It remains to show that

$$(16.6) \quad \lim_{k \rightarrow \infty} \frac{1}{k!} \left(\frac{k}{z} \right)^{k+1} \int_{\delta}^{\Delta} e^{-ku/z} u^k \phi(u) du = \phi(z)$$

uniformly in D . To establish this point we must alter the path of integration. We replace the segment of the real axis by a curve C composed of three arcs

$$\begin{aligned} C_1: & \quad r = \delta & (0 \leq \theta \leq \psi), \\ C_2: & \quad \theta = \psi & (\delta \leq r \leq \Delta), \\ C_3: & \quad r = \Delta & (0 \leq \theta \leq \psi). \end{aligned}$$

We have here supposed that $\psi \geq 0$. If $\psi < 0$, replace the inequalities defining C_1 and C_3 by $\psi \leq \theta \leq 0$. Since the integrand of (16.6) is analytic in the region bounded by the line segment $\delta \leq u \leq \Delta$ and the curve C , we do not alter the value of the integral by changing the path of integration as indicated. It is to be noted that the curve C changes as z varies in D .

We now show that

$$\lim_{k \rightarrow \infty} \frac{1}{k!} \left(\frac{k}{z} \right)^{k+1} \int_{C_i} e^{-ku/z} u^k \phi(u) du = 0 \quad (i = 1, 3)$$

uniformly in D . We prove it for $i=1$, the proof for $i=3$ being similar. We have

$$\begin{aligned} & \left| \frac{1}{k!} \left(\frac{k}{z} \right)^{k+1} \int_{C_1} e^{-ku/z} u^k \phi(u) du \right| \\ & \leq \frac{1}{k!} \left(\frac{k}{\rho_0} \right)^{k+1} \int_0^{\psi_1} e^{-[k\delta \cos \psi_1]/\rho_0} \delta^{k+1} |\phi(\delta e^{i\theta})| d\theta \\ & \leq \frac{k^{k+1}}{k!} \frac{\delta l^k}{\rho_0} \int_0^{\psi_1} |\phi(\delta e^{-i\theta})| d\theta. \end{aligned}$$

This upper bound is independent of z in D and approaches zero with $1/k$.

We have thus reduced our problem to that of showing that

$$\lim_{k \rightarrow \infty} \frac{1}{k!} \left(\frac{k}{z} \right)^{k+1} \int_{\delta}^{\Delta} e^{-[krs i \psi]/z} r^k e^{i \psi (k+1)} \phi(r e^{i \psi}) dr = \phi(z)$$

uniformly in D . But by Theorem 6

$$\lim_{k \rightarrow \infty} \frac{k^{k+1}}{k!} \frac{1}{\rho^{k+1}} \int_{\delta}^{\Delta} e^{-kr/\rho} r^k dr = 1$$

uniformly* in the interval $\rho_0 \leq r \leq \rho_1$. Hence we must show that

$$\lim_{k \rightarrow \infty} \frac{k^{k+1}}{k!} \frac{1}{\rho^{k+1}} \int_{\delta}^{\Delta} e^{-kr/\rho} r^k [\phi(re^{i\psi}) - \phi(z)] dr = 0$$

uniformly in D . Make the change of variable $v = r/\rho$. The integral becomes

$$I(k) = \frac{k^{k+1}}{k!} \int_{\delta/\rho}^{\Delta/\rho} e^{-kv} v^k [\phi(vz) - \phi(z)] dv.$$

Given an arbitrary positive ϵ we shall show that we can determine a number k_1 independent of z in D such that

$$|I(k)| < \epsilon \quad (k > k_1).$$

We first observe that

$$|I(k)| \leq \frac{k^{k+1}}{k!} \int_{\delta/\rho_1}^{\Delta/\rho_0} e^{-kv} v^k |\phi(vz) - \phi(z)| dv.$$

Since $\phi(z)$ is uniformly continuous in z we can determine a number ζ such that

$$|\phi(z') - \phi(z'')| < \epsilon/3 \quad (|z' - z''| < \zeta)$$

provided only that z' and z'' are in D . Set $\eta = \zeta/\rho_1$. Then if $|1-v| < \eta$, we have

$$|z - vz| < \rho_1 \eta = \zeta.$$

Hence if z is in D and $|1-v| < \eta$ it follows that

$$(16.7) \quad |\phi(z) - \phi(vz)| < \epsilon/3.$$

We now have

$$|I(k)| \leq I_1(k) + I_2(k) + I_3(k),$$

where

$$I_1(k) = \frac{k^{k+1}}{k!} \int_{\delta/\rho_1}^{1-\eta} e^{-kv} v^k |\phi(vz) - \phi(z)| dv,$$

and where $I_2(k)$ and $I_3(k)$ are similar integrals with intervals of integration $(1-\eta, 1+\eta)$ and $(1+\eta, \Delta/\rho_0)$ respectively.

* Take the function $\phi(t)$ of Theorem 6 equal to zero for $0 \leq t \leq \delta$ and for $\Delta \leq t < \infty$, and equal to unity for $\delta < t < \Delta$. Since $\delta < \rho_0 < \rho_1 < \Delta$, the interval (ρ_0, ρ_1) is an interval of continuity of $\phi(t)$.

By (16.7)

$$|I_2(k)| \leq \frac{k^{k+1}}{k!} \frac{\epsilon}{3} \int_{1-\eta}^{1+\eta} e^{-kv} v^k dv < \frac{\epsilon}{3}.$$

If z is in D , and if $\delta/\rho_1 \leq v \leq \Delta/\rho_0$, then vz surely lies in the closed region

$$\rho_0 \delta / \rho_1 \leq \rho \leq \rho_1 \Delta / \rho_0, \quad |\psi| \leq \psi_1.$$

Denote the maximum of $|\phi(z)|$ in that region by N . Then

$$(16.8) \quad |I_1(k)| \leq \frac{k^{k+1}}{k!} e^{-k(1-\eta)} (1-\eta)^k 2N,$$

$$(16.9) \quad |I_3(k)| \leq \frac{k^{k+1}}{k!} e^{-k(1+\eta)} (1+\eta)^k 2N.$$

If we determine k_1 so large that the right-hand members of (16.8) and (16.9) are each less than $\epsilon/3$ for $k > k_1$, we have

$$|I(k)| < \epsilon \quad (k > k_1).$$

This completes the proof of the theorem.

17. The zeros of the determining function. The applicability of our inversion formula in the complex domain enables us to extend the study of the zeros of the determining function made in Part III to the case when these zeros are no longer on the real axis. We first take the case in which the generating function is analytic at infinity and prove

THEOREM 33. *Let the function $f(s)$ be analytic at infinity and have the representation*

$$f(s) = \int_0^\infty e^{-sz} \phi(z) dz.$$

If $\phi(z)$ has n zeros not at $z=0$ in the region $|z| < l$, then there exists an integer k_1 such that $f^{(k)}(s)$ has n zeros not at $z=\infty$ in the region $|s| > k_1/l$ for $k \geq k_1$.

For, suppose that in addition to the n zeros of $\phi(z)$ which are not at the origin there are m zeros at the origin. Then $\phi(z)$ has $n+m$ zeros in the region $|z| < l$. By the Corollary of Theorem 31

$$\lim_{k \rightarrow \infty} L_{k,s}[f(s)] = \lim_{k \rightarrow \infty} \frac{(-1)^k}{k!} f^{(k)}\left(\frac{k}{s}\right) \left(\frac{k}{s}\right)^{k+1} = \phi(z)$$

uniformly in the region $|z| \leq l$. Hence by a theorem of E. Rouché* there will exist an integer k_1 such that for $k \geq k_1$ the function $L_{k,s}[f(s)]$, which is surely

* See for example Pólya and Szegő, loc. cit., vol. 1, p. 122, No. 194.

analytic at $z=0$ if suitably defined there, will have exactly $n+m$ zeros in the region $|z| < l$. If $f(s)$ and $\phi(z)$ have the power series developments (15.2) and (15.3), then

$$a_0 = a_1 = \cdots = a_{m-1} = 0.$$

Consequently the function $f^{(k)}(s)s^{k+1}$ will have m zeros at infinity and $L_{k,s}[f(s)]$ will have m zeros at $z=0$ and hence n zeros in the region $|z| < l$ not at $z=0$. But $f^{(k)}(k/z)$ has the same zeros as $L_{k,s}[f(s)]$ except at $z=0$. It follows that $f^{(k)}(s)$ has n zeros not at $z=0$ in the region $|s| > k/l \geq k_1/l$. This completes the proof of the theorem.

If $f(s)$ is no longer analytic at infinity we are not at liberty to suppose that $\phi(z)$ is entire. If $\phi(z)$ is analytic in the half-plane $x > 0$ we may still obtain results concerning its zeros. We prove

THEOREM 34. *Let the function $\phi(\rho e^{i\psi})$ be analytic in the half-plane $-\pi/2 < \psi < \pi/2$, and let*

$$f(s) = \int_0^\infty e^{-s\rho} \phi(\rho) d\rho,$$

the integral converging for some value of s . If $\phi(\rho e^{i\psi})$ has n zeros in the region

$$0 < \rho_1 < \rho < \rho_2, |\psi| < \psi_2 < \pi/2,$$

then for all k sufficiently large $f^{(k)}(s)$ will have n zeros in the region

$$k/\rho_2 < \rho < k/\rho_1, |\psi| < \psi_2.$$

By Theorem 32 we have

$$\lim_{k \rightarrow \infty} L_{k,s}[f(s)] = \lim_{k \rightarrow \infty} \frac{(-1)^k}{k!} f^{(k)}\left(\frac{k}{z}\right) \left(\frac{k}{z}\right)^{k+1} = \phi(z)$$

uniformly in the region

$$\rho_1 \leq \rho \leq \rho_2, |\psi| \leq \psi_2.$$

By Rouché's Theorem there exists a number k_1 such that for $k \geq k_1$ the function $L_{k,s}[f(s)]$ and hence also $f^{(k)}(k/z)$ has exactly n zeros in the region

$$\rho_1 < \rho < \rho_2, |\psi| < \psi_2.$$

We obtain the result of the theorem by replacing z by k/s .

We turn now to the complex analogue of Theorem 24.

THEOREM 35. *Let the function $\phi(z)$ be analytic in the half-plane $x > 0$, and let*

$$f(s) = \int_0^\infty e^{-sz} \phi(z) dz,$$

the integral converging for some value of s . If $\phi(z)$ has a zero at a point z_0 , $x_0 > 0$, then for k sufficiently large $f^{(k)}(s)$ will have a zero at a point s_k such that

$$\lim_{k \rightarrow \infty} \frac{s_k}{k} = \frac{1}{z_0}.$$

Let the zero z_0 of $\phi(z)$ be of order λ . With z_0 as center describe a circle of radius η so small that no other zero of $\phi(z)$ lies in it and such that the circle lies entirely in the half-plane $x > 0$. Then by the Theorem of Rouché we can determine an integer k_1 so large that for $k > k_1$ the function $f^{(k)}(k/z)$ will have exactly λ zeros (distinct or coincident) inside this circle. Denote any one of them by z_k . Then $f^{(k)}(s)$ will vanish at the point $s_k = k/z_k$. Then

$$\left| \frac{k}{s_k} - z_0 \right| < \eta.$$

Since η was arbitrarily small it follows that

$$\lim_{k \rightarrow \infty} \frac{k}{s_k} = z_0.$$

In a similar way we could show that if $f(s)$ is analytic at infinity and if $\phi(z)$ vanishes at a point z_0 not the origin, there would exist a zero s_k of $f^{(k)}(s)$ for k sufficiently large such that k/s_k approaches z_0 as k becomes infinite.

We illustrate the result by several examples. The first example following Theorem 24 gives us an interesting example of the present theory in case $a = b > 0$. Then $\phi(t)$ has a zero at a but no change of sign there, so that the real theory fails. We find that $f^{(k)}(s)$ has the complex zeros

$$s_{k,1} = [(k+1) + i(k+1)^{1/2}]/a,$$

$$s_{k,2} = [(k+1) - i(k+1)^{1/2}]/a.$$

Hence $f^{(k)}(s)$ will have no real zeros however large k may be. But it will have two complex zeros for all k and

$$\lim_{k \rightarrow \infty} \frac{s_{k,j}}{k} = \frac{1}{a} \quad (j = 1, 2).$$

The same analysis holds if $a < 0$ or if a is complex, and provides an example to illustrate the case in which the zeros of $\phi(z)$ may be in the half-plane $x < 0$. Of course $f(s)$ is analytic at infinity.

PART V

THE MOMENT PROBLEM

18. The problem. In this part we shall consider the infinite system of equations

$$(18.1) \quad \mu_n = \int_0^1 t^n d\alpha(t) \quad (n = 0, 1, 2, \dots),$$

where $\alpha(t)$ is of bounded variation in the interval $(0,1)$ and $\alpha(1)=0$. The relation of this system of equations to the integral equation (1.1) becomes evident if the substitution $t=e^{-u}$ is made:

$$\mu_n = \int_0^\infty e^{-nu} d\{-\alpha(e^{-u})\} \quad (n = 0, 1, 2, \dots).$$

Here the variable n , running through a discrete set, replaces the continuous variable x of (1.1). We should expect then to be able to determine the function $\alpha(t)$ in terms of the sequence $\{\mu_n\}$ by use of an operator similar to (3.1), (3.2), replacing the derivatives of $f(x)$ in that expression by the differences of $\{\mu_n\}$ and the integral sign by a summation sign. We shall find that the expected analogy is complete and that we can also obtain in a similar way an operator analogous to $L_t[f(x)]$ which, when applied to a sequence

$$\mu_n = \int_0^1 t^n \phi(t) dt,$$

will yield the function $\phi(t)$.

19. An extension of the Laplace method. In the following section we shall need an extension of the classical Laplace method for the asymptotic evaluation of a definite integral. We state the result that we shall need as a

LEMMA. Let the function $h(x)$ be of class C'' in the interval $a \leq x \leq b$ and satisfy the conditions

$$h'(b) = 0, h''(b) < 0, h(x) < h(b) \quad (a \leq x < b);$$

let the functions $\phi_1(x), \phi_2(x), \dots$ be of class C' in $a \leq x \leq b$ and satisfy the conditions

$$\begin{aligned} |\phi_k(x)| &\leq \psi(x) & (a \leq x \leq b), \\ |\phi_k'(x)| &\leq M & (a \leq x \leq b; k = 1, 2, \dots), \\ |\phi_k(b)| &\geq 1 & (k = 1, 2, \dots), \end{aligned}$$

where $\psi(x)$ is integrable in (a, b) and M is a constant independent of k . Then

$$\int_a^b \phi_k(x) e^{kh(x)} dx \sim \phi_k(b) e^{kh(b)} \left(\frac{-\pi}{2kh''(b)} \right)^{1/2}.$$

This reduces to the classical result if the functions $\phi_k(x)$ are all equal to a single function $\psi(x)$. Since the proof is much the same as in the classical case we omit it here.*

20. A preliminary limit. We begin with a consideration of the limit of

$$H_k(t) = \sum_{i=n+1}^{\infty} \frac{(i+k)!}{i!k!} t_0^i (1-t_0)^{k+1} \quad \left(n = \left[\frac{kt}{1-t} \right], 0 < t_0 < 1 \right)$$

as k becomes infinite. Here, as in the remainder of the paper, the notation $[u]$ means the largest integer contained in u . Taylor's series with exact remainder for the function $(1-x)^{-k-1}$ gives

$$(1-t_0)^{-k-1} = \sum_{i=0}^n \frac{(k+i)!}{k!i!} t_0^i + \frac{(n+k+1)!}{n!k!} \int_0^{t_0} \frac{(t_0-x)}{(1-x)^{n+k+2}} dx.$$

Hence

$$H_k(t) = (1-t_0)^{k+1} \frac{(n+k+1)!}{n!k!} \int_0^{t_0} \frac{(t_0-x)^n}{(1-x)^{n+k+2}} dx,$$

or, if we set $u = (t_0-x)/(1-x)$,

$$(20.1) \quad H_k(t) = \frac{(n+k+1)!}{n!k!} \int_0^{t_0} u^n (1-u)^k du.$$

Let us first consider $H_k(t_0)$. Set

$$v_0 = t_0/(1-t_0), \alpha = kv_0 - n.$$

Then α depends on k and satisfies the relation

$$0 \leq \alpha < 1.$$

Since

$$\int_0^1 u^n (1-u)^k du = \frac{n!k!}{(n+k+1)!},$$

we have

$$\frac{1}{H_k(t_0)} - 1 = \frac{\int_{t_0}^1 \{u^{v_0}(1-u)\}^k u^{-\alpha} du}{\int_0^{t_0} \{u^{v_0}(1-u)\}^k u^{-\alpha} du}.$$

By use of the Lemma of the previous section we shall show that this quantity approaches unity and hence that $H_k(t_0)$ approaches $1/2$ as k becomes infinite. We first obtain an asymptotic expression for the integral

* See, for example, Pólya and Szegő, *Aufgaben und Lehrsätze aus der Analysis*, vol. I, p. 80, problem 212.

$$(20.2) \quad \int_0^{t_0} \{u^{v_0}(1-u)\}^k u^{-\alpha} du.$$

Let δ be any positive quantity less than t_0 . Then

$$(20.3) \quad \begin{aligned} \int_0^\delta \{u^{v_0}(1-u)\}^k u^{-\alpha} du &< \{\delta^{v_0}(1-\delta)\}^{k-1} \int_0^\delta u^{v_0-1}(1-u) du \\ &= \{t_0^{v_0}(1-t_0)\}^k O(\beta^k), \end{aligned}$$

where β is a positive constant less than unity and independent of k . We now apply the Lemma to the integral

$$\int_0^{t_0} \{u^{v_0}(1-u)\}^k u^{-\alpha} du.$$

For the application we have

$$h(x) = v_0 \log x + \log(1-x),$$

$$h'(x) > 0 \quad (x < t_0),$$

$$h'(t_0) = 0, \quad h''(t_0) = -\frac{1}{t_0(1-t_0)^2} < 0,$$

$$-\phi_k(x) = x^{-\alpha}, \quad |\phi_k'(x)| < 1/\delta^2 = M,$$

$$\phi_k(t_0) = t_0^{-\alpha} > 1.$$

Hence we conclude that

$$\int_{t_0}^{t_0} \{u^{v_0}(1-u)\}^k u^{-\alpha} du \sim t_0^{n+(1/2)}(1-t_0)^{k+1}(\pi/(2k))^{1/2} \quad (k \rightarrow \infty).$$

By virtue of (20.3) we see that the integral (20.2) has this same asymptotic expression. Similar reasoning shows that

$$\int_{t_0}^1 \{u^{v_0}(1-u)\}^k u^{-\alpha} du \sim t_0^{n+(1/2)}(1-t_0)^{k+1}(\pi/(2k))^{1/2} \quad (k \rightarrow \infty)$$

so that

$$\lim_{k \rightarrow \infty} H_k(t_0) = \frac{1}{2}.$$

Next consider the case $t > t_0$. Set

$$v = t/(1-t).$$

Since the function $u^{v_0}(1-u)$ is increasing in the interval $0 \leq u \leq t_0$ (with its maximum at $u=t_0$), it follows that

$$\int_0^{t_0} u^n (1-u)^k du < \{t_0^n (1-t_0)\}^{k-1},$$

$$H_k(t) < \frac{\Gamma(vk+k+2)}{\Gamma(k+1)\Gamma(kv+1)} \{t_0^v (1-t_0)\}^{k-1}.$$

By use of Stirling's formula we can show that the right-hand side of this inequality approaches zero with $1/k$. Thus

$$\lim_{k \rightarrow \infty} H_k(t) = \lim_{k \rightarrow \infty} \left(\frac{vk+k+1}{e} \right)^{vk+k+1} \left(\frac{e}{k} \right)^k \left(\frac{e}{vk} \right)^{vk} \{t_0^v (1-t_0)\}^{k-1} \left(\frac{v+1}{2k\pi v} \right)^{1/2}.$$

The function of k on the right may be regarded as the k th term of an infinite series whose test ratio is

$$\left(\frac{v+1}{v} \right)^v (v+1)t_0(1-t_0).$$

We can show that this is less than unity. Introducing t and setting $t-t_0=\eta$ we must show that

$$(20.4) \quad \left(1 - \frac{\eta}{t} \right)^{t/(1-t)} \left(1 + \frac{\eta}{1-t} \right) < 1.$$

Employing a familiar inequality we have

$$\begin{aligned} \left(1 - \frac{\eta}{t} \right)^t &< 1 - \eta, \\ \left(1 + \frac{\eta}{1-t} \right)^{1-t} &< 1 + \eta, \\ \left(1 - \frac{\eta}{t} \right)^t \left(1 + \frac{\eta}{1-t} \right)^{1-t} &< 1 - \eta^2 < 1, \end{aligned}$$

from which (20.4) follows at once. Hence the series in question converges and

$$\lim_{k \rightarrow \infty} H_k(t) = 0 \quad (t > t_0).$$

Finally, if $0 < t < t_0$, we write

$$1 - H_k(t) = \frac{(n+k+1)!}{n!k!} \int_{t_0}^1 u^n (1-u)^k du,$$

$$1 - H_k(t) < \frac{\Gamma(vk+k+2)}{\Gamma(k+1)\Gamma(kv+1)} \{t_0^v (1-t_0)\}^k \frac{1}{t_0}.$$

We treat the right-hand side of this inequality as before. We are again led to

prove (20.4), but now η is a negative quantity greater than -1 . The inequalities employed are not affected by this change so that

$$\lim_{k \rightarrow \infty} [1 - H_k(t)] = 0.$$

Consequently we have proved

THEOREM 36. *If*

$$H_k(t) = \sum_{i=n+1}^{\infty} \frac{(i+k)!}{i!k!} t_0^i (1-t_0)^{k+1} \quad \left(n = \left[\frac{kt}{1-t} \right] \right),$$

$$= \frac{(n+k+1)!}{n!k!} \int_0^{t_0} u^n (1-u)^k du,$$

then

$$\lim_{k \rightarrow \infty} H_k(t) = \begin{cases} 1 & (0 < t < t_0), \\ \frac{1}{2} & (t = t_0), \\ 0 & (t_0 < t < 1). \end{cases}$$

We have really solved here the system of moment equations

$$\mu_n = \frac{t_0^{n+1}}{n+1} = \int_0^1 t^n \phi(t) dt$$

and found that

$$\phi(t) = \lim_{k \rightarrow \infty} \frac{(n+k+1)!}{n!k!} (-1)^k \Delta^k \mu_n, \quad n = \left[\frac{kt}{1-t} \right].$$

21. Inversion of the general sequence of moments. We now introduce the following operators.

DEFINITION. An operator $S_t\{\mu_n\}$ is defined by the equations

$$S_{k,t}\{\mu_n\} = -\mu_{\infty} - \sum_{i=n+1}^{\infty} \frac{(i+k)!}{i!k!} (-1)^{k+1} \Delta^{k+1} \mu_i, \quad n = \left[\frac{kt}{1-t} \right],$$

$$S_t\{\mu_n\} = \lim_{k \rightarrow \infty} S_{k,t}\{\mu_n\}.$$

DEFINITION. An operator $L_t\{\mu_n\}$ is defined by the equations

$$L_{k,t}\{\mu_n\} = \frac{(n+k+1)!}{n!k!} (-1)^k \Delta^k \mu_n, \quad n = \left[\frac{kt}{1-t} \right],$$

$$L_t\{\mu_n\} = \lim_{k \rightarrow \infty} L_{k,t}\{\mu_n\}.$$

We shall now show that $S_t\{\mu_n\}$ inverts the sequence (18.1).

THEOREM 37. If the function $\alpha(t)$ is of bounded variation in the interval $(0,1)$ with $\alpha(1)=0$, and if

$$\mu_n = \int_0^1 t^n d\alpha(t) \quad (n = 0, 1, 2, \dots),$$

then

$$S_t\{\mu_n\} = \frac{\alpha(t+) + \alpha(t-)}{2} \quad (0 < t < 1).$$

An integration by parts gives us

$$\mu_n = -n \int_0^1 t^{n-1} \alpha(t) dt \quad (n = 1, 2, \dots).$$

Then

$$\nu_n = -\frac{\mu_{n+1}}{n+1} = \int_0^1 t^n \alpha(t) dt \quad (n = 0, 1, 2, \dots).$$

We first show that $L_t\{\nu_n\}$ is well defined. We have

$$L_{k,t}\{\nu_n\} = \frac{(n+k+1)!}{n!k!} \int_0^1 t^n (1-t)^k \alpha(t) dt, \quad n = \left[\frac{kt}{1-t} \right].$$

Let t_0 be an arbitrary point in the interval $0 < t < 1$. Set

$$g(t) = \begin{cases} -1 & (0 < t < t_0), \\ -\frac{1}{2} & (t = t_0), \\ 0 & (t_0 < t < 1). \end{cases}$$

Form the function

$$\psi(t) = g(t) \{ \alpha(t_0+) - \alpha(t_0-) \} + \alpha(t_0+),$$

so that the function

$$\phi(u) = \alpha(u) - \psi(u)$$

has the property that

$$\phi(t_0+) = \phi(t_0-) = 0.$$

Now set up the integral

$$I = \frac{(n+k+1)!}{n!k!} \int_0^1 y^n (1-y)^k \phi(y) dy \quad \left(n = \left[\frac{kt_0}{1-t_0} \right] \right).$$

We divide the interval of integration into three parts $(0, 1-\eta)$, $(1-\eta, 1+\eta)$, $(1+\eta, 1)$, denoting the corresponding integrals by I_1 , I_2 , I_3 respectively. Given an arbitrary positive ϵ we determine η so small that

$$|\phi(y)| = |\alpha(y) - \psi(y)| < \epsilon/3 \quad (0 < |y - t_0| \leq \eta).$$

Then we have

$$\begin{aligned} |I_1| &\leq \frac{(n+k+1)!}{n!k!} \int_0^1 y^n(1-y)^k \frac{\epsilon}{3} dy = \frac{\epsilon}{3}, \\ |I_2| &\leq \frac{(n+k+1)!}{n!k!} (t_0 - \eta)^{(k-1)t_0/(1-t_0)} (1-t_0 + \eta)^{k-1} M, \\ |I_3| &\leq \frac{(n+k+1)!}{n!k!} (t_0 + \eta)^{(k-1)t_0/(1-t_0)} (1-t_0 - \eta)^{k-1} M, \end{aligned}$$

where M is a suitably chosen constant. We have already seen in §20 that the right-hand members of the last two inequalities approach zero with $1/k$, so that it is clearly possible to determine k_0 so large that

$$|I| < \epsilon$$

for $k > k_0$. But

$$(21.1) \quad I = L_{k,t}\{\nu_n\} + \{\alpha(t_0+) - \alpha(t_0-)\} \frac{(n+k+1)!}{n!k!} \int_0^{t_0} y^n(1-y)^k dy - \alpha(t_0+),$$

so that by allowing k to become infinite we obtain

$$\begin{aligned} 0 &= L_t\{\nu_n\} + \frac{\alpha(t_0+) - \alpha(t_0-)}{2} - \alpha(t_0+), \\ L_t\{\nu_n\} &= \frac{\alpha(t_0+) + \alpha(t_0-)}{2}. \end{aligned}$$

To evaluate the limit of the second term on the right-hand side of (21.1) we have employed Theorem 36.

It remains to show that

$$L_{k,t}\{\nu_n\} = S_{k,t}\{\mu_n\}.$$

To do this we prove first that

$$\lim_{i \rightarrow \infty} \frac{(i+k+1)!}{(k+1)!i!} \Delta^{k+1} \mu_{i+1} = \lim_{i \rightarrow \infty} \frac{(i+k+1)!}{(k+1)!i!} \int_0^1 t^{i+1}(1-t)^{k+1} d\alpha(t) = 0$$

and that μ_∞ exists. Introduce the function

$$\omega(t) = \alpha(t) - \alpha(1-t) \quad (0 \leq t \leq 1).$$

Then

$$I_{i,k} = \int_0^1 t^{i+1}(1-t)^k d\alpha(t) = \int_0^1 t^{i+1}(1-t)^k d\omega(t) \quad (k = 0, 1, 2, \dots).$$

In particular

$$\begin{aligned}\mu_{i+1} &= \int_0^1 t^{i+1} d\alpha(t) = \int_0^1 t^{i+1} d\omega(t) \\ &= -\alpha(1-) - (i+1) \int_0^1 t^i \omega(t) dt \quad (i = 0, 1, 2, \dots).\end{aligned}$$

Given an arbitrary positive ϵ , we determine a number δ such that

$$|\omega(t)| < \epsilon/2 \quad (1-\delta \leq t < 1).$$

Then

$$\begin{aligned}\left| (i+1) \int_0^1 t^i \omega(t) dt \right| &\leq \left| (i+1) \int_0^{1-\delta} t^i \omega(t) dt \right| + (i+1) \int_{1-\delta}^1 t^i \frac{\epsilon}{2} dt \\ &\leq (1-\delta)^{i+1} + \epsilon/2,\end{aligned}$$

where N is an upper bound for $|\omega(t)|$ in $(0,1)$. We can now determine i so large that $(1-\delta)^{i+1} N$ is less than $\epsilon/2$, so that

$$\mu_\infty = -\alpha(1-).$$

Next consider

$$I_{i,k} = \int_0^1 t^{i+1} (1-t)^k d\omega(t) = \int_0^1 t^{i+1} d\beta(t) = -(i+1) \int_0^1 t^i \beta(t) dt,$$

where

$$\beta(t) = \int_t^1 (1-x)^k d\omega(x) \quad (0 \leq t < 1),$$

$$\beta(1) = 0.$$

Since $\beta(t)$ is continuous at $t=1$, it follows that

$$\beta(t) = o((1-t)^k) \quad (t \rightarrow 1).$$

Hence it is easily seen that

$$I_{i,k} = o\left\{ (i+1) \int_0^1 t^i (1-t)^k dt \right\} = o\left\{ \frac{(i+1)k!}{(i+k+1)!} \right\} \quad (i \rightarrow \infty).$$

Consequently we have proved that

$$(21.2) \quad \lim_{i \rightarrow \infty} \frac{(i+k+1)!}{(k+1)!i!} (-1)^{k+1} \Delta^{k+1} \mu_{i+1} = 0 \quad (k = 0, 1, 2, \dots).$$

With this fact at our disposal we shall be able to show that the series

$$(21.3) \quad \mu_\infty + \sum_{i=n}^{\infty} \frac{(i+k+1)!}{k!(i+1)!} (-1)^{k+1} \Delta^{k+1} \mu_{i+1}$$

converges and has the sum

$$(21.4) \quad \sum_{i=0}^k \frac{(n+i)!}{i!n!} (-1)^i \Delta^i \mu_{n+1}.$$

We proceed by induction. For $k=0$ the relation reduces to

$$\mu_{\infty} - \sum_{i=n}^{\infty} \Delta \mu_{i+1} = \mu_{n+1}.$$

The series converges since μ_i is known to approach a limit as i becomes infinite, and partial summation shows the equation to be true. Now assume that (21.3) is equal to (21.4) and prove that the same equation holds when k is replaced by $k+1$. Then

$$(21.5) \quad \sum_{i=0}^{k+1} \frac{(n+i)!}{n!i!} (-1)^i \Delta^i \mu_{n+1} = \mu_{\infty} + \sum_{i=n}^{\infty} \frac{(i+k+1)!}{k!(i+1)!} (-1)^{k+1} \Delta^{k+1} \mu_{i+1} \\ + \frac{(n+k+1)!}{(k+1)!n!} (-1)^{k+1} \Delta^{k+1} \mu_{n+1}.$$

We observe that

$$\frac{(i+k+2)!}{(k+1)!(i+1)!} - \frac{(i+k+1)!}{(k+1)!i!} = \frac{(i+k+1)!}{k!(i+1)!}$$

and apply partial summation to the right-hand side of (21.5). By virtue of (21.2) we thus obtain

$$\sum_{i=0}^{k+1} \frac{(n+i)!}{n!i!} (-1)^i \Delta^i \mu_{n+1} = \mu_{\infty} + (-1)^{k+2} \sum_{i=n}^{\infty} \frac{(i+k+2)!}{(k+1)!(i+1)!} \Delta^{k+2} \mu_{i+1}.$$

The induction is complete.

But (21.3) is $-S_{k,i}\{\mu_n\}$. Also (21.4) is $-L_{k,i}\{\nu_n\}$. For,

$$\Delta^k \left\{ \frac{\mu_{n+1}}{n+1} \right\} = \sum_{i=0}^k \binom{k}{i} \Delta^i \mu_{n+1} \Delta^{k-i} \left\{ \frac{1}{n+i+1} \right\} \\ = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} \Delta^i \mu_{n+1} \frac{(n+i)!(k-i)!}{(n+k+1)!} \\ - \frac{(n+k+1)!}{n!k!} (-1)^k \Delta^k \nu_n \\ = \sum_{i=0}^k \frac{(n+i)!}{i!n!} (-1)^i \Delta^i \mu_{n+1} = -L_{k,i}\{\nu_n\}.$$

Hence

$$S_t\{\mu_n\} = \lim_{k \rightarrow \infty} S_{k,t}\{\mu_n\} = \lim_{k \rightarrow \infty} L_{k,t}\{\nu_n\} = \frac{\alpha(t+) + \alpha(t-)}{2},$$

and the proof of the theorem is complete.

In the course of the proof we have demonstrated

THEOREM 38. *If the function $\phi(t)$ is of bounded variation in the interval $(0,1)$ with $\phi(1)=0$ and if*

$$\mu_n = \int_0^1 t^n \phi(t) dt \quad (n = 0, 1, 2, \dots),$$

then

$$L_t\{\mu_n\} = \frac{\phi(t+) + \phi(t-)}{2} \quad (0 < t < 1).$$

We next establish the stronger result contained in

THEOREM 39. *If the function $\phi(t)$ is integrable in $(0,1)$ and if*

$$\mu_n = \int_0^1 t^n \phi(t) dt \quad (n = 0, 1, 2, \dots),$$

then

$$L_t\{\mu_n\} = \phi(t)$$

almost everywhere in $(0,1)$.

As we observed in §4

$$\int_t^x |\phi(u) - \phi(t)| du = o(|x - t|)$$

for almost all values of t in $(0,1)$. Let t_0 be such a value of t , and form the integrals

$$I_k = \frac{(n+k+1)!}{n!k!} \int_0^{t_0} y^n (1-y)^k \{\phi(y) - \phi(t_0)\} dy,$$

$$J_k = \frac{(n+k+1)!}{n!k!} \int_{t_0}^1 y^n (1-y)^k \{\phi(y) - \phi(t_0)\} dy, \quad n = \left[\frac{kt_0}{1-t_0} \right].$$

It will be sufficient to show that I_k and J_k approach zero with $1/k$; for,

$$L_{k,t_0}\{\mu_n\} = \frac{(n+k+1)!}{n!k!} \int_0^1 y^n (1-y)^k \phi(y) dy,$$

$$L_{t_0}\{\mu_n\} = \phi(t_0) + \lim_{k \rightarrow \infty} \{I_k + J_k\}.$$

Since the proof is similar for the two integrals we give only that for I_k . Set

$$\gamma(t) = \int_{t_0}^t \frac{|\phi(y) - \phi(t_0)|}{y} dy.$$

It is easy to see by integration by parts that $\gamma(t)$ is also $o(t_0 - t)$ as t approaches t_0 . Hence to an arbitrary positive ϵ there corresponds a number δ such that

$$(21.6) \quad |\gamma(y)| < (t_0 - y)\epsilon/2 \quad (0 < t_0 - \delta < y < t_0).$$

Introduce the integrals

$$I'_k = \frac{(n+k+1)!}{n!k!} \int_{t_0-\delta}^{t_0} y^n(1-y)^k \{\phi(y) - \phi(t_0)\} dy,$$

$$I''_k = \frac{(n+k+1)!}{n!k!} \int_0^{t_0-\delta} y^n(1-y)^k \{\phi(y) - \phi(t_0)\} dy,$$

so that

$$I_k = I'_k + I''_k.$$

If we write

$$\alpha = \frac{kt_0}{1-t_0} - \left[\frac{kt_0}{1-t_0} \right],$$

we have

$$I'_k = \frac{(n+k+1)!}{n!k!} \int_{t_0-\delta}^{t_0} y^{kt_0/(1-t_0)}(1-y)^k y^{-\alpha} \{\phi(y) - \phi(t_0)\} dy,$$

$$\left| \frac{n!k!}{(n+k+1)!} I'_k \right| \leq \int_{t_0-\delta}^{t_0} y^{kt_0/(1-t_0)}(1-y)^{k-1} |\phi(y) - \phi(t_0)| dy$$

$$= -\gamma(t_0 - \delta)(t_0 - \delta)^{kt_0/(1-t_0)}(1 - t_0 + \delta)^k$$

$$- \int_{t_0-\delta}^{t_0} \gamma(y) d\{y^{kt_0/(1-t_0)}(1-y)^k\}.$$

Noting that the function

$$y^{kt_0/(1-t_0)}(1-y)^k$$

is increasing in the interval $(t_0 - \delta, t_0)$ and applying (21.6) we obtain

$$\frac{n!k!}{(n+k+1)!} |I'_k| < \frac{\epsilon}{2} \delta(t_0 - \delta)^{kt_0/(1-t_0)}(1 - t_0 + \delta)^k$$

$$+ \frac{\epsilon}{2} \int_{t_0-\delta}^{t_0} (t_0 - y) d\{y^{kt_0/(1-t_0)}(1-y)^k\}$$

$$= \frac{\epsilon}{2} \int_{t_0-\delta}^{t_0} y^{kt_0/(1-t_0)}(1-y)^k dy < \frac{\epsilon}{2} \int_0^1 y^n(1-y)^k dy,$$

$$|I'_k| < \epsilon/2.$$

We turn next to I_k'' . It may evidently be written as

$$I_k'' = \frac{(n+k+1)!}{n!k!} \int_0^{t_0-\delta} y^{kt_0/(1-t_0)} (1-y)^k y^{-a} \{\phi(y) - \phi(t_0)\} dy.$$

Let k_0 be an integer so large that

$$\frac{k_0 t_0}{1-t_0} > 1.$$

Then

$$y^{k_0 t_0/(1-t_0)} y^{-a} (1-y)^{k_0} < 1 \quad (0 \leq y \leq 1),$$

and

$$\frac{n!k!}{(n+k+1)!} |I_k''| \leq \int_0^{t_0-\delta} y^{(k-k_0)t_0/(1-t_0)} (1-y)^{k-k_0} |\phi(y) - \phi(t_0)| dy.$$

The function

$$y^{(k-k_0)t_0/(1-t_0)} (1-y)^{k-k_0}$$

is increasing in the interval $(0, t_0)$, so that

$$|I_k''| \leq \frac{(n+k+1)!}{n!k!} (t_0 - \delta)^{(k-k_0)t_0/(1-t_0)} (1-t_0 + \delta)^{k-k_0} M,$$

where

$$M = \int_0^{t_0-\delta} |\phi(y) - \phi(t_0)| dy.$$

But the right-hand side of this inequality tends to zero with $1/k$, as we proved in §20, so that we can determine k_1 so large that

$$|I_k''| < \epsilon/2 \quad (k > k_1),$$

and

$$|I_k| < \epsilon.$$

The theorem is thus established.

As illustrations of Theorems 39 and 41 it is interesting to show by direct evaluation of the limits concerned that

$$L_t \left\{ \frac{1}{n+2} \right\} = t \quad (0 < t < 1),$$

$$S_t \left\{ \frac{1}{n+2} \right\} = \frac{t^2 - 1}{2} \quad (0 < t < 1).$$

22. Uniqueness theorems. We are now in a position to establish

THEOREM 40. If the sequence $\{\mu_n\}$ satisfies the inequalities

$$(22.1) \quad |\Delta^k \mu_n| < \frac{Mn!k!}{(n+k+1)!} \quad (n = 0, 1, 2, \dots; k = 0, 1, 2, \dots)$$

then

$$\mu_m = \lim_{k \rightarrow \infty} \int_0^1 t^m L_{k,t} \{\mu_n\} dt \quad (m = 0, 1, 2, \dots).$$

By definition of the operator $L_{k,t} \{\mu_n\}$ we have

$$I_k(m) = \int_0^1 t^m L_{k,t} \{\mu_n\} dt = (-1)^k \int_0^1 t^m \frac{(n+k+1)!}{n!k!} \Delta^k \mu_n dt,$$

where

$$n = \left[\frac{kt}{1-t} \right].$$

Set

$$(22.2) \quad \begin{aligned} H_k(m) &= (-1)^k \int_{m/(k+m)}^1 t^m \frac{(n+k+1)!}{n!k!} \Delta^k \mu_n dt, \\ J_k(m) &= (-1)^k \int_0^{m/(k+m)} t^m \frac{(n+k+1)!}{n!k!} \Delta^k \mu_n dt, \end{aligned}$$

so that

$$I_k(m) = H_k(m) + J_k(m).$$

Our theorem will then be established if we can show that

$$\lim_{k \rightarrow \infty} H_k(m) = \mu_m, \quad \lim_{k \rightarrow \infty} J_k(m) = 0 \quad (m = 0, 1, 2, \dots).$$

In the integral (22.2) make the change of variable

$$u = kt/(1-t),$$

so that

$$(22.3) \quad H_k(m) = (-1)^k \int_m^\infty \left(\frac{u}{k+u} \right)^m \frac{k}{(k+u)^2} \frac{([u]+k+1)!}{[u]!k!} \Delta^k \mu_{[u]} du.$$

Next we observe that

$$(22.4) \quad \mu_m = \sum_{p=m}^{\infty} \frac{(p+k-m-1)!}{(k-1)!(p-m)!} (-1)^k \Delta^k \mu_p.$$

This result is easily proved by induction, making use of (22.1).

Now write the summation (22.4) as an integral as follows:

$$(22.5) \quad \mu_m = (-1)^k \int_m^\infty \frac{([u] + k - m - 1)!}{(k-1)!([u] - m)!} \Delta^k \mu_{[u]} du.$$

Then

$$H_k(m) - \mu_m = (-1)^k \int_m^\infty \Delta^k \mu_{[u]} \left\{ \left(\frac{u}{k+u} \right)^m \frac{k}{(k+u)^2} \frac{([u] + k + 1)!}{[u]!k!} - \frac{([u] + k - m - 1)!}{(k-1)!([u] - m)!} \right\} du.$$

By virtue of (22.1)

$$|H_k(m) - \mu_m| \leq M \int_m^\infty \left(\frac{u}{k+u} \right)^m \frac{k}{(k+u)^2} \left| 1 - \frac{[u]}{u} \frac{[u]-1}{u} \cdots \frac{[u]-m+1}{u} \frac{k+u}{[u]+k+1} \frac{k+u}{[u]+k} \cdots \frac{k+u}{[u]+k-m} \right| du.$$

But

$$\frac{u+k}{[u]+k+j} = 1 + \frac{u-[u]-j}{[u]+k+j} = 1 + O\left(\frac{1}{k}\right) \quad (k \rightarrow \infty)$$

uniformly for $m \leq u < \infty$. Also

$$\left| \frac{[u]-j}{u} - 1 \right| = \left| \frac{[u]-u-j}{u} \right| \leq \frac{N}{u} \quad (j = 0, 1, 2, \dots, m-1),$$

where N is a constant independent of u and of j . Hence for $m \geq 1$ we have

$$\begin{aligned} H_k(m) - \mu_m &= O \left\{ \int_m^\infty \frac{k}{(k+u)^2} \frac{1}{k} du \right\} \quad (k \rightarrow \infty) \\ &= O \left(\frac{1}{k} \right) \quad (k \rightarrow \infty), \end{aligned}$$

so that

$$(22.6) \quad \lim_{k \rightarrow \infty} H_k(m) = \mu_m \quad (m = 1, 2, 3, \dots).$$

For $m=0$ we have from (22.3)

$$\begin{aligned} H_k(0) &= (-1)^k \sum_{p=0}^{\infty} \frac{(p+k+1)!}{p!k!} \Delta^k \mu_p \int_p^{p+1} \frac{k}{(k+u)^2} du \\ &= (-1)^k \sum_{p=0}^{\infty} \frac{(p+k-1)!}{(k-1)!p!} \Delta^k \mu_p. \end{aligned}$$

But reference to (22.4) shows that this is μ_0 , and (22.6) holds also for $m=0$.

It remains to show that

$$(22.7) \quad \lim_{k \rightarrow \infty} J_k(m) = 0 \quad (m = 0, 1, 2, \dots).$$

By virtue of (22.1) we have

$$|J_k(m)| \leq \int_0^{m/(k+m)} t^m M dt \leq \frac{Mm}{k+m},$$

so that (22.7) is evident. The theorem is thus established.

We turn next to a corresponding result for Stieltjes integrals.

THEOREM 41. *If*

$$(22.8) \quad \left| \sum_{p=m}^{\infty} \frac{(p+k)!}{p!k!} \Delta^{k+1}\mu_p \right| < M \quad (m = 0, 1, 2, \dots; k = 0, 1, 2, \dots),$$

then

$$\mu_m - \mu_{\infty} = \lim_{k \rightarrow \infty} \int_0^1 t^m dS_{k,1}\{\mu_n\} \quad (m = 0, 1, 2, \dots),$$

provided $S_{k,1}\{\mu_n\}$ is defined as $-\mu_{\infty}$.

To prove this result we rewrite the sum (22.8) as follows:

$$(-1)^k \sum_{p=m}^{\infty} \frac{(p+k-1)!}{p!(k-1)!} \Delta^k \mu_p = (-1)^k \sum_{p=m}^{\infty} \left\{ \frac{(p+k)!}{k!p!} - \frac{(p+k-1)!}{k!(p-1)!} \right\} \Delta^k \mu_p.$$

Now

$$\begin{aligned} (-1)^k \sum_{p=m}^q \frac{(p+k-1)!}{p!(k-1)!} \Delta^k \mu_p &= (-1)^k \sum_{p=m}^q \frac{(p+k)!}{k!p!} \Delta^k \mu_p \\ &\quad - (-1)^k \sum_{p=m}^q \frac{(p+k-1)!}{k!(p-1)!} \Delta^k \mu_p \end{aligned}$$

for any integer $q > m$. Changing the summation variable in the last of these sums we have

$$\begin{aligned} (-1)^k \sum_{p=m}^q \frac{(p+k-1)!}{p!(k-1)!} \Delta^k \mu_p \\ = (-1)^k \sum_{p=m}^q \frac{(p+k)!}{k!p!} \Delta^k \mu_p - (-1)^k \sum_{p=m+1}^{q+1} \frac{(p+k)!}{k!p!} \Delta^k \mu_{p+1} \end{aligned}$$

$$\begin{aligned}
&= (-1)^{k+1} \sum_{p=m+1}^q \frac{(p+k)!}{k!p!} \Delta^{k+1} \mu_p \\
&\quad + (-1)^k \frac{(m+k)!}{k!m!} \Delta^k \mu_m - (-1)^k \frac{(q+k+1)!}{k!(q+1)!} \Delta^k \mu_{q+1}.
\end{aligned}$$

Now let q become infinite. Since the summations on both sides of the equation approach limits by hypothesis, it follows that

$$\lim_{q \rightarrow \infty} \frac{(q+k)!}{k!q!} \Delta^k \mu_{q+1}$$

exists ($k=0, 1, 2, \dots$). We can show, in fact, that all these limits are zero except perhaps that corresponding to $k=0$. For, suppose

$$\lim_{q \rightarrow \infty} q \Delta \mu_q = B > 0.$$

Then for $q \geq q_0$ we have

$$q \Delta \mu_q > B/2,$$

and

$$\mu_{q_0+n+1} - \mu_{q_0} = \sum_{q=q_0}^{q_0+n} \Delta \mu_q > \frac{B}{2} \sum_{q=q_0}^{q_0+n} \frac{1}{q}.$$

As n becomes infinite the right-hand side of this inequality becomes positively infinite, so that μ_q can approach no limit as q becomes infinite, contrary to the fact established above. If $B < 0$ we deduce a contradiction by applying the foregoing proof to the sequence $\{-\mu_q\}$. Proceeding by induction suppose it has been established that

$$\lim_{q \rightarrow \infty} \frac{(q+k)!}{k!q!} \Delta^k \mu_{q+1} = 0;$$

let us show the same to be true when k is replaced by $k+1$. We have

$$(22.9) \quad (q+1) \Delta \left\{ \frac{(q+k)!}{k!q!} \Delta^k \mu_{q+1} \right\} = \frac{(q+k+1)!}{k!q!} \Delta^{k+1} \mu_{q+1} + \frac{(q+k)!}{q!(k+1)!} \Delta^k \mu_{q+1}.$$

By assumption the right-hand side approaches a limit. It follows that

$$q \Delta \left\{ \frac{(q+k)!}{k!q!} \Delta^k \mu_{q+1} \right\}$$

approaches the same limit. Since

$$(22.10) \quad \frac{(q+k)!}{k!q!} \Delta^k \mu_{q+1}$$

also approaches a limit we may apply the argument used for the case $k=0$ to the sequence (22.10). It follows that the limit of this sequence must be zero and hence by (22.9) that

$$\lim_{q \rightarrow \infty} \frac{(q+k+1)!}{(k+1)!q!} \Delta^{k+1} \mu_{q+1} = 0.$$

We showed in §21 that

$$(22.11) \quad \frac{(n+k+1)!}{n!k!} (-1)^k \Delta^k \left\{ \frac{\mu_{n+1}}{n+1} \right\} \\ = \mu_{\infty} + (-1)^{k+1} \sum_{i=n}^{\infty} \frac{(i+k+1)!}{k!(i+1)!} \Delta^{k+1} \mu_{i+1}$$

provided that

$$\lim_{i \rightarrow \infty} \frac{(i+k+1)!}{(k+1)!i!} \Delta^{k+1} \mu_{i+1} = 0.$$

But we have just established this latter result. Then by (22.2) and (22.8) we have

$$\left| \frac{(n+k+1)!}{n!k!} \Delta^k \left\{ \frac{\mu_{n+1}}{n+1} \right\} \right| < |\mu_{\infty}| + M.$$

By Theorem 40 it follows that

$$\frac{\mu_{m+1}}{m+1} = \lim_{k \rightarrow \infty} \int_0^1 t^m L_{k,t} \left\{ \frac{\mu_{n+1}}{n+1} \right\} dt.$$

By (22.11) we see that

$$L_{k,t} \left\{ \frac{\mu_{n+1}}{n+1} \right\} = -S_{k,t} \{ \mu_n \}.$$

Hence

$$\begin{aligned} \frac{\mu_{m+1}}{m+1} &= - \lim_{k \rightarrow \infty} \int_0^1 t^m S_{k,t} \{ \mu_n \} dt \\ &= - \lim_{k \rightarrow \infty} \frac{t^{m+1}}{m+1} S_{k,t} \{ \mu_n \} \Big|_{t=0}^{t=1} + \lim_{k \rightarrow \infty} \frac{1}{m+1} \int_0^1 t^{m+1} dS_{k,t} \{ \mu_n \}, \\ \mu_{m+1} &= \lim_{k \rightarrow \infty} \int_0^1 t^{m+1} dS_{k,t} \{ \mu_n \} + \mu_{\infty}. \end{aligned}$$

This completes the proof of the theorem.

23. Hausdorff's theorem. Just as our inversion operator enabled us to give a proof of Bernstein's theorem so will the present inversion operator enable us to give a proof of a familiar theorem of Hausdorff.

THEOREM 42. *A necessary and sufficient condition that the equations*

$$\mu_n = \int_0^1 t^n d\alpha(t) \quad (n = 0, 1, 2, \dots)$$

should have a bounded non-decreasing solution $\alpha(t)$ is that the sequence $\{\mu_n\}$ should be completely monotonic:

$$(-1)^k \Delta^k \mu_n \geq 0 \quad (k = 0, 1, 2, \dots; n = 0, 1, 2, \dots).$$

The necessity of the condition follows at once from the equation

$$(-1)^k \Delta^k \mu_n = \int_0^1 t^n (1-t)^k d\alpha(t).$$

To prove the sufficiency apply the operator S_t to the sequence $\{\mu_n\}$. We must first show that $S_{k,t}\{\mu_n\}$ exists. To do this we show that for a completely monotonic sequence μ_n we have

$$(23.1) \quad \lim_{n \rightarrow \infty} \frac{(n+k)!}{n!k!} \Delta^k \mu_{n+1} = c_k \quad (k = 0, 1, 2, \dots).$$

We use induction. The result is immediate for $k=0$, for the sequence $\{\mu_n\}$ is non-negative non-increasing. Next form the sequence

$$\nu_n = \mu_{n+1} - (n+1)\Delta\mu_{n+1} \quad (n = 0, 1, 2, \dots).$$

This is also a non-negative sequence. Moreover,

$$\begin{aligned} \Delta\nu_n &= \Delta\mu_{n+1} - (n+2)\Delta^2\mu_{n+1} - \Delta\mu_{n+1} \\ &= -(n+2)\Delta^2\mu_{n+1}. \end{aligned}$$

This is not greater than zero, so that the sequence $\{\nu_n\}$ is also non-increasing, and must therefore approach a limit. Since μ_{n+1} has a limit, the same must be true of $(n+1)\Delta\mu_{n+1}$. We proceed by induction. Suppose that (23.1) has been established for $k < m$. Form the sequence

$$\begin{aligned} (23.2) \quad \nu_n &= \mu_{n+1} - (n+1)\Delta\mu_{n+1} + \frac{(n+1)(n+2)}{2!} \Delta^2\mu_{n+1} - \dots \\ &\quad + (-1)^m \frac{(n+m)!}{m!n!} \Delta^m \mu_{n+1}. \end{aligned}$$

Simple computation gives

$$\Delta \nu_n = (-1)^n \frac{(n+m+1)!}{m!(n+1)!} \Delta^{m+1} \mu_{n+1} \leq 0.$$

It follows that ν_n is a non-negative, non-increasing sequence and hence must approach a limit. Thus every term in (23.2) except the last is known to have a limit. It follows that (23.1) holds for $k=m$, and the induction is complete.

In the proof of Theorem 41 we showed that the existence of the limits c_k implied that they were all zero except perhaps c_0 . This, in turn, implies the convergence of the series

$$(23.3) \quad (-1)^{k+1} \sum_{i=n}^{\infty} \frac{(i+k+1)!}{k!(i+1)!} \Delta^{k+1} \mu_{i+1}.$$

Moreover

$$(23.4) \quad \begin{aligned} \sum_{i=n}^{\infty} \frac{(i+k+1)!}{k!i!} (-1)^{k+1} \Delta^{k+1} \mu_{i+1} \\ = \sum_{i=n}^{\infty} \frac{(i+k+2)!}{(k+1)!(i+1)!} (-1)^{k+2} \Delta^{k+2} \mu_{i+1}. \end{aligned}$$

The derivation of this formula involved (23.1) as well as the convergence of the series. But the series on the left surely converges for $k=0$ since the limit of μ_n exists. Hence by induction we see that (23.3) converges for all k .

Having verified that the operator $S_{k,t}\{\mu_n\}$ exists we prove next that it defines a non-decreasing function. We have

$$S_{k,t}\{\mu_n\} = -\mu_{\infty} - \sum_{i=n}^{\infty} \frac{(i+k+1)!}{k!(i+1)!} (-1)^{k+1} \Delta^{k+1} \mu_{i+1}, \quad n = \left[\frac{kt}{1-t} \right].$$

As t increases n is non-decreasing, so that

$$\sum_{i=n}^{\infty} \frac{(i+k+1)!}{k!(i+1)!} (-1)^{k+1} \Delta^{k+1} \mu_{i+1}$$

is surely non-increasing. That is, $S_{k,t}\{\mu_n\}$ is non-decreasing. The same must be true of $S_t\{\mu_n\}$. Clearly

$$\begin{aligned} -S_{k,t}\{\mu_n\} &\leq -S_{k,0}\{\mu_n\} = \mu_{\infty} + (-1)^{k+1} \sum_{i=0}^{\infty} \frac{(i+k+1)!}{k!(i+1)!} \Delta^{k+1} \mu_{i+1} \\ &\leq \mu_{\infty} + (-1)^{k+1} \sum_{i=0}^{\infty} \frac{(i+k+1)!}{k!(i+1)!} \Delta^{k+1} \mu_{i+1} + (-1)^{k+1} \Delta^{k+1} \mu_0 \\ &= \mu_{\infty} + \sum_{i=0}^{\infty} \frac{(i+k)!}{k!i!} (-1)^{k+1} \Delta^{k+1} \mu_i = \mu_0. \end{aligned}$$

This latter result is certainly exact for $k=0$ and is easily seen to be true in general by induction using (23.4). Hence the inequalities (22.8) are satisfied by the given completely monotonic sequence $\{\mu_n\}$. Hence by Theorem 41

$$\mu_m - \mu_\infty = \int_0^1 t^m dS_{k,t}\{\mu_n\}.$$

Now we employ Helly's theorem precisely as we did in the proof of Theorem 17 to select from the bounded sequence of non-decreasing functions $S_{k,t}\{\mu_n\}$ a sub-set which approaches a non-decreasing function $\beta(t)$. As in that proof we see that

$$\mu_m - \mu_\infty = \int_0^1 t^m d\beta(t).$$

Here $\beta(1) = -\mu_\infty$. We define $\alpha(t)$ as follows:

$$\alpha(t) = \beta(t) \quad (0 \leq t < 1),$$

$$\alpha(1) = 0.$$

The function $\alpha(t)$ remains non-decreasing since $-\mu_\infty \leq 0$ and

$$\mu_m = \int_0^1 t^m d\alpha(t) \quad (m = 0, 1, 2, \dots).$$

This completes the proof of the theorem.†

The present methods are powerful in the discussion of what sequences are moment sequences. We use them to prove one further result of Hausdorff.

† Compare F. Hausdorff, *Momentprobleme für ein endliches Intervall*, Mathematische Zeitschrift, vol. 16 (1923), pp. 220-248. It is of interest to compare Hausdorff's method of approximation to the function $\alpha(t)$ with our own. His k th approximating function $\chi_k(t)$ is a step-function which vanishes at $t=0$ and has a jump of amount

$$(-1)^{k-n} \binom{k}{n} \Delta^{k-n} \mu^n$$

at the point n/k ($n=0, 1, 2, \dots, k$). Our k th approximating function, $S_{k,t}\{\mu^n\}$, is also a step-function vanishing at the origin and with jump of amount

$$(-1)^{k+1-n} \binom{n+k}{n} \Delta^{k+1-n} \mu^n$$

at the points $n/(n+k)$ ($n=0, 1, 2, \dots$). Thus the function $\chi_k(t)$ has $(k+1)$ jumps which depend on differences of order less than or equal to k . On the other hand the function $S_{k,t}\{\mu^n\}$ has infinitely many jumps which cluster about the point $t=1$, the amount of the jumps depending on the $(k+1)$ th differences only. We note further that the function $L_{k,t}\{\mu^n\}$ is also a step-function with jumps at the same points $n/(n+k)$, the amounts of the jumps depending on differences of order k only.

THEOREM 43. *A necessary and sufficient condition that*

$$(23.5) \quad \mu_n = \int_0^1 t^n d\alpha(t) \quad (n = 0, 1, 2, \dots)$$

where $\alpha(t)$ is of bounded variation in $(0,1)$ is that for a suitable constant M

$$(23.6) \quad \sum_{p=m}^{\infty} \frac{(p+k)!}{p!k!} |\Delta^{k+1}\mu_p| < M \quad (m = 0, 1, 2, \dots; k = 0, 1, 2, \dots).$$

We first prove the necessity of the condition. Assume that μ_n is defined by (23.5) with $\alpha(t)$ of bounded variation in $(0,1)$. Denote by $V(t)$ the total variation of $\alpha(x)$ in the interval $0 \leq x \leq t$. Then by Theorem 42 the sequence

$$\nu_n = \int_0^1 t^n dV(t) \quad (n = 0, 1, 2, \dots)$$

is completely monotonic. Hence

$$\begin{aligned} |\Delta^{k+1}\mu_p| &\leq (-1)^{k+1} \Delta^{k+1}\nu_p, \\ \sum_{p=m}^{\infty} \frac{(p+k)!}{p!k!} |\Delta^{k+1}\mu_p| &\leq \sum_{p=0}^{\infty} \frac{(p+k)!}{p!k!} (-1)^{k+1} \Delta^{k+1}\nu_p = \nu_0 - \nu_{\infty} = M. \end{aligned}$$

This completes the proof of the necessity.

We turn to the sufficiency of the condition. By condition (23.6) we see at once that Theorem 41 is applicable to the sequence $\{\mu_n\}$ so that

$$(23.7) \quad \mu_m - \mu_{\infty} = \lim_{k \rightarrow \infty} \int_0^1 t^m dS_{k,t}\{\mu_n\}.$$

Simple computation shows that the function $S_{k,t}\{\mu_n\}$ is a function of bounded variation and that its total variation in the interval $(0,1)$ is not greater than M .

We now employ the theorem of Helly used in the proof of Theorem 19 to select a convergent sequence of functions from the sequence $S_{k,t}\{\mu_n\}$. The limit function $\alpha(t)$ will itself be a function of bounded variation and by (23.7) we have

$$\mu_m - \mu_{\infty} = \int_0^1 t^m d\alpha(t).$$

If $\alpha(t)$ is redefined at $t=1$ to be zero there, we have

$$\mu_m = \int_0^1 t^m d\alpha(t) \quad (m = 0, 1, 2, \dots)$$

and the proof of the theorem is complete.

24. **The changes of sign in a moment sequence.** The operators S and L are useful in discussing the changes of sign in a sequence

$$\mu_n = \int_0^1 t^n d\alpha(t) \quad (n = 0, 1, 2, \dots)$$

in terms of the changes of trend in the function $\alpha(t)$ or in terms of the changes of sign in the derivative $\alpha'(t)$, if it exists. We begin by proving

THEOREM 44. *If the function $\alpha(t)$ has m changes of trend in the interval $(0,1)$ then the sequence*

$$(24.1) \quad \mu_n = \int_0^1 t^n d\alpha(t) \quad (n = 0, 1, 2, \dots)$$

can have at most m changes of sign.

For, set

$$\begin{aligned} \mu(x) &= \int_0^1 t^x d\alpha(t) = \int_0^\infty e^{-xt} d\beta(t), \\ \beta(t) &= -\alpha(e^{-t}). \end{aligned}$$

Since $\mu(x)$ is continuous (analytic in fact), and since $\mu(n) = \mu_n$, the function $\mu(x)$ would have more than m zeros if the sequence (24.1) had more than m changes of sign. But this is impossible by Theorem 22. We next establish

THEOREM 45. *If the function $\alpha(t)$ is a normalized function of bounded variation with $\alpha(1) = \alpha(1-) = 0$ and with m changes of trend in $(0,1)$, then the sequence*

$$(24.2) \quad (-1)^k \Delta^k \mu_n = \int_0^1 t^n (1-t)^k d\alpha(t) \quad (n = 0, 1, 2, \dots)$$

has exactly m changes of sign for all k sufficiently large.

We prove exactly as in the proof of Theorem 23 that corresponding to two adjoining intervals (t_{i-1}, t_i) , (t_i, t_{i+1}) of the intervals referred to in the definition of number of changes of trend (in the first of which $\alpha(t)$ is increasing, in the second of which, decreasing) there are three points ξ, η, ζ such that

$$\alpha(\xi) < \alpha(\eta) > \alpha(\zeta) \quad (t_{i-1} \leq \xi < \eta < \zeta \leq t_{i+1}).$$

We then determine k_0 so large that for $k > k_0$ we have

$$\left[\frac{\xi k}{1 - \xi} \right] < \left[\frac{k\eta}{1 - \eta} \right] < \left[\frac{k\zeta}{1 - \zeta} \right]$$

and such that

$$S_{k,\xi}\{\mu_n\} < S_{k,\eta}\{\mu_n\} > S_{k,\zeta}\{\mu_n\}$$

for $k > k_0$. This is possible by Theorem 37. Hence

$$\begin{aligned} (-1)^{k+1} \sum_{i=n_\xi}^{n_\eta-1} \frac{(i+k+1)!}{k!(i+1)!} \Delta^{k+1} \mu_{i+1} &> 0, \quad n_\xi = \left[\frac{k\xi}{1-\xi} \right], \quad n_\eta = \left[\frac{k\eta}{1-\eta} \right], \\ (-1)^{k+1} \sum_{i=n_\eta}^{n_\zeta-1} \frac{(i+k+1)!}{k!(i+1)!} \Delta^{k+1} \mu_{i+1} &< 0, \quad n_\zeta = \left[\frac{k\zeta}{1-\zeta} \right], \quad k > k_0. \end{aligned}$$

There is at least one term in each series, so that there must be at least one change of sign in the sequence $\{\Delta^{k+1} \mu_n\}$ ($n=0, 1, 2, \dots$) between the terms n_ξ and n_ζ . We are thus able to show as in the proof of Theorem 23 that the sequence (24.1) has at least m changes of sign for k sufficiently large. That it can not have more follows from Theorem 44.

COROLLARY 1. *If the function $\alpha(t)$ has a maximum (minimum) at a point $t=t_0$, then for k sufficiently large the sequence*

$$(-1)^k \Delta^k \mu_n = \int_0^1 t^n (1-t)^k d\alpha(t) \quad (n=0, 1, 2, \dots)$$

will have a change of sign at a term $n=n_k$ such that

$$\lim_{k \rightarrow \infty} \frac{n_k}{n_k + k} = t_0.$$

For, if ϵ is an arbitrary positive number we must show that a number k_0 exists such that the sequence $\Delta^k \mu_n$ has a change of sign at $n=n_k$, where

$$\left| \frac{n_k}{n_k + k} - t_0 \right| < \epsilon.$$

As in the proof of the theorem we see that for k sufficiently large the sequence changes sign between the terms with indices n_ξ and n_ζ where now

$$\xi = t_0 - (\epsilon/2), \quad \eta = t_0 + (\epsilon/2).$$

Suppose that a change of sign occurs at the term $n=n_k$. Then

$$k\xi/(1-\xi) - 1 < n_\xi < n_k < n_\zeta \leq k\zeta/(1-\zeta).$$

Hence

$$\begin{aligned} \frac{n_k}{n_k + k} &< \zeta < t_0 + \epsilon, \\ \xi &< \frac{n_k + 1}{n_k + k + 1}, \quad \xi - \frac{k}{(n_k + k)(n_k + k + 1)} < \frac{n_k}{n_k + k}. \end{aligned}$$

Since the left-hand side of this last inequality approaches ξ as k becomes infinite we can find a number k_0 so large that

$$t_0 - \epsilon = \xi - \frac{\epsilon}{2} < \frac{n_k}{n_k + k} \quad (k > k_0).$$

This proves the corollary.

COROLLARY 2. *If the function $\phi(t)$ is integrable in $(0,1)$ and has a change of sign at t_0 between 0 and 1, then the sequence*

$$(-1)^k \Delta^k \mu_n = \int_0^1 t^n (1-t)^k \phi(t) dt$$

has a change of sign at a term with index n_k such that

$$\lim_{k \rightarrow \infty} \frac{n_k}{n_k + k} = t_0.$$

One proves this by Corollary 1, setting

$$\alpha(t) = \int_t^1 \phi(u) du$$

and observing that $\alpha(t)$ has a maximum or minimum at t_0 . This result is a generalization of a theorem of Fekete cited in the introduction. Fekete considered only functions $\phi(t)$ which are continuous.

25. Complex variable. Hitherto we have considered only real moment sequences. It is natural to inquire whether our inversion operator is still valid for complex sequences. Let

$$(25.1) \quad \mu_n = \int_0^1 t^n \phi(t) dt$$

where now $\phi(t)$ is a complex function. We shall continue to take the path of integration as the real axis. By breaking $\phi(t)$ into real and imaginary parts we could easily show that

$$L_t \{ \mu_n \} = \phi(t)$$

for all real t between 0 and 1. However, we wish also to obtain an inversion formula which will be valid for complex t . The operator L becomes meaningless for complex t since its definition involves the greatest integer contained in $kt/(1-t)$. We may define an operator L_t^* which is applicable to any sequence defined by (25.1) as follows.

DEFINITION. An operator $L_t^* \{\mu_n\}$ is defined by the equations

$$L_{k,t}^* \{\mu_n\} = \frac{\Gamma(\omega + k + 2)}{\Gamma(\omega + 1)\Gamma(k + 1)} (-1)^k \Delta^k \mu_\omega, \quad \omega = \frac{kt}{1-t},$$

$$L_t^* \{\mu_n\} = \lim_{k \rightarrow \infty} L_{k,t}^* \{\mu_n\}.$$

In this definition the factor $\Delta^k \mu_\omega$ is taken to mean

$$\Delta^k \mu_\omega = (-1)^k \int_0^1 u^\omega (1-u)^k \phi(u) du$$

when t is not an integer. We shall be able to show that this operator inverts the sequence (25.1) not only for all real t between 0 and 1 but for all t in a circle of unit diameter with center at $t = \frac{1}{2}$. In fact we prove

THEOREM 46. If the function $\phi(u)$ is analytic in the circle

$$|u - \frac{1}{2}| < \frac{1}{2}$$

and if

$$\mu_n = \int_0^1 u^n \phi(u) du,$$

then for any t in that circle

$$L_t^* \{\mu_n\} = \phi(t).$$

If t_0 is in the circle C described in the theorem, then

$$R(t_0/(1-t_0)) > 0,$$

where the symbol R denotes "the real part of." Hence the integral

$$\int_0^1 u^{k\omega_0} (1-u)^k \phi(u) du, \quad \omega_0 = t_0/(1-t_0)$$

converges for all positive k , and we have

$$(25.2) \quad L_{k,t_0}^* \{\mu_n\} = \int_0^1 u^{k\omega_0} (1-u)^k \phi(u) du / \int_0^1 u^{k\omega_0} (1-u)^k du.$$

By the function $u^{k\omega_0}$ we mean $\exp(k\omega_0 \log u)$, the real determination of the logarithm being taken. To evaluate the limit of $L_{k,t_0}^* \{\mu_n\}$ as k becomes infinite we employ the method of O. Perron.† To make use of this method we

† O. Perron, *Über die näherungsweise Berechnung von Funktionen grosser Zahlen*, Sitzungsberichte der Akademie der Wissenschaften zu München, mathematisch-physikalischen Klasse, vol. 7 (1917), pp. 191-219.

must alter the path of integration in both integrals (25.2) so as to make it pass through the point t_0 .

Set

$$g(u) = u^{\omega_0}(1-u) = \exp \{ \omega_0 \log u + \log (1-u) \},$$

where u is in the circle C and where that determination of the logarithm is taken which is real when u is real. It is easily verified that $g'(t_0) = 0$. If we set

$$h(v) = \left(\frac{v+t_0}{t_0} \right)^{\omega_0} \left(\frac{1-v-t_0}{1-t_0} \right),$$

the power series development of $\log h(v)$ begins as follows:

$$\log h(v) = - \frac{v^2}{2t_0(1-t_0)^2} + \dots$$

Now define r and β by the equation

$$re^{i\beta} = -1 / \{ 2t_0(1-t_0)^2 \}$$

and apply Perron's result. We obtain

$$\begin{aligned} \int_0^1 u^{k\omega_0}(1-u)^k \phi(u) du &\sim (kr)^{-1/2} i e^{i\beta/2} (\pi)^{1/2} \phi(t_0), \\ \int_0^1 u^{k\omega_0}(1-u)^k du &\sim (kr)^{-1/2} i e^{i\beta/2} (\pi)^{1/2} \quad (k \rightarrow \infty). \end{aligned}$$

It follows that

$$\lim_{k \rightarrow \infty} L_k^* \{ \mu_n \} = \phi(t_0).$$

One of the hypotheses assumed by Perron, when stated for the case in hand, was that it should be possible to pass a curve from $u=0$ to $u=1$ in the circle C such that for all points of the curve except $u=t_0$

$$|g(u)| < |g(t_0)|.$$

To establish the existence of such a curve consider the level lines

$$|g(u)| = |g(t_0)|.$$

We can show that they consist of two curves intersecting only at t_0 which divide the circle into four parts. In two of these parts

$$|g(u)| < |g(t_0)|$$

and in two

$$|g(u)| > |g(t_0)|.$$

Without setting down the details of the proof we point out that it will

be quite sufficient to show that, on the boundary of C , $|g(u)|$ has only two relative maxima and only two minima ($u=0$ and $u=1$).

If $u = \rho e^{i\theta}$, simple computation shows that on the boundary of C , where $\rho = \cos \theta$,

$$|g(u)| = (\cos \theta)^a e^{-b\theta} (1 - \cos \theta) = \gamma(\theta).$$

Here a and b are the real and imaginary parts of ω_0 respectively. It will be sufficient to show that the logarithmic derivative of $\gamma(\theta)$ vanishes just twice for $-\pi/2 < \theta < \pi/2$. But

$$\frac{\gamma'(\theta)}{\gamma(\theta)} = -a \tan \theta - b + \cot(\theta/2).$$

Since $a > 0$ the curve

$$y = a \tan x + b$$

consists of two branches, one descending from $+\infty$ to $-\infty$ as x varies from $-\pi/2$ to 0, the other descending from $+\infty$ to $-\infty$ as x varies from 0 to $\pi/2$. On the other hand the curve

$$y = \cot(\theta/2)$$

consists of a single branch ascending from $-\infty$ to $+\infty$ as x varies from $-\pi/2$ to $\pi/2$. These two curves can intersect in only two points, the points where $\gamma(\theta)$ is maximum. One of these points clearly lies between $-\pi/2$ and 0, the other between 0 and $\pi/2$. The level lines through $t=t_0$ must then cut the circle at only four points, thus dividing the circle into four parts as described above. This completes the proof of the theorem.

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ON THE DISTRIBUTION OF VALUES OF BOUNDED ANALYTIC FUNCTIONS†

BY

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1. Let $f(z)$ be a regular analytic function in the unit circle $|z| < 1$. Consider any point P of the periphery. Adopting a terminology due to W. Gross§ we may associate with P three sets of points:

(i) *The cluster set $C(P)$ of $f(z)$ in P .* This is defined as the set of all those values α which $f(z)$ approaches on a sequence of points of the unit circle $|z| < 1$ converging toward P .

(ii) *The range of values $R(P)$ of $f(z)$ in P .* A value α belongs to the set $R(P)$ if, and only if, $f(z)$ assumes the value α in every neighborhood of the point P .

(iii) *The convergence set $\Gamma(P)$ of $f(z)$ in P .* The set $\Gamma(P)$ consists of all those values α which $f(z)$ approaches on a Jordan arc lying, except for one end point, in the interior of the unit circle $|z| < 1$ and terminating in the point P .

In case that the function $f(z)$ under consideration is bounded: $|f(z)| < M$ in $|z| < 1$, where M is some positive constant, from well known theorems one obtains at once additional information concerning the set $\Gamma(P)$. According to a theorem of Fatou|| the limit

$$\lim_{r \rightarrow 1} f(re^{i\theta}) = f^*(e^{i\theta}) \quad (z = re^{i\theta})$$

exists for all values of θ in the interval $0 \leq \theta \leq 2\pi$ save perhaps for a set of measure zero of values of θ . The function $f^*(e^{i\theta})$ will henceforth be denoted by us as the *boundary function* of $f(z)$. In our terminology, Fatou's theorem may be stated in the following form: *The convergence set $\Gamma(P)$ of a bounded analytic function $f(z)$ in the unit circle $|z| < 1$ contains at least one point for almost all points P of the circumference $|z| = 1$.*

† Presented to the Society, October 29, 1932; received by the editors September 27, 1932, and, in revised form, February 18, 1933. The author is indebted to Professor J. D. Tamarkin for the suggestion of basing the proofs of Theorems 2-5 on Theorem 1, thus simplifying the original proofs.

‡ A part of these investigations was carried out while the author was a National Research Fellow.

§ W. Gross, *Über die Singularitäten analytischer Funktionen*, Monatshefte für Mathematik und Physik, vol. 29 (1918), pp. 3-47.

|| P. Fatou, *Séries trigonométriques et séries de Taylor*, Acta Mathematica, vol. 30 (1906), pp. 366-368.

By the preceding theorem and a theorem of Lindelöf†, moreover, it follows immediately that the convergence set $\Gamma(P)$ of a bounded analytic function $f(z)$ in the unit circle $|z| < 1$ contains one and only one point for almost all points P of the circumference $|z| = 1$, and no convergence set $\Gamma(P)$ can contain more than one point.

As regards the sets $C(P)$ and $R(P)$, it will suffice for the present to remark that the set $R(P)$ is always contained in the set $C(P)$.

The present paper will be concerned primarily with those bounded analytic functions $f(z)$ in the unit circle $|z| < 1$, for which the modulus of the boundary function $|f^*(e^{i\theta})| = 1$ for almost all values of θ in the interval $0 \leq \theta \leq 2\pi$. Throughout this paper such functions will be called of class (A). R. Nevanlinna‡ was the first to point out the interest which lies in this class of functions. There exists a wide range of well known functions which belong to the class which we propose to study. Of these we shall mention the following three groups:

(i) If a_1, a_2, \dots, a_{m-k} are points interior to the unit circle, all functions of the form

$$(1.1) \quad R(z) = e^{i\alpha z^k} \prod_{i=1}^{m-k} \frac{z - a_i}{1 - \bar{a}_i z},$$

where k and m are positive integers and α is any real number. These functions yield the most general $(1, m)$ conformal correspondence of the unit circle with itself.§

(ii) All infinite products of the form

$$(1.2) \quad B(z) = \prod_{i=1}^{\infty} \frac{1 - z/a_i}{1 - \bar{a}_i z} |a_i|,$$

where the points a_i are interior to the unit circle and satisfy the condition

$$(1.3) \quad \prod_{i=1}^{\infty} |a_i| > 0.$$

It has been shown by W. Blaschke|| that, when condition (1.3) is satisfied,

† E. Lindelöf, *Sur un principe général de l'analyse*, Acta Societatis Scientiarum Fennicae, vol. 46 (1915), No. 4, pp. 1-35.

‡ R. Nevanlinna, *Über beschränkte analytische Funktionen*, Annales Academiæ Scientiarum Fennicae, (A), vol. 32 (1929), No. 7, p. 64.

§ For a detailed account of these products, cf. T. Radó, *Zur Theorie der mehrdeutigen konformen Abbildungen*, Acta Litterarum ac Scientiarum Regiæ Universitatis Hungaricæ Francisco-Josephinae, vol. 1 (1922), pp. 55-64; as well as G. Julia, *Principes Géométriques d'Analyse*, part 1, Paris, 1930, pp. 54-59.

|| W. Blaschke, *Eine Erweiterung des Satzes von Vitali über Folgen analytischer Funktionen*, Leipziger Berichte, vol. 67 (1915), pp. 194-200; also G. Julia, loc. cit. in preceding footnote.

the product $B(z)$ converges uniformly in every closed subregion lying wholly interior to the unit circle $|z| < 1$, thus defining there an analytic function $B(z)$, which is commonly called a *Blaschke product*. It was shown by F. Riesz† that the boundary function $B^*(e^{i\theta})$ of a Blaschke product $B(z)$ has the modulus 1 in almost all points of the circumference $|z| = 1$. This proves that Blaschke products belong to the class of functions under consideration. In a previous paper‡ the author showed that if the numbers a_i in (1.2) converge to a single point P of the circumference $|z| = 1$, then the range of values $R(P)$ of the Blaschke product $B(z)$ consists of all the points of the unit circle $|\alpha| < 1$, with the possible exception of at most one point. In this paper a further study is made of the ranges of values of Blaschke products under more general distributions of the zeros $\{a_i\}$.

(iii) Consider the unit circle $|w| < 1$ and a set S of points closed relatively to the circle $|w| < 1$. On removing all points of S from the unit circle $|w| < 1$, at least one in general infinitely connected region Σ is obtained. According to general existence theorems of conformal mapping§ it is known that there exists an infinitely multiple-valued function $z = \phi(w)$ analytic in Σ and assuming in it every value from the interior of the unit circle $|z| < 1$ once and only once. This function is said to map Σ conformally on the circle $|z| < 1$. As will be proved in this paper, the inverse function $w = f(z)$ of the mapping function $z = \phi(w)$ belongs under appropriate restrictions on the set S to the class of functions under consideration and may be represented as a linear function of a Blaschke product.

Finally, in connection with this work mention should be made of recent papers by G. Hössjer and J. L. Doob.||

2. We begin by establishing an integral representation for all functions of class (A) in the unit circle $|z| < 1$. Let $f(z)$ be a function of class (A) in $|z| < 1$ and denote its zeros (if they exist) by $a_1, a_2, \dots, a_i, \dots$. Since $f(z)$ is bounded, by a theorem of Blaschke¶ its zeros satisfy the inequality $\prod_{i=1}^{\infty} |a_i| > 0$. In accordance with the result stated in §1, page 202, we may form the Blaschke product

† F. Riesz, *Über die Randwerte einer analytischen Funktion*, Mathematische Zeitschrift, vol. 18 (1923), p. 94.

For further results concerning Blaschke products cf. J. L. Walsh, *Interpolation and functions analytic interior to the unit circle*, these Transactions, vol. 34 (1932), pp. 523-556.

‡ W. Seidel, *On the cluster values of analytic functions*, these Transactions, vol. 34 (1932), p. 17.

§ L. Bieberbach, *Lehrbuch der Funktionentheorie*, vol. II, 1931, chapter I, pp. 1-84.

|| G. Hössjer, *Über die Randwerte beschränkter Funktionen*, Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Franciscus-Josephinae, vol. 5 (1930), p. 55.

J. L. Doob, *The boundary values of analytic functions*, these Transactions, vol. 34 (1932), pp. 153-170.

¶ Leipziger Berichte, loc. cit., pp. 194-200.

$$B(z) = \prod_{i=1}^{\infty} \frac{1 - z/a_i}{1 - \bar{a}_i z} |a_i|,$$

extended over the zeros a_i . Hence, by the Riesz decomposition theorem[†] if we define the function $g(z)$ by the relation $f(z) = B(z) g(z)$, we find that $g(z)$ is of class (A) and different from zero in the circle $|z| < 1$. Consider now the function $h(z) = \log g(z)$ which, if we select a definite branch of the logarithm, becomes single-valued and analytic in the circle $|z| < 1$. Furthermore, $\Re h(z) \leq 0$ in $|z| < 1$. Consequently, applying a result of Herglotz[‡] to the function $h(z)$, we obtain for it the representation

$$(2.1) \quad h(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta) + i\beta,$$

where $\sigma(\theta)$ is a monotonic non-increasing function of θ in the interval $-\pi \leq \theta \leq \pi$ and β is some real number. There also exists the following relation[§] between the function $h(z)$ and the derivative $\sigma'(\theta)$ of $\sigma(\theta)$:

$$(2.2) \quad \lim_{z \rightarrow e^{i\theta}} \Re h(z) = \sigma'(\theta),$$

for all values of θ in the interval $-\pi \leq \theta \leq \pi$ for which $\sigma(\theta)$ possesses a derivative, the approach $z \rightarrow e^{i\theta}$ being made along any path which is non-tangential to the circle $|z| < 1$. Since $g(z)$ is of class (A) in $|z| < 1$, the left-hand side of equation (2.2), and therefore $\sigma'(\theta)$, is equal to zero almost everywhere. This proves the following theorem:

THEOREM 1. *Let $f(z)$ be a function of class (A) in the unit circle $|z| < 1$. Then*

$$(2.3) \quad f(z) = e^{i\beta} B(z) \exp \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta) \right],$$

where $B(z)$ is the Blaschke product extended over the zeros of $f(z)$, $\sigma(\theta)$ is a monotonic non-increasing function of θ in the interval $-\pi \leq \theta \leq \pi$ whose derivative $\sigma'(\theta) = 0$ almost everywhere in $-\pi \leq \theta \leq \pi$, and β is a real constant.||

It is obvious, conversely, that every function $f(z)$ of the form (2.3) is of class (A) in $|z| < 1$.

[†] Mathematische Zeitschrift, loc. cit.

[‡] G. Herglotz, *Über Potenzreihen mit positivem, reellen Teil im Einheitskreise*, Leipziger Berichte, vol. 63 (1911), pp. 501-511; p. 508.

[§] Cf. G. C. Evans, *The Logarithmic Potential*, 1927, pp. 40-43.

|| This is a special case of a result obtained by V. Smirnov, *Sur les valeurs limites des fonctions régulières à l'intérieur d'un cercle*, Journal de la Société Physico-Mathématique de Leningrad, vol. 2 (1929), pp. 22-37.

3. As an immediate consequence of Theorem 1 we prove the following theorem:

THEOREM 2. *Let $f(z)$ be a function of class (A), not a constant, in the unit circle $|z| < 1$. If $f(z) \neq \alpha$ ($|\alpha| < 1$), in the whole circle $|z| < 1$, then there exists at least one radius $\theta = \theta_0$ such that*

$$\lim_{r \rightarrow 1} f(re^{i\theta_0}) = \alpha.$$

We may assume without loss of generality that $\alpha = 0$. For if that is not the case, we prove the theorem for the function

$$\frac{f(z) - \alpha}{1 - \bar{\alpha}f(z)}$$

which is likewise of class (A) and is different from zero in the whole unit circle. Since $f(z)$ has no zeros in $|z| < 1$, formula (2.3) reduces to

$$(3.1) \quad f(z) = e^{i\theta} \exp \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta) \right],$$

where $\sigma(\theta)$ satisfies the conditions of Theorem 1. The function $\sigma(\theta)$ is not identically a constant, for otherwise $f(z)$ would be constant. Hence, if $\sigma(\theta)$ is continuous, there exists a point $\theta = \theta_0$ in the interval $-\pi \leq \theta \leq \pi$ at which $\sigma(\theta)$ possesses a derivative equal to $-\infty$.† An easy modification of Evans' proof‡ shows that under these conditions the Poisson-Stieltjes integral

$$u(r, \phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \phi)} d\sigma(\theta)$$

approaches the limit $-\infty$ along the radius $\phi = \theta_0$. This proves that $|f(z)|$, which is given by the formula

$$|f(z)| = e^{u(r, \phi)}, \quad z = re^{i\phi},$$

approaches the limit zero along the radius $\phi = \theta_0$.

Suppose now that $\sigma(\theta)$ is not continuous in the interval $-\pi \leq \theta \leq \pi$. Then, since $\sigma(\theta)$ is non-increasing, it admits the following representation:

$$\sigma(\theta) = S(\theta) + \Phi(\theta) + \omega(\theta).$$

Here all functions $S(\theta)$, $\Phi(\theta)$, $\omega(\theta)$ are non-increasing; $S(\theta)$ is continuous and $S'(\theta) = 0$ almost everywhere in the interval $-\pi \leq \theta \leq \pi$, $\Phi(\theta)$ is absolutely continuous, and $\omega(\theta)$ is a step function. Since $\sigma'(\theta) = 0$ almost everywhere in

† Schlesinger and Plessner, *Lebesguesche Integrale und Fouriersche Reihen*, Berlin and Leipzig, 1926, §43.

‡ *The Logarithmic Potential*, loc. cit.

$-\pi \leq \theta \leq \pi$, $\Phi(\theta) \equiv 0$. If $S(\theta)$ is not constant, by the theorem mentioned in the first footnote on page 205, $S'(\theta) = -\infty$ at a non-denumerable set of points. Among them there will be at least one point at which $\omega(\theta)$ is continuous. Hence, we may apply to this point the modification of Evans' proof already indicated. There only remains now the case when $S(\theta)$ is constant. In that case $\omega(\theta)$ is certainly not constant. It will have at least one point of discontinuity which, without loss of generality, we may assume to be $\theta=0$. The Poisson-Stieltjes integral on the radius $\theta=0$,

$$u(r, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r \cos \theta} d\omega(\theta),$$

will be of the form

$$(3.2) \quad J_0 \frac{1+r}{1-r} + \sum_{k=1}^{\infty} J_k \frac{1-r^2}{1+r^2-2r \cos \theta_k},$$

where $\theta = \theta_k$ are the points of discontinuity of $\omega(\theta)$ at which $\omega(\theta)$ has the negative jumps J_k , while J_0 is the jump of $\omega(\theta)$ at the point $\theta=0$. We now choose any positive number ϵ less than $|J_0|$. To this number there corresponds a positive integer n such that

$$\sum_{k=n+1}^{\infty} |J_k| < \epsilon.$$

The sum in (3.2) will now be decomposed in the following manner:

$$(3.3) \quad u(r, 0) = J_0 \frac{1+r}{1-r} + \sum_{k=1}^n J_k \frac{1-r^2}{1+r^2-2r \cos \theta_k} + \sum_{k=n+1}^{\infty} J_k \frac{1-r^2}{1+r^2-2r \cos \theta_k}.$$

We may disregard the sum

$$\sum_{k=1}^n J_k \frac{1-r^2}{1+r^2-2r \cos \theta_k}$$

since it tends to 0 as r tends to 1. The two remaining terms in (3.3) are algebraically less than

$$-(|J_0| - \epsilon) \frac{1+r}{1-r}$$

which tends to $-\infty$ as r tends to 1.

This completes the proof of Theorem 2.

It is evident that Theorem 2 still holds if the function $f(z)$ is allowed to assume the value α only a finite number of times in the circle $|z| < 1$.

4. By essentially the same method one may prove the following extension of Theorem 2:

THEOREM 3. *Let $f(z)$ be a bounded analytic function, not a constant, in the circle $|z| < 1$:*

$$|f(z)| < 1.$$

Denoting by $f^(e^{i\theta})$ the boundary function of $f(z)$, let*

$$|f^*(e^{i\theta})| = 1$$

for almost all values of θ in the interval $A: (0 \leq \alpha_1 < \theta < \alpha_2 < 2\pi)$. If the value α ($|\alpha| < 1$) is omitted by $f(z)$, then the set of singularities of $f(z)$ on the arc A is the closed cover of the set of points defined by the solutions of the equation $f^(e^{i\theta}) = \alpha$.*

We may assume, as in the proof of the preceding theorem, that $\alpha = 0$. The representation (3.1) is still valid for the function $f(z)$, where $\sigma(\theta)$ is a monotonic non-increasing function in the interval $-\pi \leq \theta \leq \pi$, the relation $\sigma'(\theta) = 0$, however, holding almost everywhere in the open interval $\alpha_1 < \theta < \alpha_2$. If the function $f(z)$ is analytic on the whole arc A , then $|f^*(e^{i\theta})| = 1$ for all points of A . Therefore, the equation $f^*(e^{i\theta}) = 0$ has no solutions in the interval $\alpha_1 < \theta < \alpha_2$. It is immediately evident that the set of singularities of $f(z)$ on the arc A contains all points of the closed cover of the set defined by the equation $f^*(e^{i\theta}) = 0$. In fact, every point $z = e^{i\theta}$ of the arc A for which $f^*(e^{i\theta}) = 0$ is a singular point of $f(z)$. We therefore have to show now that if P is a singular point of $f(z)$ lying on the arc A , then either $f^*(P) = 0$ or the points defined by the solutions of the equation $f^*(e^{i\theta}) = 0$ have P as a limit point. Denote by Δ an arbitrarily small arc of $|z| = 1$ which contains the point P in its interior. The function $\sigma(\theta)$ cannot remain constant on the arc Δ . Indeed, if $\sigma(\theta)$ were constant on Δ , it would follow from equation (3.1) that

$$f(z) = e^{i\theta} \exp \left[\frac{1}{2\pi} \int_{C\Delta} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta) \right],$$

where $C\Delta$ denotes the arc of the circle $|z| = 1$ complementary to the arc Δ . This shows that $f(z)$ is analytic on Δ . Hence, if $\sigma(\theta)$ is continuous on A , there must exist a point of the arc Δ at which $\sigma'(\theta) = -\infty$. Since Δ was taken arbitrarily small, this means that either $\sigma'(P) = -\infty$ or the points at which $\sigma'(\theta) = -\infty$ have P as a limit point. But now if $\sigma'(\theta) = -\infty$, for some θ , then $f^*(e^{i\theta}) = 0$ for the same value of θ . If $\sigma(\theta)$ is not continuous on A , we can reason in a manner similar to that used in the proof of Theorem 2. This proves the theorem.

5. Dr. J. L. Doob kindly pointed out to the author that, as a consequence of Theorem 3, we may prove the following theorem which is a sharper form of Nevanlinna's theorem to be mentioned in §8, page 211:

THEOREM 4. *Let $w=f(z)$ be a bounded analytic function, not a constant, in the circle $|z|<1$:*

$$|f(z)| < 1.$$

Let $0 \leq \alpha_1 < \theta < \alpha_2 < 2\pi$, $r=1$, be an arc A of the circumference $|z|=1$ such that

$$\lim_{r \rightarrow 1} f(re^{i\theta}) = f^*(e^{i\theta}), \quad |f^*(e^{i\theta})| = 1,$$

for almost all values of θ in the interval $\alpha_1 < \theta < \alpha_2$. Then if all these limit values $f^(e^{i\theta})$ of modulus one are represented by a set of points E on the circumference $|w|=1$, either the set E is the whole circumference $|w|=1$, or $f(z)$ may be continued analytically beyond the arc A .*

Let $\alpha = e^{i\lambda}$, where λ is a real constant, be an arbitrary point of the circumference $|w|=1$ and let the arc A contain in its interior a singular point P of $f(z)$. We wish to prove that there exists a point $e^{i\theta_0}$ on the arc A for which $f^*(e^{i\theta_0}) = \alpha$.

Consider the function

$$(5.1) \quad \phi(z) = e^{(f(z)+\alpha)/(f(z)-\alpha)}.$$

This function is analytic and bounded in the circle $|z|<1$, and

$$\lim_{r \rightarrow 1} \phi(re^{i\theta}) = \phi^*(e^{i\theta}), \quad |\phi^*(e^{i\theta})| = 1,$$

for almost all values of θ in the interval $\alpha_1 < \theta < \alpha_2$. Furthermore, the point P is a singular point of the function $\phi(z)$. For, since P is a singular point of $f(z)$ according to Theorem 8, to be proved in §10, there exists at least one value c ($|c|<1$) such that the equation $f(z)=c$ has infinitely many solutions z_1, z_2, \dots interior to the unit circle $|z|<1$ and converging to the point P . Hence, it follows from equation (5.1) that

$$\phi(z_n) = e^{(c+\alpha)/(c-\alpha)} \quad (n = 1, 2, \dots).$$

If P were a regular point of $\phi(z)$, the function $\phi(z)$, and hence also $f(z)$, would be constant. Now, according to (5.1) $\phi(z) \neq 0$ in the circle $|z|=1$. Hence, by Theorem 3 there exists at least one radius $\theta = \theta_0$ terminating in a point $z = e^{i\theta_0}$ of the arc A such that

$$\lim_{r \rightarrow 1} \phi(re^{i\theta_0}) = \phi^*(e^{i\theta_0}) = 0.$$

Hence, $f^*(e^{i\theta_0}) = \alpha$, as was to be proved.

6. Theorem 2 may be made to yield the following theorem:

THEOREM 5. *Let $f(z)$ be a function of class (A) in the unit circle $|z| < 1$. Then, either $f(z)$ is a rational function of the form*

$$(6.1) \quad f(z) = e^{i\beta z^k} \prod_{i=1}^{m-k} \frac{z - a_i}{1 - \bar{a}_i z},$$

where k and m are positive integers, β a real number, and a_i complex numbers of modulus less than one, giving the most general $(1, m)$ conformal representation of the unit circle into itself; or each value α ($|\alpha| < 1$) belongs to the cluster set $C(P)$ of $f(z)$ in at least one point P of the circumference $|z| = 1$.

Suppose some number a ($|a| < 1$) belongs to no cluster set $C(P)$ of $f(z)$. Forming the function

$$(6.2) \quad \phi(z) = \frac{f(z) - a}{1 - \bar{a}f(z)},$$

we see that $\phi(z)$ is analytic and bounded in $|z| < 1$:

$$|\phi(z)| < 1,$$

and since $f(z)$ is of class (A), the boundary function $\phi^*(e^{i\theta})$ satisfies the relation

$$(6.3) \quad |\phi^*(e^{i\theta})| = 1$$

for almost all values of θ in the interval $0 \leq \theta \leq 2\pi$. Furthermore, the value 0 belongs to no cluster set $C(P)$ of $\phi(z)$. Hence, there exist at most a finite number of points z_1, z_2, \dots, z_m of $|z| < 1$ at which $\phi(z)$ vanishes. Letting, therefore,

$$(6.4) \quad \phi(z) = \prod_{k=1}^m \frac{z - z_k}{1 - \bar{z}_k z} \psi(z),$$

we have by Schwarz's Lemma† that $\psi(z)$ is analytic and bounded in $|z| < 1$ satisfying there the inequality

$$|\psi(z)| < 1.$$

According to (6.3) its boundary function $\psi^*(e^{i\theta})$ is 1 in modulus:

$$(6.5) \quad |\psi^*(e^{i\theta})| = 1$$

for almost all values of θ in the interval $0 \leq \theta \leq 2\pi$. Finally, $\psi(z) \neq 0$ in $|z| < 1$ and 0 belongs to no cluster set $C(P)$ of $\psi(z)$. Since, as shown by (6.5), $\psi(z)$

† G. Julia, loc. cit., p. 67.

is of class (A), it follows from Theorem 2, that $\psi(z)$ is identically a constant of modulus one. It is now easily seen with the aid of the relation (6.4) that $f(z)$ admits the representation (6.1).

7. We now consider a certain extension of Schwarz's reflection principle which will be found useful in the sequel. It may be formulated in the following manner:

THEOREM 6. *Let $f(z)$ be a bounded analytic function in the unit circle $|z| < 1$:*

$$|f(z)| < 1.$$

Let $0 \leq \alpha_1 < \theta < \alpha_2 < 2\pi$, $r=1$, be an arc A of the periphery of the unit circle such that

$$\lim_{r \rightarrow 1} |f(re^{i\theta})| = 1$$

for almost all values of θ in the interval $\alpha_1 < \theta < \alpha_2$. Then, either $f(z)$ may be continued analytically beyond the arc A or every value α ($|\alpha| < 1$) belongs to at least one of the cluster sets $C(P)$ of $f(z)$ for some point P of A .

The theorem need merely be proved for the value $\alpha=0$. Indeed, let us assume that the theorem is true for $\alpha=0$ and let $\beta \neq 0$ be some other value such that $|\beta| < 1$. We wish to show that unless $f(z)$ may be continued analytically beyond the arc A , the value β is contained in at least one of the sets $C(P)$. For suppose that were not the case. Consider the function

$$\phi(z) = \frac{f(z) - \beta}{1 - \bar{\beta}f(z)}.$$

Then, clearly $|\phi(z)| < 1$ and for almost all values of θ in the interval $\alpha_1 < \theta < \alpha_2$ we have

$$\lim_{r \rightarrow 1} |\phi(re^{i\theta})| = 1.$$

Furthermore, we know that $\phi(z)$ may not be continued analytically beyond the arc A and the value 0 is contained in none of the sets $C(P)$ formed for the function $\phi(z)$. This, however, contradicts our hypothesis, according to which the theorem was true for the value $\alpha=0$.

In order to prove the theorem for $\alpha=0$, we observe that if the value 0 is contained in none of the sets $C(P)$ formed for the function $f(z)$, there exists a region R lying in the interior of the unit circle and with the arc A as part of its boundary, in which $1 > |f(z)| > \rho$, where ρ is some positive number. The remainder of the proof is analogous to the standard proof of Schwarz's reflection principle. The obvious modifications can be easily supplied by the reader.

8. As corollary to Theorem 6 we obtain the following result:

COROLLARY. Let $f(z)$ be a bounded analytic function in the unit circle $|z| < 1$: $|f(z)| < 1$.

Let $0 \leq \alpha_1 < \theta < \alpha_2 < 2\pi$, $r=1$, be an arc A of the periphery of the unit circle such that

$$\lim_{r \rightarrow 1} |f(re^{i\theta})| = 1$$

for almost all values of θ in the interval $\alpha_1 < \theta < \alpha_2$. Then, if P is an arbitrary, interior point of A , either $f(z)$ is analytic in P or the cluster set $C(P)$ formed for $f(z)$ is the closed unit circle $|\alpha| \leq 1$.

Suppose P is not a point of analyticity of the function $f(z)$. The corollary follows at once if one applies Theorem 6 to a sequence of arcs Δ_n of the periphery containing the point P whose lengths tend to zero with $1/n$.

This corollary sharpens Theorem 5 in that in the second case of that theorem every point P of the periphery $|z|=1$ is either a point of analyticity of $f(z)$ or $C(P)$ is the closed unit circle $|\alpha| \leq 1$.

Finally it may be mentioned in passing that by means of Theorem 6 one may easily prove the following theorem of R. Nevanlinna†:

THEOREM OF R. NEVANLINNA. Let $f(z)=w$ be a bounded analytic function in the unit circle $|z| < 1$: $|f(z)| < 1$. Let $\alpha_1 < \theta < \alpha_2$, $r=1$, be an arc A of the periphery of the unit circle such that

$$\lim_{r \rightarrow 1} f(re^{i\theta}) = f^*(e^{i\theta}), \quad |f^*(e^{i\theta})| = 1,$$

for almost all values of θ in the interval $\alpha_1 < \theta < \alpha_2$. Then, if these limit values $f^*(e^{i\theta})$ are represented by a set of points E on the periphery of the unit circle $|w|=1$, either the set E is measurable and of measure 2π , or $f(z)$ may be continued analytically beyond the arc A .

9. Again let $w=f(z)$ be a function of class (A) in the circle $|z| < 1$. Denote by $z=\phi(w)$ the inverse function of $w=f(z)$. This function is in general infinitely many-valued. In the theorem that follows, we establish a connection between the non-algebraic singularities of the function $\phi(w)$ and the convergence values of the function $f(z)$ in precisely the same manner that Hurwitz and Iversen‡ established such a connection for functions meromorphic in the finite plane. Before stating the theorem, we shall give Bieberbach's definition of a singular point:

† R. Nevanlinna, *Annales Academiae Scientiarum Fennicae*, loc. cit., p. 28.

‡ A. Hurwitz, *Sur les points critiques des fonctions inverses*, Paris Comptes Rendus, vol. 143 (1906), pp. 877-879, and vol. 144 (1907), pp. 63-65; F. Iversen, *Recherches sur les Fonctions Inverses des Fonctions Méromorphes*, Thèse, Helsingfors, 1914, p. 13.

If the members of a chain of regular elements $\mathfrak{P}(w-a)$ of the function $z=\phi(w)$ are obtainable from one another by direct continuation in such a manner that their centers have a single limit point and their radii of convergence tend to zero, then this chain is said to define a singular point. If $w=\alpha$ is the coordinate of the limit point, the singular point is said to lie over the point $w=\alpha$ of the w -plane. If the singular point is not algebraic in character, it is said to be non-algebraic.[†]

The theorem which we shall find useful in the sequel is as follows:

THEOREM 7. Let $w=f(z)$ be a function of class (A) in the unit circle $|z| < 1$. If to some number α ($|\alpha| < 1$), there exists a radius $\theta = \theta_0$ for which

$$\lim_{r \rightarrow 1} f(re^{i\theta_0}) = \alpha,$$

then the inverse function $z=\phi(w)$ of $w=f(z)$ has a non-algebraic singularity over the point $w=\alpha$. And, conversely, if $\phi(w)$ has a non-algebraic singularity over the point $w=\alpha$, then there exists at least one radius $\theta = \theta_0$, for which

$$\lim_{r \rightarrow 1} f(re^{i\theta_0}) = \alpha.$$

Since the proof of this theorem is almost identical with that of Iversen's theorem, we shall omit it here.

10. We now turn to the study of the ranges of values $R(P)$ of our functions. As an immediate consequence to the corollary of Theorem 6, we state the following theorem:

THEOREM 8. Let $w=f(z)$ be a bounded analytic function in the unit circle $|z| < 1$:

$$|f(z)| < 1.$$

Let $0 \leq \alpha_1 < \theta < \alpha_2 < 2\pi$, $r=1$, be an arc A of the periphery of the unit circle such that

$$\lim_{r \rightarrow 1} |f(re^{i\theta})| = 1$$

for almost all values θ in the interval $\alpha_1 < \theta < \alpha_2$. Then, if P is a singular point of $f(z)$ lying in the interior of the arc A , the range of values $R(P)$ is a set of points everywhere dense in the unit circle $|w| < 1$.

Consider a sequence of circles $\{C_n\}$ about the point P as center with radii tending to zero. Denote by V_n the set of values which $f(z)$ assumes in that part of the circle C_n , exclusive of its periphery, which lies in the circle

[†] For further details concerning these matters, cf. L. Bieberbach, *Lehrbuch der Funktionentheorie*, vol. 1 (1923), pp. 207-217.

$|w| < 1$. Each of the sets V_n is open and by the corollary to Theorem 6 is everywhere dense in the circle $|w| < 1$. Furthermore, we have

$$R(P) = \prod_{n=1}^{\infty} V_n.$$

Hence, by a well known theorem†, the set $R(P)$ is dense in the unit circle $|w| < 1$.

11. Under more restrictive hypotheses it is possible to give a sharper statement of Theorem 8.

THEOREM 9. *If $w=f(z)$ is a bounded analytic function in the unit circle $|z| < 1$ whose boundary function $f^*(e^{i\theta})$ satisfies the condition $|f^*(e^{i\theta})| = 1$ for all values of θ in the interval $0 \leq \theta \leq 2\pi$ except perhaps in a denumerable set, then either $f(z)$ is of the form (1.1) or it assumes infinitely often all values of the unit circle $|w| < 1$ except perhaps for a denumerable set of values.*

According to Theorem 2, if $f(z)$ omits (or assumes a finite number of times) a value w ($|w| < 1$), then there exists at least one radius $\theta = \theta_0$, such that

$$\lim_{r \rightarrow 1} f(re^{i\theta_0}) = w.$$

Hence, unless the set of values w which is omitted (or assumed only a finite number of times) is denumerable, the function $|f^*(e^{i\theta})| < 1$ for a non-denumerable set of values in the interval $0 \leq \theta \leq 2\pi$.

The following still sharper theorem was obtained by the author‡ in another paper:

Let $w=f(z)$ be a bounded analytic function in the circle $|z| < 1$: $|f(z)| < 1$. Let $\{n_k\}$ be an infinite sequence of points interior to the unit circle converging toward $z=1$ in which $f(z)$ vanishes and let A be an arc of the unit circle, $-\alpha < \theta < \alpha$, $z=e^{i\theta}$, containing $P: (z=1)$, on which $f(z)$ is continuous except for P and assumes values of modulus one. Then $R(P)$ is the unit circle $|w| < 1$ with the exception of at most one point.

As a corollary to this theorem, we may state the following result:

THEOREM 10. *Let $w=f(z)$ be of class (A) in the unit circle $|z| < 1$. If $f(z)$ omits two or more values of modulus less than one, the set of singularities of $f(z)$ on the circumference $|z| = 1$ is perfect.*

† Cf. C. Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig and Berlin, 1927, p. 63, Theorem 5.

‡ These Transactions, loc. cit., p. 17.

If $f(z)$ is analytic on the boundary $|z|=1$, then $f(z)$ is necessarily a rational function of the form (1.1). Since, however, the function (1.1) assumes all values of modulus less than one, it follows that $f(z)$ possesses a singularity P on the circumference $|z|=1$. This singularity P cannot be isolated. For if that were the case, there would exist by Theorem 8 a number c ($|c|<1$), and a sequence of points $z=n_k$ interior to the circle $|z|<1$ and converging toward P such that $f(n_k)=c$. If we now form the function $(f(z)-c)/(1-\bar{c}f(z))$ and apply the theorem just quoted, we obtain a contradiction. Since the set of singularities of $f(z)$ is closed and cannot have isolated points, it is perfect.

12. We shall now investigate the set Σ of those values which functions $f(z)$ of class (A) assume infinitely often in the unit circle.

A function of class (A) may omit one value. This is evidently true of the function $f(z)=e^{(z+1)/(z-1)}$ which omits the value 0.

Furthermore, even a Blaschke product[†], being of class (A), may omit one value. This is readily shown by establishing directly from the Weierstrass product formula the following identity:

$$(12.1) \quad \frac{\exp\left[\frac{z+1}{z-1}\right] - e^{-1}}{1 - e^{-1} \exp\left[\frac{z+1}{z-1}\right]} = -z \prod'_{n=-\infty}^{\infty} \left[\frac{1 - \frac{z}{\pi ni / (\pi ni + 1)}}{1 + \frac{\pi ni}{-\pi ni + 1} z} \left| \frac{\pi ni}{\pi ni + 1} \right| \right],$$

the prime on the product sign indicating that the factor for $n=0$ is omitted. The function on the left evidently omits the value $-e^{-1}$. The function on the right is a Blaschke product of the form (1.2), where

$$a_n = \frac{\pi ni}{\pi ni + 1}.$$

Thus the Blaschke product in (12.1) omits the value $-e^{-1}$. Since $w=e^{(z+1)/(z-1)}$ is the inverse function of that function which maps the punctured circle $0<|w|<1$ on the unit circle $|z|<1$, we find that the inverse function of this mapping function may be represented as a linear function of the Blaschke product $B(z)$ in (12.1):

$$w = \frac{B(z) + e^{-1}}{1 + e^{-1}B(z)}.$$

13. This last result will now be extended in the following manner:

[†] For Blaschke products cf. §1 of this paper.

THEOREM 11. *Let $f(z)$ be a function of class (A) in the unit circle $|z| < 1$. Let the value α ($|\alpha| < 1$) which is assumed infinitely often by $f(z)$ in the circle $|z| < 1$ be not a convergence value of $f(z)$. That is, for no radius $\theta = \theta_0$ does the equation*

$$(13.1) \quad \lim_{r \rightarrow 1} f(re^{i\theta_0}) = \alpha$$

hold. Then $f(z)$ is equal to a linear function of a Blaschke product:

$$f(z) = \frac{e^{i\delta} B(z) + \alpha}{1 + \bar{\alpha} e^{i\delta} B(z)}.$$

For consider the function

$$(13.2) \quad f_1(z) = \frac{f(z) - \alpha}{1 - \bar{\alpha} f(z)}.$$

By hypothesis $f_1(z)$ has infinitely many zeros in points $a_1, a_2, \dots, a_i, \dots$. By Theorem 1 $f_1(z)$ can be represented in the following manner:

$$(13.3) \quad f_1(z) = e^{i\delta} B(z) \exp \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta) \right],$$

where δ is a real constant, $B(z)$ the Blaschke product extended over the zeros of $f_1(z)$, and $\sigma(\theta)$ a monotonic non-increasing function of θ in the interval $-\pi \leq \theta \leq \pi$ for which $\sigma'(\theta) = 0$ in almost all points of $-\pi \leq \theta \leq \pi$. Now if $\sigma(\theta)$ is not constant in the whole interval $-\pi \leq \theta \leq \pi$, by the reasoning of §3 there exists a value θ_0 of θ such that $f_1^*(e^{i\theta_0}) = 0$ or $f^*(e^{i\theta_0}) = \alpha$. This contradicts the non-existence of the relation (13.1). Hence, $\sigma(\theta)$ is a constant and formula (13.3) reduces to

$$f_1(z) = e^{i\delta} B(z).$$

14. Theorem 11 leads immediately to the following result:

THEOREM 12. *Let $f(z)$ be a function of class (A) in the circle $|z| < 1$. Then either $f(z)$ is a linear function of a Blaschke product, or to every number α ($|\alpha| < 1$) there exists at least one radius $\theta = \theta_0$ on which $f(z)$ tends to the value α :*

$$\lim_{r \rightarrow 1} f(re^{i\theta_0}) = \alpha.$$

By Theorem 7 in the second case, the Riemann surface of the inverse function $z = \phi(w)$ of $w = f(z)$ has non-algebraic singularities over every point of the unit circle $|w| < 1$.

If $f(z)$ is not a linear function of a Blaschke product, then according to Theorem 11, every value α ($|\alpha| < 1$) which is assumed infinitely often by

$f(z)$ is also a convergence value of $f(z)$. But we know from Theorem 2 that every value $\alpha (|\alpha| < 1)$ which is omitted (or assumed only a finite number of times) is a convergence value of $f(z)$. Hence every value $\alpha (|\alpha| < 1)$ is a convergence value of $f(z)$. The author has been unable to determine whether or not functions of the second kind may actually exist.

15. We may now return to the study of the sets Σ of values which functions of class (A) assume infinitely often in the unit circle $|z| < 1$. We have seen in §12 that there are functions of class (A) which omit one value of modulus less than one. It is an easy matter to obtain *functions of class (A) which omit a non-denumerable infinity of values of modulus less than one*.

In order to construct functions of this kind, consider a set S of interior points of the circle $|w| < 1$. Let us assume that the set S is closed relatively to the circle and that it has the following property: to every $\epsilon > 0$ there corresponds a sequence of circles $\{C_n\}$ which cover the set S and whose radii δ_n satisfy the inequality

$$\sum_{n=1}^{\infty} \frac{1}{\log^+(1/\delta_n)} < \epsilon.$$

A set with this property is said to be of *logarithmic measure zero*. If the points of the set S are removed from the interior of the circle $|w| < 1$, there remains an open set of points. If this open set is connected, we denote the resulting region by R_S . If this open set is not connected (as is a priori conceivable), we denote by R_S an arbitrary one of the regions into which the open set is decomposed by S . It will be shown a little later that R_S is always a dense set in the circle $|w| < 1$, thus proving that the open set in question is always connected.

By the fundamental theorem of conformal mapping there exists a function $z = \phi(w)$ which maps the region R_S conformally on the unit circle $|z| < 1$ of the z -plane. This function $\phi(w)$ is infinitely multiple-valued in the region R_S if R_S is multiply connected. The Riemann surface \Re of $\phi(w)$ is the universal covering surface† of the region R_S . Such a covering surface is unramified relatively to R_S and consequently can have non-algebraic singularities only over the points of the set S . Let us now consider the inverse function $w = f(z)$ of $z = \phi(w)$. This function $f(z)$ is always single-valued, analytic, and bounded in the circle $|z| < 1: |f(z)| < 1$. The boundary function $f^*(e^{i\theta})$ is defined by Fatou's theorem in almost all points of the circumference $r = 1$, $0 \leq \theta \leq 2\pi$. Denote by E the set of all those points of the circumference at which $f^*(e^{i\theta})$ assumes a value belonging to the set S . Since S is of logarithmic

† For a detailed account of the subject of conformal mapping of multiply connected regions, cf. L. Bieberbach, *Lehrbuch der Funktionentheorie*, vol. 2, 2d edition, 1931, chapters I and IV, as well as H. Weyl, *Die Idee der Riemannschen Fläche*, 1913, chapter I, especially §9.

measure zero, it readily follows from a theorem of R. Nevanlinna† that the set E is linearly measurable and of measure zero. Furthermore, it may be shown that the equation $f^*(e^{i\theta}) = \alpha$, where α is any complex number of modulus less than one, has a solution if, and only if, the point $w = \alpha$ is a point of the set S . Hence, $|f^*(e^{i\theta})| < 1$ for a set of values of θ which is of measure zero. Thus, for almost all values of θ we have $|f^*(e^{i\theta})| = 1$. This shows that the function $f(z)$ is of class (A). On the other hand, it is immediately apparent from the definition of $f(z)$ that $f(z)$ omits in the circle $|z| < 1$ all values corresponding to points of the set S . This proves our initial assertion.

Incidentally, we notice that, according to Theorem 5, $f(z)$, being of class (A) and possessing singularities on the boundary $|z| = 1$, assumes a set of values everywhere dense in the circle $|z| < 1$. This proves that the region R_S is everywhere dense in the circle $|w| < 1$. Hence, if the set S is removed from the interior of the circle $|w| < 1$, there remains a single connected region R_S .

With the aid of Theorem 11 we now prove the following theorem:

THEOREM 13. *Let S be a set of interior points of the circle $|w| < 1$ which is closed relatively to the circle and of logarithmic measure zero. Denote by R_S the region which is obtained by removing the points of the set S from the interior of the circle $|w| < 1$. Let $z = \phi(w)$ be an arbitrary function which maps R_S conformally on the circle $|z| < 1$. Then the inverse function $w = f(z)$ of $z = \phi(w)$ may be represented as a linear function of a Blaschke product:*

$$(15.1) \quad f(z) = \frac{e^{i\delta} B(z) + \beta}{1 + \bar{\beta} e^{i\delta} B(z)}, \quad \delta \text{ real}, \quad |\beta| < 1.$$

From the discussion which has preceded the statement of this theorem, it is evident that $f(z)$ is of class (A) in the circle $|z| < 1$ and assumes infinitely often in this circle every value of modulus less than one which corresponds to an interior point of the region R_S . Furthermore, no such value α can be a convergence value of $f(z)$, for the Riemann surface \mathfrak{R}_S is unramified relatively to R_S . Consequently, according to Theorem 11, whose hypotheses are satisfied by the function $f(z)$, the inverse function $f(z)$ of the mapping function $\phi(w)$ may be represented as a linear function of a Blaschke product $B(z)$ of the form (15.1). Since $f(z)$ clearly omits all values of S , this theorem also shows that to any set S of logarithmic measure zero and closed relatively to the circle $|w| < 1$ there corresponds a Blaschke product, or a linear function of a Blaschke product, which assumes all values of modulus less than one infinitely often in the unit circle $|z| < 1$ save those belonging to the set S which it omits.

† R. Nevanlinna, *Über die Randwerte von analytischen Funktionen*, Commentarii Mathematici Helvetici, vol. 2 (1930), pp. 237-244.

16. Theorem 13 is of some interest in the light of a recent remarkable investigation of Besicovitch.[†] Among other things Besicovitch generalizes Weierstrass's theorem on the behavior of single-valued analytic functions in the neighborhood of an isolated essential singularity to the case of a non-isolated one. His theorem is as follows:

THEOREM OF BESICOVITCH. *If the set E of essential singularities of a single-valued analytic function $f(z)$ is of linear measure zero, then the set of values of $f(z)$ in the neighborhood of each of the points of E is everywhere dense on the complex plane.*

It is natural to inquire whether or not it is possible to extend Picard's theorem in an analogous manner. Theorem 13 gives us the means of answering this question negatively, thus showing that the hypothesis of an *isolated* essential singularity, while not necessary for Weierstrass's theorem, cannot be dropped in Picard's theorem. In fact, we shall prove the following assertion:

There exist single-valued functions $f(z)$ analytic in the whole z -plane except for a set E of essential singularities of linear measure zero and omitting a non-denumerable set Σ of values of logarithmic measure zero.

Let us consider an arbitrary, closed, non-denumerable set of points S of logarithmic measure zero containing the origin $w=0$ and lying in the interior of some circle $|w| < \rho < 1$. On removing the points of the set S from the circle $|w| < 1$, we obtain a region R_s . Let $w=f(z)$ be the inverse function of that function $z=\phi(w)$ which maps R_s conformally on the circle $|z| < 1$, carrying three preassigned points of the circumference $|w|=1$ into three pre-assigned points of the circumference $|z|=1$. According to the result of §15, the boundary function $f^*(e^{i\theta})$ has the property that

$$(16.1) \quad |f^*(e^{i\theta})| = 1$$

for almost all values θ in the interval $0 \leq \theta \leq 2\pi$.

We shall prove now that every point $z=e^{i\theta_0}$ for which $|f^*(e^{i\theta_0})|=1$ is a point of analyticity of $f(z)$. To this purpose, consider the radius $\theta=\theta_0$ and the image L of this radius by $w=f(z)$ on the universal covering surface \mathfrak{R}_s of the region R_s . It is clear that the projection of the curve L on the w -plane tends to a point $w=e^{i\theta_0}$ of modulus one. We shall now show that the curve L from a certain point on lies wholly on one sheet of the Riemann surface \mathfrak{R}_s . If this were not the case, then L would have to wind infinitely many times around branch points. Since, however, \mathfrak{R}_s has no branch points in the ring $\rho < |w| < 1$, the curve L either would have to wind infinitely often in the ring $\rho < |w|$

[†] A. S. Besicovitch, *On sufficient conditions for a function to be analytic*, Proceedings of the London Mathematical Society, (2), vol. 32 (1931), pp. 1-9.

<1 or would have to penetrate the circle $|w| < \rho$ infinitely many times. In either case, its projection would not tend to a single limit point. Hence, the curve L from a certain point on lies on a single sheet of \mathfrak{R}_s and terminates in a definite boundary point P of \mathfrak{R}_s which lies over the point $w = e^{i\theta}$. Now, the function $z = \phi(w)$ is clearly analytic in the point P and $\phi'(P) \neq 0$, since it is an interior point of the free analytic curve $|w| = 1$, that is, of a curve whose points are not limit points of boundary points not belonging to the curve. Hence, $f(z)$ is analytic in the point $z = e^{i\theta\phi}$. Consequently, by virtue of (16.1) $f(z)$ is analytic and of modulus one in almost all points of the circumference $|z| = 1$ which form an open, everywhere dense set.

By Schwarz's reflection principle $f(z)$ may be continued analytically across points of this open set on the circumference $|z| = 1$ in accordance with the functional relation

$$f(1/\bar{z}) = (1/\overline{f(z)})$$

Since the set S was so chosen as to contain the origin $w = 0$ this defines a function $f(z)$ analytic in the whole plane except for a perfect nowhere dense set of linear measure zero of essential singularities on the circumference $|z| = 1$. Furthermore, if we denote by S' the image set of S by an inversion in the unit circle, it is evident that the function $f(z)$ omits all values belonging to the set $\Sigma = S + S'$.

17. It is natural to inquire whether Theorem 13 still holds when the set S of logarithmic measure zero is replaced by a set of linear measure zero. The answer hinges on whether or not the inverse of the mapping function is of class (A). In the sequel we shall prove that in general the question is to be answered in the negative. The sets S with which we shall deal can be characterized as follows:

- (1) S is closed.
- (2) If the points of S are removed from the plane, the remaining set of points is connected.
- (3) There exists a function $s(\rho)$ which is positive, continuous, and monotonically increasing for $\rho > 0$ such that the integral

$$\int_0^k \frac{s(\rho)}{\rho} d\rho$$

is finite for some positive value of k , and there exists a positive number ϵ such that for every sequence of circles $\{C_v\}$ with the radii ρ_v which cover S the inequality

$$\sum_{v=1}^{\infty} s(\rho_v) > \epsilon$$

is satisfied. All sets S satisfying conditions 1, 2, and 3 will be said to possess the property (K). Given a set S and a function satisfying condition 3, denote by $C(\lambda)$ the greatest lower bound of sums $\sum_{i=1}^n s(\rho_i)$ corresponding to a sequence of circles $\{C_i\}$ of radii ρ_i which cover S and such that $\rho_i \leq \lambda$. The function $C(\lambda)$ is non-increasing. Hence, $\lim_{\lambda \rightarrow 0} C(\lambda) = m_s S$ exists and will be called the s -measure of the set S . Thus, if $s(\rho) = \rho^\alpha$ ($\alpha > 0$), condition 3 states that S is of positive α -dimensional measure (in particular, $\alpha = 1$ gives linear measure and $\alpha = 2$ superficial measure). The s -measure of a set, however, may also be defined by means of the function

$$s(\rho) = \frac{1}{\left(\log^+ \frac{1}{\rho}\right)^{1+\eta}} \quad (\eta > 0),$$

or more generally by means of the function

$$s(\rho) = \frac{1}{\log^+ \frac{1}{\rho} \log_2^+ \frac{1}{\rho} \cdots \left(\log_m^+ \frac{1}{\rho}\right)^{1+\eta}} \quad (\eta > 0).$$

It is evident that a set S with the property (K) may be of linear measure zero.

In view of a later application which is to be made of the following result, we state it in the form of a lemma:

LEMMA. *Let S be a set of points possessing the property (K) and lying in the interior of a circle $|w| < \lambda < 1$. If the set S is removed from the interior of the circle $|w| < 1$, there remains an open connected region R_S . Denote by $z = \phi(w)$ a function which maps R_S conformally on the unit circle $|z| < 1$. Then the inverse function $w = f(z)$ of $z = \phi(w)$ is not of class (A).*

The function $w = f(z)$ is bounded in the unit circle $|z| < 1$: $|f(z)| < 1$. Consequently, by Fatou's theorem,

$$\lim_{r \rightarrow 1} f(re^{i\theta}) = f^*(e^{i\theta}) \quad (z = re^{i\theta})$$

exists for almost all values of θ in the interval $0 \leq \theta \leq 2\pi$. It is to be shown that†

$$mE(|f^*(e^{i\theta})| < 1) > 0.$$

We shall assume that

$$(17.1) \quad mE(|f^*(e^{i\theta})| < 1) = 0,$$

and derive a contradiction.

† The notation $mE(|f^*(e^{i\theta})| < 1)$ is used here to denote the linear measure of that set of points of the circumference $|z| = 1$ for which $|f^*(e^{i\theta})| < 1$.

We may remove the set S and the point at infinity, $w = \infty$, from the whole w -plane, and thus obtain a new region Δ which we shall map conformally on the unit circle $|t| < 1$ by means of a function $t = \Phi(w)$. Consider the inverse function $w = F(t)$ which is single-valued and analytic in the circle $|t| < 1$. R. Nevanlinna proved that $F(t)$ may be represented as the quotient of two functions each of which is analytic and bounded in the circle $|t| < 1$.† Hence, as may be easily seen from a theorem of F. and M. Riesz‡,

$$(17.2) \quad \lim_{\rho \rightarrow 1} F(\rho e^{i\tau}) = F^*(e^{i\tau}) \quad (t = \rho e^{i\tau})$$

exists for almost all values of τ in the interval $0 \leq \tau \leq 2\pi$. Nevanlinna further shows that each finite convergence value (17.2), when represented as a point in the w -plane, is a point of the set S . Hence

$$(17.3) \quad mE(|F^*(e^{i\tau})| < \lambda) = 2\pi,$$

where by hypothesis $\lambda < 1$.

Consider now the set G of all those points of the circle $|t| < 1$ in which $|F(t)| < 1$. This set of points is evidently open and, as we shall show now, connected. If the set G is not connected, it may be decomposed into two or more connected open subsets G_1, G_2, \dots . Let G_1 and G_2 be any two of these subsets and let t_1 be a point of G_1 and t_2 a point of G_2 . Let l be any continuous path in the circle $|t| < 1$ connecting t_1 and t_2 . Then, there exists at least one point t_0 on l for which $|F(t_0)| \geq 1$. But now $F(t)$ maps the unit circle $|t| < 1$ in a one-to-one manner and conformally on the universal covering surface \mathcal{R}_Δ of the region Δ . Consequently, the curve l is mapped on a curve L lying on the surface \mathcal{R}_Δ joining two points w_1 and w_2 which lie over two interior points of the circle $|w| < 1$ and passing through at least one point w_0 which lies over some point of the closed region $|w| \geq 1$. We know, however, from the definition of the region Δ that the surface \mathcal{R}_Δ is unramified over the region $\lambda < |w| < \infty$. Hence, we may deform the curve L continuously into a curve L' which joins the points w_1 and w_2 and lies wholly over the interior of the circle $|w| < 1$. Hence, the curve l may be deformed continuously into a curve l' which joins the two points t_1 and t_2 and on which $|F(t)| < 1$. Hence, the two sets G_1 and G_2 could not have been distinct. This proves that G is a connected open set.

The region G cannot be wholly contained in any circle $|t| < r < 1$. Furthermore, as follows easily from the maximum principle of analytic func-

† R. Nevanlinna, *Commentarii Mathematici Helvetici*, loc. cit., pp. 250-252.

‡ F. and M. Riesz, *Über die Randwerte einer analytischen Funktion*, *Compte Rendu du Quatrième Congrès des Mathématiciens Scandinaves* (1920), pp. 28-30.

tions, the region G is simply connected. Consider now that connected part \mathfrak{R}_Δ' of the surface \mathfrak{R}_Δ into which the region G is mapped by the function $w = F(t)$. From the definition of G it follows that \mathfrak{R}_Δ' is that part of the surface \mathfrak{R}_Δ which lies over the region R_s . Furthermore, from the simple connectivity of G follows the simple connectivity of \mathfrak{R}_Δ' . Hence, \mathfrak{R}_Δ' is a simply connected, unramified, and unbounded covering surface of the region R_s . It follows from this at once that \mathfrak{R}_Δ' is the universal covering surface \mathfrak{R}_{R_s} of R_s . To sum up the preceding, *we have obtained a simply connected subregion G of the circle $|t| < 1$ which is mapped in a one-to-one manner and conformally by $w = F(t)$ on the universal covering surface \mathfrak{R}_{R_s} of the region R_s .*

18. Denote by $w = f(z)$ the inverse function of some function $z = \phi(w)$ which maps the region R_s conformally on the circle $|z| < 1$. The function $w = f(z)$ maps, therefore, the circle $|z| < 1$ in a one-to-one manner and conformally on the universal covering surface \mathfrak{R}_{R_s} of the region R_s . Consequently, by virtue of the result of §17 *the function $t = \Phi(f(z))$ establishes a one-to-one conformal map between the circle $|z| < 1$ and the subregion G of $|t| < 1$.*

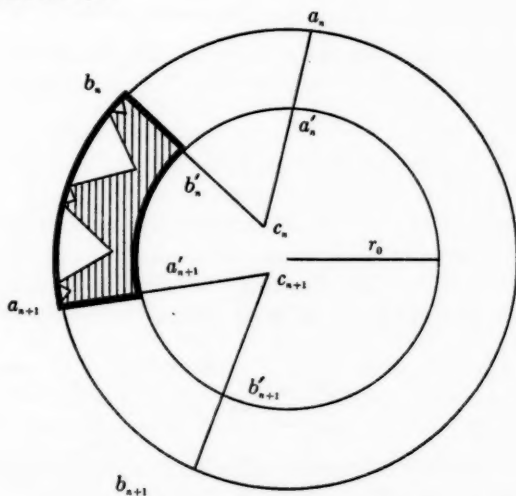
Before completing the proof of the lemma, it is necessary to make some additional remarks about the region G . If the point $t = e^{i\tau_0}$ is such that

$$(18.1) \quad \lim_{\rho \rightarrow 1} F(\rho e^{i\tau_0})$$

exists, then by virtue of the fact that $F(t)$ can be represented as the quotient of two bounded functions and by Lindelöf's theorem it follows that the limit (18.1) exists uniformly in every angle smaller than 180° whose vertex lies in the point $e^{i\tau_0}$ and whose bisector falls along the radius joining the origin $t=0$ with the point $e^{i\tau_0}$. From (17.3) it follows by applying Egoroff's well known theorem that there exists a perfect set Ω of positive measure on the circumference $|t|=1$ and a positive number $r_0 < 1$ such that if $e^{i\tau}$ is an arbitrary point of Ω and $A(e^{i\tau})$ an angle of 60° whose vertex lies in the point $e^{i\tau}$ and whose bisector falls on the radius joining the origin $t=0$ with the point $t=e^{i\tau}$, then $|F(t)| < 1$ for every point t of the angle $A(e^{i\tau})$ whose distance $|t|$ from the origin is greater than r_0 . That part of the angle $A(e^{i\tau})$ whose points lie at a distance not less than r_0 from the origin will be denoted by $B(e^{i\tau})$. Since G was defined as the totality of those points of the circle $|t| < 1$ for which $|F(t)| < 1$, it follows that the region $B(e^{i\tau})$ lies wholly within G for every point $e^{i\tau}$ belonging to the set Ω .

Now, the set $C\Omega$ complementary to Ω is open and consists of denumerably many open arcs $a_n b_n$ of the periphery $|t|=1$, no two arcs having any points in common. Through each end point a_n and b_n draw a line forming an angle of 30° with the corresponding radius. These two lines intersect in a

point c_n . We thus obtain denumerably many circular sectors $a_n b_n c_n$. Consider now a point t ($|t| < 1$), whose distance $|t|$ from the origin $t=0$ is greater than r_0 and which lies outside of all the sectors $a_n b_n c_n$. Corresponding to this point there exists a point $e^{i\tau}$ of the set Ω such that the region $B(e^{i\tau})$ contains t in its interior or on its boundary. In either case t is an interior point of G . There exist at most a finite number of sectors, which we number $a_1 b_1 c_1, \dots, a_M b_M c_M$, falling in part within the circle $|t| \leq r_0$. If $n \leq M$, denote by a'_n the intersection of $\overline{a_n c_n}$ with $|t| = r_0$ and by b'_n the intersection of $\overline{b_n c_n}$ with $|t| = r_0$. Let us assume that the sectors $a_1 b_1 c_1, \dots, a_M b_M c_M$ have been so numbered that the points $a_1, b_1, a_2, b_2, \dots, a_M, b_M$ describe the circumference $|t| = 1$ in counter-clockwise sense. Then, at least one of the arcs $b_1 a_2, b_2 a_3, \dots, b_{M-1} a_M, b_M a_1$ contains a subset of positive measure of the set Ω . Denote any such arc by $b_n a_{n+1}$. Consider now the contour $b_n b'_n a'_{n+1} a_{n+1}$ as shown in the figure. If we remove all the circular sectors which fall within the region bounded by the contour $b_n b'_n a'_{n+1} a_{n+1}$, we obtain a new region G' which is a subregion of G . The boundary of the region G' is by construction a closed rectifiable Jordan curve and it has a set of points of positive measure in common with the circumference $|t| = 1$, namely that subset of Ω which lies on the arc $b_n a_{n+1}$.



19. We have assumed that (17.1) holds; that is, that the function $f(z)$ is of class (A). Consider some point P of the subset of Ω which lies on the arc $b_n a_{n+1}$. Let γ be an arc which lies in the region G' except for its end point P . The function $w = F(t)$ maps it on an arc γ' lying on the Riemann surface

\Re_{R_s} and terminating in some point which lies over a point of the set S . The function $z = \phi(w)$ maps γ' on an arc γ'' which terminates in some point P'' of $|z| = 1$ belonging to the set $E(|f^*(e^{i\theta})| < 1)$ which by (17.1) is of measure zero.

Consider now a function $\nu(z)$ defined and bounded in the unit circle $|z| < 1$: $|\nu(z)| < 1$, and such that it does not tend to any limit along all paths terminating in points of the set $E(|f^*(e^{i\theta})| < 1)$. That such functions exist is known from a result of Lusin and Priwaloff.† We have seen in the beginning of §18 that the function $t = \Phi(f(z))$ maps the circle $|z| = 1$ in a one-to-one manner on the region G . Denote by $z = g(t)$ the inverse function and consider the new function $\mu(t) = \nu(g(t))$. It follows from the last paragraph that $\mu(t)$ approaches no limit along any path of the region G (and in particular of the region G') terminating in points of that subset of Ω which lies on the arc $b_n a_{n+1}$. We thus obtain a bounded analytic function $\mu(t)$ in a region G' bounded by a single rectifiable Jordan curve and a set of boundary points of G' of positive linear measure such that $\mu(t)$ tends to no limit along any path in G' terminating in a point of the set. This, however, contradicts a known theorem.‡ Thus the assumption (17.1) leads to a contradiction. Consequently, $mE(|f^*(e^{i\theta})| < 1) > 0$, and $f(z)$ is not of class (A).

20. By means of the lemma just established we prove

THEOREM 14. *Let $f(z)$ be a function which is analytic and bounded in the unit circle $|z| < 1$: $|f(z)| < 1$. Let $A: \alpha_1 < \theta < \alpha_2$, $r = 1$, be an arc of the circumference $|z| = 1$ such that*

$$(20.1) \quad \lim_{r \rightarrow 1} f(re^{i\theta}) = f^*(e^{i\theta}), \quad |f^*(e^{i\theta})| = 1,$$

for almost all values of θ in the interval $\alpha_1 < \theta < \alpha_2$. Then either $f(z)$ is analytic on the arc A or $f(z)$ assumes in the circle $|z| < 1$ every value α ($|\alpha| < 1$) save perhaps for a set S of such values possessing the following property: For every function $s(\rho)$ positive, continuous, and monotonically increasing in $\rho > 0$ such that the integral

$$\int_0^k \frac{s(\rho)}{\rho} d\rho \quad (k > 0)$$

is finite for some k , and for any positive ϵ there exists a sequence of circles $\{C_r\}$ with radii ρ_r covering the set and such that $\sum_{r=1}^{\infty} s(\rho_r) < \epsilon$.

† N. Lusin and J. Priwaloff, *Sur l'unicité et la multiplicité des fonctions analytiques*, Annales Scientifiques de l'Ecole Normale Supérieure, (3), vol. 42 (1925), pp. 157-159.

‡ V. V. Golubev, *Single-valued analytic functions with perfect singular sets* (in Russian), Moscow, 1916, p. 44. Cf. also F. and M. Riesz, loc. cit., p. 40.

Let the function $f(z)$ be not analytic on the arc A . If the set S of values in the w -plane which $w=f(z)$ omits does not satisfy the condition of Theorem 14, then the set possesses the property (K) defined in §17. Thus, for some function $s(\rho)$ the s -measure of the set S is positive. Consider a monotonically increasing sequence of positive numbers $\{r_n\}$ which tend to 1 in the limit. Denote by S_1 that part of S which lies in the circle $|z| < r_1$, and in general by S_n that part of S which lies in the ring $r_{n-1} \leq |z| < r_n$. Then $S = \sum_{n=1}^{\infty} S_n$. Furthermore, it follows immediately from the definition of s -measure that

$$m_s S \leq m_s S_1 + m_s S_2 + \cdots + m_s S_m + \cdots$$

Hence, if the s -measure of S is positive, then the s -measure of at least one set S_n is likewise positive. Consider any such set S_n . It possesses the property (K) and lies in the interior of some circle $|w| < \lambda < 1$. If the set S_n is removed from the interior of the circle $|w| < 1$, there remains an open connected region R_{S_n} . Denote by $t = \phi(w)$ some function which maps the region R_{S_n} on the unit circle $|t| < 1$. By the lemma stated in §17 the inverse function $w = \psi(t)$ of $t = \phi(w)$ is not of class (A) . That is to say, the relations

$$(20.2) \quad \lim_{\sigma \rightarrow 1} \psi(\sigma e^{i\tau}) = \psi^*(e^{i\tau}), \quad |\psi^*(e^{i\tau})| < \lambda < 1, \quad t = \sigma e^{i\tau},$$

hold for a set of positive measure of values of τ in the interval $0 \leq \tau \leq 2\pi$.

Consider now the function $t = \phi[f(z)]$. Since $f(z)$ does not assume any value of the set S_n and since $\phi(w)$ has the points of S_n as its only singularities, it follows that the function $\phi[f(z)]$ may be continued analytically along every path lying in the interior of the unit circle $|z| < 1$. By the monodromy theorem, therefore, the function $\phi[f(z)]$ is analytic and single-valued in the circle $|z| < 1$ as soon as some definite branch of $\phi(w)$ has been selected. It is furthermore immediately evident that

$$|\phi[f(z)]| < 1$$

in the circle $|z| < 1$. Next, consider any radius $\theta = \theta_0$ terminating in a point of the arc A on which $f(z)$ tends to a limit and $|f(z)|$ tends to the value 1. Such a radius $\theta = \theta_0$ is mapped by $w = f(z)$ on a continuous curve p_0 in the circle $|w| < 1$ terminating in a point P_0 of the periphery $|w| = 1$. But now the function $t = \phi(w)$ maps the curve p_0 on a curve π_0 in the circle $|t| < 1$ terminating in a point Π_0 of the periphery $|t| = 1$. Since by hypothesis the relations (20.1) hold for almost all points of the arc A , this proves that the function $\phi[f(z)]$ likewise satisfies the conditions

$$(20.3) \quad \lim_{r \rightarrow 1} \phi[f(re^{i\theta})] = \phi^*[f^*(e^{i\theta})], \quad |\phi^*[f^*(e^{i\theta})]| = 1,$$

for almost all values of θ in the interval $\alpha_1 < \theta < \alpha_2$; that is, for almost all

points of the arc A . Furthermore, by Lindelöf's theorem, if Π_0 is the point $e^{i\tau_0}$, then

$$\lim_{\sigma \rightarrow 1} |\psi(\sigma e^{i\tau_0})| = 1.$$

Thus we see that if $e^{i\theta_0}$ is a point for which $|f^*(e^{i\theta_0})| = 1$, there corresponds to it a point $e^{i\tau_0} = \phi^*[f^*(e^{i\theta_0})]$ such that $|\psi^*(e^{i\tau_0})| = 1$. Consider now the set E of all those points $e^{i\theta}$ for which (20.3) holds. Then, by Nevanlinna's theorem, stated in §8, the set E' of points

$$e^{i\tau} = \phi^*[f^*(e^{i\theta})],$$

where $e^{i\theta}$ describes the set E , is of measure 2π . Hence, the relation

$$|\psi^*(e^{i\tau})| = 1$$

is satisfied in almost all points of the circumference $|t| = 1$. This, however, contradicts the fact that the relations (20.2) hold for a set of positive measure on the circumference $|t| = 1$. Thus, our assumption that for some function $s(\rho)$ the s -measure of the set S is positive has led to a contradiction. This proves the theorem.

As an immediate corollary of Theorem 14, we state the following extension of Schwarz's reflection principle:

Let $w = f(z)$ be a bounded analytic function in the unit circle $|z| < 1$:

$$|f(z)| < 1.$$

Let $A: \alpha_1 < \theta < \alpha_2, r = 1$, be an arc of the circumference $|z| = 1$ such that

$$\lim_{r \rightarrow 1} f(re^{i\theta}) = f^*(e^{i\theta}), \quad |f^*(e^{i\theta})| = 1,$$

for almost all values of θ in the interval $\alpha_1 < \theta < \alpha_2$. Let $s(\rho)$ be a positive, continuous, monotonically increasing function for $\rho > 0$ such that the integral

$$\int_0^k \frac{s(\rho)}{\rho} d\rho \quad (k > 0)$$

is finite for some k . If $f(z)$ does not assume in the unit circle $|z| < 1$ values which form a set of positive s -measure in the circle $|w| < 1$, then $f(z)$ may be continued analytically beyond the arc A in accordance with the functional relation

$$f\left(\frac{1}{\bar{z}}\right) = \frac{1}{\overline{f(z)}}.$$

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CONVERSES OF GAUSS' THEOREM ON THE ARITHMETIC MEAN*

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1.1. Introduction. Let T denote a domain (open continuum) in the plane. Let u be a function, continuous in T , and equal, at each point P of T , to the arithmetic mean of its values on each circle with center at P and lying, with its interior, in T . According to Gauss' theorem, this is the property of any function u harmonic in T . On the other hand, Koebe† showed, that conversely, any function with the stated properties is harmonic in T . It is with some extensions of this theorem of Koebe we are concerned.

Undoubtedly, one of the simplest proofs of Koebe's theorem is as follows. Let c be any circle lying with its interior C in T . Let m be the minimum of u in $C+c$. Then, since u is continuous in $C+c$, either u assumes the value m on c , or else there is a point P of C such that $u = m$ at P , and $u > m$ at all points of $C+c$ nearer to c than is P . But, by the mean-value property, the second of these two alternatives is impossible. Thus, u assumes its lower bound in $C+c$ on c . Applying this result to $u-v$ and to $v-u$, where v is the function harmonic in C , continuous on $C+c$, and equal to u on C , we conclude that u is harmonic in C , and therefore in T .

This reasoning can evidently be applied in case the circles attached to each point P , on which the arithmetic mean of u is equal to the value of u at their center P , constitute any infinite set whose radii have 0 as lower limit. It fails, of course, if these radii are bounded away from 0. The question then arises as to what can be said about a function which coincides at each point P with its arithmetic mean on some single circle about P as center. In what follows we shall be concerned with functions satisfying a condition of this type.

1.2. All our theorems are valid both in the plane and in space. The proofs themselves are independent of the number of dimensions if "circle" is interpreted as "sphere" and "plane" as "space" in the case of three dimensions. We shall always suppose that T is a bounded domain. Results of a similar character are readily obtained by inversions for unbounded domains, containing or not the point at infinity, provided these domains do not, with their

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† Koebe, 6. Numbers in heavy type refer to the bibliography at the end of this paper.

boundaries, fill out the extended plane. The results of some of the theorems can easily be extended to domains not having this latter property. This does not, however, seem to be the case for those of others. In Theorem V, for example, a condition equivalent to boundedness seems to be essential.

We denote by t the boundary of T . We associate with each point P of T a normal* domain $C(P)$, containing P , and consisting wholly of points of T . We denote by $c(P)$ the boundary of $C(P)$. This boundary may coincide wholly or in part with t . We denote by τ the set of those points common to t and some $c(P)$.

We denote, for a function u continuous in $T+\tau$, by $A\{u(P)\}$, or by $A(u)$, the value at P of the function harmonic in $C(P)$, continuous in $C(P)+c(P)$, and equal to u on $c(P)$. Our theorems are concerned chiefly with a continuous function u satisfying $u(P)=A(u)$ in T . This condition is, of course, a generalization of the condition that u be equal at each point P to the arithmetic mean of its values on some circle about P as center. In fact, when $c(P)$ is a circle about P as center, then $A\{u(P)\}$ is the arithmetic mean of u on $c(P)$.

2.1. On the bounds of a function satisfying $u=A(u)$. We first apply the reasoning instrumental in the preceding proof of Koebe's theorem. We obtain

THEOREM I. *Let u be continuous in $T+\tau$. Then, if $u(P) \geq A(u)$ in T , it follows that u tends to its lower bound in T on a sequence of points in T tending to a point of t . Similarly, if $u \leq A(u)$, then u tends to its upper bound in T on a sequence of points in T tending to a point of t .*

We need consider only the first part of this theorem. Let m denote the lower bound of u in T ; and let λ be the set of points in $T+\tau$ on which $u=m$. If λ is void, or if λ contains any point of t , the conclusion follows from the continuity of u in $T+\tau$. Let us suppose then that λ is not void and that it lies wholly in T . Let P be a point of λ . Then, by a familiar property of harmonic functions, and our hypothesis on u , it follows that all the points of $c(P)$ are points of λ . Thus, $u=m$ on a point nearer to t than is P . We conclude that the distance from λ to t is 0; for if it were positive the set λ would be closed and we could select, contrary to fact, a point P of λ such that no other point of λ lies nearer to t than does P . This proves the theorem.

2.2. A different form of reasoning gives a somewhat stronger result than that embodied in Theorem I. It can be shown in fact that, under the first hypotheses of that theorem, if u attains in T its lower bound, u attains that

* A domain C is normal if the Dirichlet problem for C admits a solution for every assigned function continuous on the boundary of C .

For a set of references on the Dirichlet problem and on harmonic functions in general, the reader is referred to Kellogg, 2 and 4.

bound in $T+\tau$ in every neighborhood of every point of t . This is a consequence of the following theorem. We note also, as another corollary of this theorem, that if u is continuous in $T+t$ and satisfies $u(P) = A(u)$ in T , then u cannot assume in T both its bounds without reducing to a constant.

THEOREM II. *Let u be continuous in $T+\tau$. Then, if u attains in T its lower bound m in T , and if $u(P) \geq A(u)$ in T , it follows that every point of T is a point of some $C(P)$ on the boundary of which $u = m$ identically.*

Let P_1 be a point of T at which $u = m$. Then plainly $u = m$ on $c(P_1)$. Now let P_2 be any second point of T . If $u(P_2) = m$, then $u = m$ on $c(P_2)$ and there is nothing to prove. In the contrary case, let α be a polygonal line, lying in T , and having its end points at P_1 and P_2 . Consider the points of α at which $u = m$. This set is a non-null closed set. We therefore can select the last point P^* , in the sense P_1 to P_2 , at which $u = m$. On $c(P^*)$ we have $u = m$. Accordingly, $c(P^*)$ cannot cut α between P^* and P_2 . Thus, P_2 is a point of $C(P^*)$. We conclude the truth of the theorem.

2.3. The question now arises whether, in contradistinction to non-constant harmonic functions, a non-constant function having the generalized mean-value property can attain one or both of its bounds without reducing to a constant. The answer is that in general such a function can assume both its bounds. Consider in fact the following example.

Let O be any fixed point. Let T be the interior of the unit circle about O as center. Let b_n , $n = 2, 3, \dots$, be the circle of radius $1 - 1/n$ about O . Let $u(P) = 0$ for $OP < 1/2$ and for P on b_2, b_4, \dots ; and let $u(P) = 1$ for P on b_3, b_5, \dots . Finally, let u be harmonic in the region bounded by b_n, b_{n+1} and assume continuously the values defined for u on b_n and b_{n+1} . Then, noting that, if P is on b_n , $n = 2, 3, \dots$, we can take for $C(P)$ the interior of b_{n+2} , and that if P is not on one of the circles b_n we can take for $C(P)$ the interior of any sufficiently small circle about P , we see that u is continuous in T and has the generalized mean-value property. In addition, we see that u assumes in T its lower bound 0 and its upper bound 1.

2.4. Considerably more information can be obtained in regard to the question raised in the preceding paragraph. It can be shown that a function u of the prescribed type cannot attain both its bounds if the domains $C(P)$ are of a sufficiently restricted character. We consider in the next theorem a set of domains satisfying the following conditions:

- (a) the diameter $\delta(P)$ of $C(P)$ tends to 0 as P tends to any point of t ;
- (b) the boundary $c(P)$ has at least one point in common with $c(Q)$ if there is a point of $c(P)$ exterior and a point of $c(P)$ interior to $C(Q)$.

The second of these two conditions is satisfied, of course, if each $c(P)$ is

a connected set. Both conditions are satisfied if each $c(P)$ is a circle about P as center.

THEOREM III. *Let the domains $C(P)$ satisfy the conditions (a) and (b). Let u be continuous in $T + \tau$ and let $u(P) = A(u)$ in T . Then, if the bounds of u in T are distinct, both these bounds cannot be attained by u in T .*

To prove this we show that the contrary assumption, that u attains in T both its bounds, $m < M$, implies a contradiction. We first select a point P_1 at which $u = M$, and then choose a point Q of t such that the segment P_1Q lies in T except for its extremity Q . We note that Q is not a point of $c(P)$ for any P in T ; for if it were we should have, contrary to hypothesis, $m = M$, because u would be continuous at Q and would tend, according to our remark in §2.2, to M on one sequence, and to m on another sequence, of points tending to Q .

Consider now $c(P_1)$. On this set we have $u \equiv M$. Further, $c(P_1)$ has at least one point in common with P_1Q . We denote by P_2 one such point, observing that P_2 is a point of T .

Consider next the set of points on P_2Q at which $u = m$. Since P_2 is a point of some $C(P)$ on the boundary of which $u \equiv m$, this set is not void. In addition, if we adjoin to this set the point Q , the resulting set is closed. Accordingly, there is a first point, starting from P_2 , of P_2Q at which $u = m$. We denote this point by P_3 , observing that P_3 is a point of T . Now at P_2 , $u = M$, and on $c(P_2)$, $u \equiv m < M$. It follows from this, and our choice of P_3 , that P_3 is a point of $C(P_2)$. We deduce further, on applying the fact that $u \equiv M$ on $c(P_1)$ and the fact that the domains $C(P)$ satisfy condition (b), that $c(P_1)$ is contained in $C(P_3)$. Thus,

$$\delta(P_1) \leq \delta(P_3).$$

We continue this process. We select a point P_4 of intersection of P_3Q with $c(P_3)$, noting that P_4 is a point of T . We next select the first point, starting from P_4 , of P_4Q at which $u = M$. We denote this point by P_5 , observing that $C(P_4)$ contains P_3 . We find also that $C(P_5)$ contains $c(P_3)$; and accordingly that

$$\delta(P_1) \leq \delta(P_3) \leq \delta(P_5).$$

Proceeding in this manner we obtain an infinite sequence of points P_1, P_2, \dots , lying on P_1Q and in T . We have $P_1P_2 < P_1P_3 < \dots < P_1Q$, and $u(P_1) = u(P_2) = u(P_3) = \dots = M$, and $u(P_3) = u(P_4) = u(P_5) = \dots = m$. We find also that

$$(2.41) \quad \delta(P_1) \leq \delta(P_3) \leq \dots$$

We arrive now at the desired contradiction. It follows from (2.41), and the fact that the domains $C(P)$ satisfy the condition (a), that the points P_n do not tend to Q . Accordingly, these points have a limit point P in T . But this is impossible; for u is continuous at P , and is therefore distinct from at least one of its bounds in some neighborhood of P .

2.5. It is conceivable that, under the restrictions placed upon u in Theorem III, u can attain neither of its bounds in T without reducing to a constant. This however is not the case. An example illustrating this point follows.

We take for T the interior of the unit circle about some fixed point O as center. We shall so define u that it assumes its lower bound in T everywhere in T except in certain circles k_1, k_2, \dots . The domain $C(P)$ will be for each P the interior of a circle about P as center.

Let Q be a point on the boundary of T . Let P_1, P_2, \dots be points on OQ such that

$$0 < OP_1 < OP_2 < \dots < OQ, \quad \lim_{n \rightarrow \infty} P_n = Q.$$

About P_n as center we now construct a circle d_n of radius ρ_n , choosing ρ_n so that each d_n lies in T , and is exterior to every other d_m of the set. The circle k_n , then, shall be the circle about P_n with radius

$$r_n = \rho_n(1 - OP_{n+1})/3.$$

The circles k_n have the following properties. Each is interior to T and each is exterior to every other circle of the set. Further, if P is interior to k_n , then the circle $e_1(P)$ with P as center and radius $2r_n$ lies exterior to k_n and interior to d_n , whereas the circle $e_2(P)$ with center at P and passing through P_{n+1} lies in T . We denote by K_n the interior of k_n .

Turning now to the definition of u , we first choose a set of positive numbers B_1, B_2, \dots . We take $B_1 = 1$. We then choose B_2 so that the arithmetic mean on $e_2(P')$ of the function $u_2(P)$, defined as

$$B_2(r_2 - PP_2)$$

for P in K_2 and as 0 elsewhere, exceeds B_1 independently of the position of P' in K_1 . This is possible since the mean in question exceeds a constant multiple of B_2 for all P' in K_1 . Continuing in this manner we choose, in general, B_{n+1} so large that the arithmetic mean on $e_2(P')$ of the function $u_{n+1}(P)$, defined as

$$B_{n+1}(r_{n+1} - PP_{n+1})$$

for P in K_{n+1} and as 0 elsewhere, exceeds B_n independently of the position of P' in K_n .

We now define $u(P)$ as

$$B_n(r_n - PP_n)$$

for P in K_n , $n = 1, 2, \dots$, and as 0 at all other points of T . It is evident then that u is continuous in T and assumes in T its lower bound, 0. It is clear also that, if P is exterior to all the circles k_n , then u has the mean-value property at P with respect to all sufficiently small circles about P . It remains to consider the points of $k_n + K_n$.

Suppose first that P is on k_n . For such a point we can take for $C(P)$ the interior of the circle with P as center and with radius $2r_n$; for then $C(P) + c(P)$ lies in T and u is 0 at P and on $c(P)$.

Suppose now that P is in K_n . Consider the circles $e_1(P)$ and $e_2(P)$. On $e_1(P)$ the arithmetic mean $A_1(P)$ of u is 0; and on $e_2(P)$ the arithmetic mean $A_2(P)$ of u exceeds B_n because of our choice of the B_m . Thus,

$$A_1(P) < u(P) < A_2(P), \text{ since } 0 < u(P) < B_n.$$

Now as η varies from η_1 , the radius of e_1 , to η_2 , the radius of e_2 , the circle of radius η about P as center remains in T , and the arithmetic mean of u on this circle varies continuously from A_1 to A_2 . Hence we can select an η so that the arithmetic mean of u on the corresponding circle is exactly $u(P)$. The interior of this circle we can take for $C(P)$. The function u has then the mean-value property at P . Accordingly, u has all the asserted properties.

3.1. Sufficient conditions that u be harmonic in T . We turn now to a study of the conditions under which a function possessing the generalized mean-value property is necessarily harmonic. One result in this direction is readily obtained as a corollary of Theorem I.

THEOREM IV. Let u be continuous in $T + \tau$, and satisfy $u(P) = A(u)$ in T . Then, if there exists a function v , harmonic in T , such that

$$\lim_{P \rightarrow Q} \{u(P) - v(P)\} = 0 \quad (P \text{ in } T)$$

at every point Q of t , it follows that u is harmonic in T .

We note, in fact, that $u - v$ is continuous in $T + t$ if defined as 0 on t . Accordingly, v is continuous on $T + \tau$ if defined as u on τ . It follows that v , and therefore that $u - v$, has the generalized mean-value property in T . The bounds of $u - v$ in T are then both 0. This proves the theorem.

3.2. The condition as to the existence of v in Theorem IV is satisfied, of

course, if we suppose that T is a normal domain, and that u is given as continuous in $T+t$. This suggests a possible result of a much deeper character, namely, that a continuous function having the generalized mean-value property is harmonic if it tends to continuous boundary values at all *regular points** of the boundary of its region of definition. In the next theorem we show that for a bounded function of this type this result is indeed true.

THEOREM V. *Let u be bounded and continuous in $T+\tau$; and let $u(P) = A(u)$ in T . In addition, let $u(P)$ approach at each regular point Q of t a limit value $f(Q)$ when P , while remaining in T , tends to Q . Then, if the values of $f(Q)$ are those of a function continuous on t , it follows that u is harmonic in T .*

The proof rests on the following lemma.

LEMMA. *Let U be continuous in $T+\tau$; and let $U(P) \geq A(U)$ in T . Let V be harmonic in T . Then $U - V$ tends to its lower bound m in T on a sequence of points in T tending to a point of t .*

The reasoning in this lemma is an extension of that in Theorem I. Let λ be the set of points in T at which $U - V = m$. If λ is void the conclusion follows from the continuity of $U - V$ in T . If λ is not void, and if $c(P)$ is for each point P of λ interior to T , the conclusion follows as in Theorem I. Let us suppose, then, that there is a point P_0 of λ such that $c(P_0)$ has a point Q in common with t . Let W be the function, harmonic in $C(P_0)$, continuous in $C(P_0) + c(P_0)$, which coincides on $c(P_0)$ with U . We observe first that

$$(3.21) \quad W - V \equiv m$$

in $C(P_0)$. In fact, we have

$$W(P_0) - V(P_0) \leq U(P_0) - V(P_0) = m;$$

so that if (3.21) failed to hold we could select a sequence of points $\{P_n\}$, $n=1, 2, \dots$, in $C(P_0)$, tending to a point P' of $c(P_0)$, such that

$$\lim_{n \rightarrow \infty} \{W(P_n) - V(P_n)\} < m.$$

But

$$\lim_{n \rightarrow \infty} U(P_n) = U(P') = \lim_{n \rightarrow \infty} W(P_n);$$

* A point Q on the boundary τ of a region R , bounded or not, is regular for τ if the sequence solution of the Dirichlet problem for R , corresponding to any set of continuous boundary values f , tends to $f(Q)$ at Q . Compare, for example, Wiener, 9, p. 128, or Kellogg, 4, p. 606. In 2, p. 326, Kellogg defines regularity (for three dimensions) by means of the Poincaré-Lebesgue barrier concept. As a consequence of a theorem due to Lebesgue, the complete proof of which can be found in the material in 2, pp. 326-328, and 4, pp. 607-609, the barrier definition is equivalent to that given above. The barrier definition is readily extensible to the plane and its equivalence to the sequence solution can be established.

and we should thus have, contrary to the definition of m , $U - V < m$ on some points of $C(P_0)$. The proof is now immediate; for, as a consequence of (3.21), we have

$$\lim_{P \rightarrow Q} \{U(P) - V(P)\} = \lim_{P \rightarrow Q} \{W(P) - V(P)\} = m \quad (P \text{ in } C(P_0)).$$

3.3. Returning now to the proof of Theorem V, we first observe that it is enough to consider the case in which T lies in the interior S of a circle s of diameter less than $1/2$. It is easily seen in fact, on applying a transformation of similitude, that, if the result holds in this case, then it holds in general.*

We let v denote the sequence solution of the Dirichlet problem for T , corresponding to the boundary values f on t ; and we let w denote either of the functions, $u - v$ or $v - u$. We observe that, as a consequence of our hypotheses and a familiar property† of the sequence solution, the lower bound m of w in T is finite. To prove the theorem we show that m cannot be negative. We assume the contrary, that m is negative, and arrive eventually at a contradiction.

Let e denote the set of points Q of t at which

$$(3.31) \quad \lim_{P \rightarrow Q} w(P) \leq m/2 \quad (P \text{ in } T).$$

On applying the preceding lemma we see that this set is not void. Plainly, it is bounded and closed. Further, its complement E with respect to the plane is a domain. Since E is open, and contains T , and T is connected, this will follow if we prove that, if Q is any point common to E and the complement of T , then Q can be joined to a point of T by a polygonal line not passing through e . For this, suppose first that Q is a point of $t - e$. In this case the conclusion is immediate; for then Q is at non-zero distance from e and at zero distance from T . Suppose on the other hand that Q is exterior to $T + t$. Let R be a point of t such that no other point of t lies nearer to Q than does R . Then R is regular for t ‡; and accordingly,

$$\lim_{P \rightarrow R} w(P) = 0 \quad (P \text{ in } T).$$

It follows that R is not a point of e . We conclude that Q can be joined to a

* This reduction is useful only in the plane case.

† The property in question is that the sequence solution lies between the extremes of the assigned boundary values. This results directly from the definition of the sequence solution. See, for example, Wiener, 10, p. 39, or Kellogg, 2, pp. 317-326. Kellogg's arguments are for three dimensions. Analogous arguments hold, however, in the plane.

‡ This is a consequence of any one of several criteria for regularity. For references see, for example, Wiener, 9, pp. 130 and 142.

point of T by a polygonal line of the required type. Accordingly, E is a domain.

We now form the conductor potential ξ of the set e .^{*} We first construct a sequence $\{E_n\}$, $n = 1, 2, \dots$, of normal, unbounded domains, nested in and approximating E .[†] In particular we construct these domains so that for each n the (finite) boundary e_n of E_n lies in S , and the set $E_n + e_n$ contains no point of e . Then, if T is a three-dimensional domain, we denote by ξ_n the function, harmonic in E_n , which vanishes at infinity and assumes continuously the boundary values 1 on e_n . On the other hand, if T is a plane domain, we first select a point O of e ; then denote by $\bar{\xi}_n$ the function, harmonic and bounded in E_n , which assumes continuously the values $\log OP$ on e_n ; and finally set

$$\xi_n(P) = \{a_n + \log OP - \bar{\xi}_n(P)\}/a_n,$$

where a_n is the value of $\bar{\xi}_n$ at infinity. In both instances we extend the definition of ξ_n over the points exterior to $E_n + e_n$ by defining it equal to 1 there. Then, at each point P of E the sequence $\{\xi_n(P)\}$ converges. The limit function is the conductor potential ξ .

In regard to ξ we now show that

$$(3.32) \quad w(P) \geq m\{1 + \xi(P)\}/2$$

for all P in T . Let n be any positive integer. We observe first that

$$(3.33) \quad \xi_n \leq 1$$

in S . This is clear in three dimensions. To see that it is true in the plane we have only to note that

$$a_n < \log 1/2 < 0,$$

as this implies that ξ_n becomes negatively infinite at infinity. We observe next that ξ_n is continuous in S . We see finally that ξ_n is superharmonic in S . In fact, if P is any point of $S \cdot E_n$, or of $S - S \cdot (E_n + e_n)$, then the value of ξ_n at P is equal to the arithmetic mean of its values on every sufficiently small circle about P as center. On the other hand, if P is a point of $S \cdot e_n$, then the value of ξ_n at P exceeds, as we see by applying (3.33), the arithmetic mean of its values on every sufficiently small circle about P as center. We conclude, as a consequence of a familiar theorem[‡] on superharmonic functions, that ξ_n is superharmonic in S .

^{*} In this connection see, for example, Wiener, 9, p. 142, and 10, p. 26, or for the three-dimensional case, Kellogg, 2, p. 330. Kellogg's treatment of the conductor potential can be extended to the plane.

[†] For such a construction see, for example, Kellogg, 2, pp. 317-323. The author is chiefly interested here in bounded three-dimensional domains. The reasoning, however, is applicable to plane and to unbounded domains.

[‡] See, for example, Kellogg, 2, p. 330.

Consider, then, the function

$$\alpha_n(P) = w(P) - m\{1 + \xi_n(P)\}/2.$$

The function $\pm u - m(1 + \xi_n)/2$ satisfies the conditions imposed upon U in our lemma. On the other hand, $\pm v$ satisfies the conditions imposed upon V . Accordingly, α_n tends to its lower bound m_n in T on a sequence of points $\{P_j\}$, $j=1, 2, \dots$, in T tending to a point Q of t . Now, if Q is a point of e , we have

$$^* \quad \lim_{j \rightarrow \infty} w(P_j) \geq m, \quad \lim_{j \rightarrow \infty} \{-m(1 + \xi_n)/2\} = -m.$$

On the other hand, if Q is a point of $t - e$, we have

$$\lim_{j \rightarrow \infty} w(P_j) \geq m/2,$$

and also, as we see on applying the fact that $\xi_n \geq 0$ in S^* ,

$$\lim_{j \rightarrow \infty} \{-m(1 + \xi_n)/2\} \geq -m/2.$$

We deduce that $m_n \geq 0$. Accordingly,

$$w \geq m(1 + \xi_n)/2$$

in T for every n . Allowing n to become infinite, we obtain (3.32).

The proof of the theorem can now easily be completed. We observe first that the capacity of e is 0.† In fact, if its capacity were positive, it would contain at least one point Q regular for the boundary of E .‡ But the point Q , being regular for the boundary of E , would be regular for t since T is contained in E and t contains Q .§ We should therefore have at a point of e

$$\lim_{P \rightarrow Q} w(P) = 0 \quad (P \text{ in } T);$$

and this is impossible. Thus the capacity of e is 0. But now, since the capacity

* It is plain that this inequality holds if T is a three-dimensional domain. To see that it holds in the plane, one need only apply the formulas given by Wiener in 9, p. 142. Essentially, it was in order to obtain this inequality that we reduced the problem in the beginning of the proof.

† For the definition of capacity see, for example, Wiener, 9, p. 143, and 10, p. 26, or Kellogg, 2, p. 330.

‡ The lemma that every bounded, closed set of positive capacity contains at least one point regular for the boundary of the unbounded domain bounded by the set is due in the plane to Kellogg, 5, and in space to Evans, 1. Evans' proof is valid in the plane.

§ This is immediate in view of the equivalence of the barrier definition to the sequence definition. See, for example, Kellogg, 2, p. 328. It also follows from Wiener's fundamental criterion on regular points. See Wiener, 9, pp. 130 and 142.

of e is 0, ξ vanishes identically in E^* and therefore in T . It follows from (3.32) that in T

$$w \geq m/2.$$

This gives us, as a consequence of the definition of m , the desired contradiction; and this completes the proof.

4.1. On a construction of the sequence solution of the Dirichlet problem. The reasoning of Theorem I is applicable in another connection. We close this paper in obtaining by means of it a theorem concerning a method by which the sequence solution of the Dirichlet problem can be constructed. This method was first considered by Lebesgue.[†] Lebesgue's results were later extended by Perkins.[‡]

In this theorem we shall suppose that the domains $C(P)$ satisfy the following conditions:

(c) if U is continuous in $T+t$, then $A\{U(P)\}$ is continuous in T ;

(d) if U is continuous in $T+t$, then

$$\lim_{P \rightarrow Q} A\{U(P)\} = U(Q) \quad (P \text{ in } T)$$

at every point Q of t .

We note that these conditions are equivalent to the following:

(e) if U is continuous in $T+t$, then the function $U_1(P)$, defined as $A\{U(P)\}$ in T , and as $U(P)$ on t , is continuous in $T+t$.

The theorem is then

THEOREM VI. Let the domains $C(P)$ satisfy conditions (c) and (d) above. Let u_0 be continuous in $T+t$; and let

$$(4.11) \quad u_n(P) = \begin{cases} A(u_{n-1}), & P \text{ in } T, \\ u_{n-1}(P), & P \text{ on } t, \end{cases} \quad n = 1, 2, \dots$$

Then at every point P of T the sequence $\{u_n(P)\}$ converges to the sequence solution $v(P)$, corresponding to the boundary values u_0 on t , of the Dirichlet problem for T . Further, the convergence is uniform on any closed subset of T .

In regard to the condition (d), we note that (d) is satisfied if the condition (a) of §2.4 holds. In fact, if U is continuous in $T+t$, we can, given P , select points P_1 and P_2 on $c(P)$, such that

$$U(P_1) \leq A\{U(P)\} \leq U(P_2).$$

* For the three-dimensional case see, for example, Kellogg, 3, p. 403. For both cases see Wiener, 9, p. 142, and 10, p. 26. Kellogg's proof can be extended to the plane.

† Lebesgue, 7.

‡ Perkins, 8.

If, now, (a) holds, and if P tends to a point Q of t , then P_1 and P_2 tend to Q , and $U(P_1)$ and $U(P_2)$ tend to $U(Q)$. Accordingly, $A\{U(P)\}$ tends to $U(Q)$. Thus, (d) can be replaced by (a) in our theorem.

Now, as pointed out before, (a) holds if for each point P of T , $C(P)$ is the interior of a circle about P . Moreover, if in addition we assume that the radius of $c(P)$ is a continuous function of P , then (c) holds. Thus it is enough to assume in the theorem that $C(P)$ is for each P the interior of a circle about P , and that the radius of $c(P)$ is a continuous function of P . A family of circles which satisfies this second condition is that in which the radius of $c(P)$ is, for each P , equal to the distance from P to t . It was with this family of circles that Lebesgue and Perkins were concerned. Lebesgue showed that, if T is a normal domain, the sequence defined in (4.11) converges to the solution, corresponding to the boundary values u_0 on T , of the Dirichlet problem for T . Perkins extended Lebesgue's result to an arbitrary domain, thereby obtaining the result of Theorem VI for Lebesgue's family of circles. In each case the method of proof is somewhat different from ours.

Another point which might be mentioned in connection with the above theorem is that, although we are apparently concerned only with the sequence solution corresponding to values on the boundary which are those of a function continuous throughout $T+t$, this is in reality the general case. Given a function continuous on t , we can, of course, always extend its definition so that the resulting function is continuous on $T+t$.

4.2. We first prove the theorem in the case that u_0 is a superharmonic polynomial. In this case we have $u_0 \geq A(u_0) = u_1 \geq m$, where m is the minimum of u_0 in $T+t$, and $u_n \geq u_{n+1} \geq m$ if $u_{n-1} \geq u_n \geq m$, $n = 1, 2, \dots$. Accordingly

$$(4.21) \quad u_0(P) \geq u_1(P) \geq \dots \geq m$$

and we can conclude at once that the sequence $\{u_n\}$ converges at each point P of $T+t$ to a limit $u(P)$. We have, then, to show that $u \equiv v$ in T , and that the convergence is uniform in any closed subset of T . Now, of these two propositions, the second follows immediately from the first. In fact, since the u_n are continuous in T and since (4.21) holds, the convergence, by a familiar theorem, is necessarily uniform in any closed subset of T if the limit function is continuous in T . Accordingly, our problem reduces to showing that $u \equiv v$ in T .

Let T_1, T_2, \dots be a set of normal domains nested in and approximating T . Let v_k be the solution of the Dirichlet problem for T_k , corresponding to the boundary values u_0 on t_k , the boundary of T_k . Let $v_k(P)$ be defined as equal to $u_0(P)$ for P exterior to T_k+t_k . Then v_k is continuous and superharmonic in the plane and we have

$$u_0(P) \geq v_k(P)$$

in T . Further, at each point P of T , the sequence $\{v_k\}$ converges to $v(P)$. We thus have

$$(4.22) \quad u_0(P) \geq v(P)$$

in T .

As a consequence of this last inequality and the lemma of §3.2, it is easily seen that

$$(4.23) \quad u - v \geq 0$$

in T . In fact, for any fixed integer $n > 0$, we have

$$u_n(P) = A(u_{n-1}) \geq A(u_n)$$

in T . Hence, since u_n is continuous in $T+t$ and v is harmonic in T , the function $u_n - v$ tends to its lower bound in T on a sequence of points $\{P_j\}$, $j=1, 2, \dots$, in T tending to a point Q of t . Now,

$$\lim_{j \rightarrow \infty} u_n(P_j) = u_0(Q), \quad \overline{\lim}_{j \rightarrow \infty} v(P_j) \leq u_0(Q),$$

the latter by (4.22). It follows that

$$u_n - v \geq 0$$

in T . Allowing n to become infinite, we see that (4.23) holds.

We have now only to prove that

$$(4.24) \quad v - u \geq 0$$

in T . For this we consider the function $v_k - u$ corresponding to some integer k . We shall show that $v_k - u$ assumes its lower bound m' in $T+t$ on a point of t . This will of course justify (4.24). In fact, we have

$$v_k - u = u_0 - u_0 = 0$$

on t , and

$$\lim_{k \rightarrow \infty} (v_k - u) = v - u$$

in T .

We first observe that, as a consequence of the continuity of the u_n and the fact that (4.21) holds, u is upper semi-continuous in $T+t$. Accordingly, since v_k is continuous in $T+t$, $v_k - u$ is lower semi-continuous there. It follows that the set λ of points in $T+t$, at which $v_k - u = m'$, is not void. To obtain our desired conclusion we prove first that, if P' is a point of $\lambda \cdot T$, then all the points of $c(P')$ are points of λ .

Let U_n , $n=0, 1, \dots$, be the function, harmonic in $C(P')$, continuous in $C(P') + c(P')$, which assumes the values u_n on $c(P')$. Let V be the function having the corresponding properties with regard to v_k . Then we have

$$U_0 \geq U_1 \geq \dots$$

in $C(P')$, and

$$\lim_{n \rightarrow \infty} U_n(P') = u(P') \geq m \quad (> -\infty).$$

It follows that the sequence $\{U_n\}$ converges in $C(P')$ to a function U harmonic in $C(P')$. Now, we have

$$V(P') - U(P') \leq v_k(P') - u(P') = m'.$$

Accordingly, either

$$(4.25) \quad V - U \equiv m'$$

in $C(P')$, or else there is a sequence of points $\{P_j\}$, $j=1, 2, \dots$, in $C(P')$, tending to a point Q of $c(P')$, such that

$$\lim_{j \rightarrow \infty} \{V(P_j) - U(P_j)\} = m' - a,$$

where a is positive. But, if the second of these two alternatives holds, we have

$$\begin{aligned} v_k(Q) - u_n(Q) &= \lim_{j \rightarrow \infty} \{V(P_j) - U_n(P_j)\} \\ (4.26) \quad &\leq \lim_{j \rightarrow \infty} \{V(P_j) - U(P_j)\} \\ &= m' - a \end{aligned}$$

for every integer $n \geq 0$. It follows that (4.25) holds; for otherwise, by (4.26), we should have

$$v_k(Q) - u(Q) < m',$$

contrary to the definition of m' . Let, then, Q be any point of $c(P')$. We have, on applying (4.25),

$$v_k(Q) - u_n(Q) = \lim_{P \rightarrow Q} \{V(P) - U_n(P)\} \leq m' \quad (P \text{ in } C(P'))$$

for every integer $n \geq 0$. It follows that $c(P')$ consists wholly of points λ .

We can readily deduce now that there is a point of λ on t . We have only to apply the reasoning of Theorem I. Since $v_k - u$ is lower semi-continuous in $T+t$ and since $T+t$ is closed, the set λ is closed. Hence, since λ is not void, and t is closed, there is a point P of λ whose distance to t is equal to the dis-

tance δ from λ to t . If, now, we assume that δ is positive we immediately get a contradiction; for there is a point of $c(P)$ nearer to t than is P and by our previous reasoning all the points of $c(P)$ are points of λ . The theorem for superharmonic polynomials is thus completely established.

4.3. Turning now to the general case, that in which u_0 is given as continuous on $T+t$, we let R denote a closed subset of T and ϵ an arbitrary positive number. To prove the theorem it is enough to show that

$$\overline{\lim}_{n \rightarrow \infty} |u_n - v| \leq \epsilon$$

uniformly in R .

Now, by the Weierstrass theorem, we can find a polynomial \bar{u} such that

$$|u_0 - \bar{u}| \leq \epsilon/2$$

everywhere in $T+t$. Next, we can write

$$\bar{u} = u_0' - u_0''$$

where u_0' and u_0'' are superharmonic polynomials. We set $u_0''' = u_0 - \bar{u}$ and consider the sequences $\{u_n'\}$, $\{u_n''\}$, $\{u_n'''\}$ built upon the continuous functions u_0' , u_0'' , u_0''' in the same way that $\{u_n\}$ is built upon u_0 .

We note first that

$$|u_n'''| \leq \epsilon/2$$

in $T+t$ since $|u_0'''| \leq \epsilon/2$ on t . We note next that, by the conclusion of the preceding paragraph, we have uniformly in R

$$\lim_{n \rightarrow \infty} u_n' = v', \quad \lim_{n \rightarrow \infty} u_n'' = v'',$$

where v' and v'' are the sequence solutions of the Dirichlet problem for T corresponding to the boundary values u_0' and u_0'' . Thus, since

$$u_n = u_n' - u_n'' + u_n''',$$

we have

$$\overline{\lim}_{n \rightarrow \infty} |u_n - v' + v''| \leq \epsilon/2$$

uniformly in R . But, denoting by v''' the sequence solution of the Dirichlet problem for T corresponding to the boundary values u_0''' , we have

$$v''' = v - v' + v'', \quad |v'''| \leq \epsilon/2$$

in T . We deduce that uniformly in R

$$\overline{\lim}_{n \rightarrow \infty} |u_n - v| \leq \overline{\lim}_{n \rightarrow \infty} |u_n - v' + v'''| + |v'''| \leq \epsilon.$$

This completes the proof.

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CONTRIBUTIONS TO THE THEORY OF FINITE FIELDS*

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The present paper contains a number of results in the theory of finite fields or higher congruences. The method may be considered as an application of the theory of p -polynomials, which I have developed in a recent paper† *On a special class of polynomials*. In this special case the p -polynomials form a commutative ring. However, this paper may be read without reference to the former investigations and one may say that the method applied is the representation of the finite field in its group ring. It should be mentioned at this point that a number of the results have direct applications in the theory of algebraic numbers.

In chapter 1 the special properties of the p -polynomials with coefficients in a finite field have been derived and the main results are the theorems that every p -polynomial has primitive roots and that every p -modulus is simple. A corollary is the theorem of Hensel, that every finite field has a basis consisting of conjugate elements. Through the introduction of a symbolic multiplication of elements in a p -modulus we make every such modulus a ring usually containing divisors of zero. The results of this first chapter I have previously given without proofs.‡

In chapter 2 various theorems of decomposition and theorems on prime polynomials belonging to a product of p -polynomials have been derived. Theorems 4 and 5 seem to be the most interesting of the results. In the next chapter these results are applied to the construction of irreducible polynomials. Theorem 1 gives a general type of irreducible polynomials. Next the complete prime polynomial decomposition of the simplest p -polynomials are given, and it is shown how most known irreducible polynomials (mod p) can be obtained in this way, thus obtaining a unified method for deriving various formerly known results. In the last paragraph one finds a new class of irreducible polynomials closely related to the linear fractional substitutions. The last chapter contains a few rudiments of the theory of finite fields considered as cyclic fields and also a particularly simple proof for the general law of reciprocity.

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† These Transactions, vol. 35 (1933), pp. 559-584.

‡ O. Ore, *Einige Untersuchungen über endliche Körper*, Proceedings 7th Scandinavian Mathematical Congress, Oslo, 1930, pp. 65-67.

CHAPTER 1. THEOREMS ON FINITE FIELDS

1. **Fundamental properties of p -polynomials.** In the following, polynomials with rational integral coefficients will be studied for a rational prime modulus p ; since almost all congruences occurring in this paper are taken with respect to this modulus, we shall, when no ambiguity is to be feared, replace congruences (mod p) by equalities.

A polynomial of the form

$$(1) \quad F(x) = a_0x^{p^n} + a_1x^{p^{n-1}} + \cdots + a_{n-1}x^p + a_nx$$

shall be called a p -polynomial. $F(x)$ is *reduced* when $a_0=1$. The polynomial

$$(2) \quad f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

is called the *polynomial corresponding to $F(x)$* ; the degree n of $f(x)$ is called the *exponent of $F(x)$* .

The system of all p -polynomials forms a modulus, but not a ring, since the ordinary product of two p -polynomials is not a p -polynomial. One finds, however, that the p th power of a p -polynomial (mod p) is again a p -polynomial; this shows that if

$$(3) \quad G(x) = b_0x^{p^m} + b_1x^{p^{m-1}} + \cdots + b_{m-1}x^p + b_mx$$

is a second p -polynomial with the corresponding polynomial

$$(4) \quad g(x) = b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m,$$

then the result of substituting $F(x)$ in $G(x)$ is also a p -polynomial $G(F(x))$. We therefore are led to the definition of a symbolic multiplication

$$(5) \quad G(x) \times F(x) = G(F(x)),$$

and a simple investigation of the symbolic product gives the following results:

THEOREM 1. *The symbolic multiplication is commutative and distributive and the polynomial corresponding to a symbolic product is equal to the product of the corresponding polynomials of the symbolic factors.*

If consequently

$$F_1(x), \dots, F_r(x)$$

are p -polynomials with the corresponding polynomials

$$f_1(x), \dots, f_r(x),$$

then the symbolic product

$$\Pi(x) = F_1(x) \times \cdots \times F_r(x)$$

has the corresponding polynomial

$$\pi(x) = f_1(x) \cdots f_r(x).$$

We shall say that $P(x)$ is a *symbolic prime polynomial*, when it is reduced and no symbolic decomposition $P(x) = A(x) \times B(x)$ exists except when one of the factors has the exponent zero. One could also have used the correspondence stated in Theorem 1 and defined $P(x)$ as a prime p -polynomial when the corresponding polynomial $p(x)$ is irreducible (mod p). This correspondence immediately shows

THEOREM 2. *The decomposition of a p -polynomial in symbolic prime factors is unique.*

One could also have concluded this from the fact that there exists a Euclid algorithm for the symbolic multiplication. When two p -polynomials $A(x)$ and $B(x)$ are given, one can find two others $Q(x)$ and $R(x)$ such that

$$(6) \quad A(x) = B(x) \times Q(x) + R(x)$$

where the exponent of $R(x)$ is smaller than the exponent of $B(x)$. From (6) the existence of a Euclid algorithm follows; there exists a greatest common symbolic factor for any two or more p -polynomials. When $A(x)$ and $B(x)$ have only the trivial symbolic common factor x , we say that $A(x)$ and $B(x)$ are *symbolically relatively prime*.

It should be observed that when $A(x)$ is symbolically divisible by $D(x)$, then $A(x)$ is also divisible by $D(x)$ in the ordinary sense and conversely. From $A(x) = Q(x) \times D(x)$ follows, namely, when $Q(x) = q_1(x) \cdot x$, that $A(x) = q_1(D(x)) \cdot D(x)$. On the other hand, let $A(x)$ be divisible by $D(x)$ in the ordinary sense; one can divide $A(x)$ symbolically by $D(x)$ and obtain

$$(7) \quad A(x) = Q(x) \times D(x) + R(x) = q_1(D(x)) \cdot D(x) + R(x).$$

Here the degree of $R(x)$ is smaller than the degree of $D(x)$, and the second equation (7) shows that $R(x) = 0$. This reasoning also shows that the symbolic Euclid algorithm will contain the same residues as the ordinary Euclid algorithm. One obtains in particular

THEOREM 3. *The greatest common symbolic factor of two p -polynomials is the same as the ordinary greatest common factor of the p -polynomials.*

When therefore $A(x)$ and $B(x)$ are symbolically relatively prime, then the ordinary greatest common factor of $A(x)$ and $B(x)$ is x and conversely. Let us also observe that in this case one can determine two p -polynomials $X(x)$ and $Y(x)$ such that

$$(8) \quad X(x) \times A(x) + Y(x) \times B(x) = x.$$

The p -polynomials of the greatest interest in the following are the well known

$$(9) \quad F_n(x) = x^{p^n} - x$$

with the corresponding polynomial

$$(10) \quad f_n(x) = x^n - 1.$$

Theorem 1 shows

THEOREM 4. *When $f_n(x)$ has the ordinary prime factor decomposition*

$$(11) \quad x^n - 1 = \phi_1(x) \cdots \phi_r(x)$$

then $F_n(x)$ has the symbolic prime factor decomposition

$$(12) \quad x^{p^n} - 1 = \Phi_1(x) \times \cdots \times \Phi_r(x),$$

where $\phi_i(x)$ ($i=1, 2, \dots, r$) is the polynomial corresponding to $\Phi_i(x)$.

2. The roots of p -polynomials. We shall now discuss the properties of the roots of a p -polynomial $F(x)$ defined by (1). Since $F(x)$ can be represented as the ordinary product of prime factors, it is obvious that the roots will belong to some finite field K . For a p -polynomial one has $F'(x) = a_n$ and this shows that $F(x)$ can only have equal roots when $a_n = 0$; this case will always be excluded in the following considerations.

Let μ and ν be roots of

$$(13) \quad F(x) = 0;$$

due to the special form of a p -polynomial one sees that $\mu \pm \nu$ is also a root of (13) and, furthermore, that the p th power μ^p will also be a root.

We shall say that a finite modulus M_p is a p -modulus, if it has the property that the p th power of every element is contained in it. This definition implies that every p -modulus lies in some finite field. We can now show

THEOREM 5. *The roots of a p -polynomial form a p -modulus and every p -modulus is the set of roots of a p -polynomial.*

The first part of the theorem follows from the remarks made above. Since a p -modulus M_p always has elements in some finite field, and since μ^p for each μ is the conjugate of μ it follows that the totality of elements of M_p will satisfy an equation with rational coefficients. In order to show that this is a p -polynomial, let

$$M_p = r_1\mu_1 + \cdots + r_n\mu_n \quad (r_i = 0, 1, \dots, p-1)$$

be a representation of M_p by a basis. Then all elements of M_p are seen to satisfy the equation

$$F(x) = \begin{vmatrix} \mu_1 & \cdots & \mu_n & x \\ \mu_1^p & \cdots & \mu_n^p & x^p \\ \vdots & \vdots & \vdots & \vdots \\ \mu_1^{p^n} & \cdots & \mu_n^{p^n} & x^{p^n} \end{vmatrix} = 0$$

and the fact that the elements μ_1, \dots, μ_n form a basis shows that the highest coefficient does not vanish.

Theorem 5 gives a correspondence between p -moduli and p -polynomials; we shall derive a few simple consequences. Let $F(x)$ be a p -polynomial with the p -modulus M_p ; when $F(x)$ is symbolically reducible,

$$F(x) = F_1(x) \times F_2(x) = F_1(F_2(x)),$$

it follows that M_p must contain as a sub-modulus the roots M_p' of $F_1(x)$ (or $F_2(x)$). Conversely, if M_p contains a sub- p -modulus M_p' corresponding to a p -polynomial $F_1(x)$, then according to §1, $F_1(x)$ must divide $F(x)$ both in the ordinary and in the symbolic sense.

THEOREM 6. *The necessary and sufficient condition that $F(x)$ be symbolically reducible is that its p -modulus M_p contain a sub- p -modulus.*

We shall say that M_p is a *prime p -modulus*, when it contains no sub- p -modulus except the zero modulus. The necessary and sufficient condition that M_p be prime is that the corresponding p -polynomial be symbolically irreducible. When M_p and N_p are two p -moduli corresponding to $F(x)$ and $G(x)$, it is easily seen that $M_p + N_p$ is also a p -modulus corresponding to the least common multiple $[F(x), G(x)]$, and that the cross-cut (M_p, N_p) is a p -modulus corresponding to the greatest common factor $(F(x), G(x))$.

Now let μ be an arbitrary element of a finite field; all elements of the form

$$(14) \quad S_p = k_0\mu + k_1\mu^p + k_2\mu^{p^2} + \cdots$$

obviously form a p -modulus and a p -modulus generated in this way by a single element shall be called *simple*. There must exist a smallest exponent a such that a relation

$$\mu^{p^a} + m_1\mu^{p^{a-1}} + \cdots + m_{a-1}\mu^p + m_a\mu = 0$$

holds, and the elements of the simple p -modulus (14) can then be represented uniquely in the form

$$(15) \quad S_p = k_0\mu + k_1\mu^p + \cdots + k_{a-1}\mu^{p^{a-1}}.$$

From the definition of a prime p -modulus it follows that every prime p -modulus is simple. It is one of the main results of this theory that

THEOREM 7. *Every p -modulus is simple.*

This theorem will be proved in §4.

3. **Polynomials belonging to a p -polynomial.** Let $\phi(x)$ be an arbitrary polynomial of degree m ; it will be shown that $\phi(x)$ always divides a p -polynomial $F(x)$. In order to find the p -polynomial $F(x)$ of smallest degree having this property, we divide the successive powers x^{p^i} by $\phi(x)$ and obtain a set of congruences

$$(16) \quad x^{p^i} \equiv a_1^{(i)} x^{m-1} + \cdots + a_m^{(i)} \pmod{\phi(x)} \quad (i = 0, 1, \dots).$$

Through linear elimination one can obtain a relation $\pmod{\phi(x)}$ between the powers x^{p^i} , eliminating $1, x, x^2, \dots, x^{m-1}$ from the right-hand side of (16). If $\nu+1$ is the first index such that there exists a linear homogeneous relation between the first $\nu+1$ polynomials on the right-hand side, then $\phi(x)$ will divide a p -polynomial $F(x)$ with the exponent ν . The construction of $F(x)$ shows that it is the p -polynomial with smallest exponent divisible by $\phi(x)$ and we shall say that $\phi(x)$ belongs to $F(x)$. The following is then easily seen:

THEOREM 8. *Every polynomial $\phi(x)$ of degree m belongs to a unique p -polynomial $F(x)$ with the exponent $\nu \leq m$. Every p -polynomial divisible by $\phi(x)$ is symbolically divisible by $F(x)$.*

Let next $F(x)$ be an arbitrary p -polynomial without equal roots, and let $f(x)$ be the corresponding polynomial. Since each prime factor of $f(x)$ divides some $x^n - 1$, it follows that there exists a smallest exponent N such that $x^N - 1$ is divisible by $f(x)$. This gives, when applied to $F(x)$,

THEOREM 9. *There exists for each p -polynomial $F(x)$ without equal roots a smallest number N such that*

$$(17) \quad x^{p^N} - x = G(x) \times F(x).$$

We shall call N the *index* of $F(x)$; every irreducible ordinary factor of $F(x)$ has then a degree dividing the index.

Since every polynomial belongs to some p -polynomial, it follows, in particular, that every prime polynomial $\phi(x)$ belongs to some $F(x)$, and it is easily seen that one can assume that $F(x)$ has no equal roots. The degree N' of $\phi(x)$ is then a divisor of the index N of $F(x)$, according to (17). On the other hand, $\phi(x)$ is a divisor of the p -polynomial

$$F_{p^{N'}}(x) = x^{p^{N'}} - x,$$

and $F(x)$ is therefore also a symbolic divisor of the p -polynomial $F_{p^{N'}}(x)$. This shows, conversely, that N is a divisor of N' , and we obtain

THEOREM 10. *An irreducible polynomial of degree N belongs to a p -polynomial with the index N , and conversely, every irreducible ordinary factor belonging to a p -polynomial with the index N has the degree N .*

At the close of these considerations I should like to make another observation. When one wishes to find the prime function decomposition (mod p) of an ordinary polynomial $f(x)$, one usually determines the smallest exponent N such that $f(x)$ divides $x^{p^N} - x$.* In order to obtain this, one can construct the system of congruences (16); instead of continuing the divisions until

$$x^{p^N} \equiv x \pmod{f(x)},$$

it is usually simpler to eliminate the powers of x on the right-hand side and find the p -polynomial $\Phi(x)$ which $f(x)$ divides. When $\Phi(x)$ corresponds to $\phi(x)$ it is only necessary to find the N for which $\phi(x)$ divides $x^N - 1$.

4. **Primitive roots.** The problem now naturally arises to find the number of irreducible polynomials belonging to a given p -polynomial $F(x)$. When $F(x)$ has the exponent N , these polynomials are all of degree N . One may state the problem in a somewhat different form. We shall say that a root μ is a *primitive root* of $F(x) = 0$ when it satisfies no p -equation of lower degree. Our problem is then equivalent to the determination of the primitive roots. Now let

$$(18) \quad F(x) = \Phi_1(x)^{e_1} \times \dots \times \Phi_r(x)^{e_r}$$

be the symbolic prime function decomposition of $F(x)$, in which the exponents signify the repetition of equal factors; the exponent of $\Phi_i(x)$ is m_i . The primitive roots of $F(x)$ are obtained when one omits all the roots of the polynomials $F(x) \times \Phi_i(x)^{-1}$ and a common argument in number theory shows that

$$(19) \quad \begin{aligned} N_F &= p^m - \sum_i p^{m-m_i} + \sum_{i,j} p^{m-m_i-m_j} + \dots \\ &= p^m \left(1 - \frac{1}{p^{m_1}}\right) \dots \left(1 - \frac{1}{p^{m_r}}\right) \end{aligned}$$

represents the number of primitive roots.

The expression (19) can also be interpreted in a different way. Let $f(x)$ be the polynomial corresponding to $F(x)$; then according to (18)

$$(20) \quad f(x) = \phi_1(x)^{e_1} \dots \phi_r(x)^{e_r}$$

* See for instance A. Arwin, *Über Kongruenzen von dem fünften und höheren Graden nach einem Primzahlmodulus*, Arkiv för Matematik, Astronomi och Fysik, vol. 14 (1918).

is the prime polynomial decomposition of $f(x)$. Now let $\Phi(f(x))$ denote the number of residues (mod p , $f(x)$) which are relatively prime to $f(x)$; one finds then for this generalized Φ -function exactly the expression (19). This gives

THEOREM 11. *When the p -polynomial $F(x)$ with the corresponding polynomial $f(x)$ has the symbolic prime function decomposition (18), then $F(x)$ has exactly*

$$(21) \quad \Phi(f(x)) = p^m \left(1 - \frac{1}{p^{m_1}}\right) \cdots \left(1 - \frac{1}{p^{m_r}}\right)$$

primitive roots; here the m_i denote the exponents of the different prime factors of $F(x)$.

This theorem permits a series of applications. It shows the following, first of all:

There exist primitive roots for all p -polynomials.

Furthermore:

The number of irreducible polynomials belonging to $F(x)$ is $(1/N)\Phi(f(x))$, where N is the index of $F(x)$.

Since there always exist prime functions of degree N dividing $F(x)$, it follows that every p -polynomial has the following property in common with $x^{p^N} - x$:

The degrees of the ordinary irreducible factors of a p -polynomial always divide the degree of the prime divisor of highest degree.

Since every p -modulus M_p forms the set of roots of a p -polynomial $F(x)$, and since $F(x)$ has primitive roots, it follows that M_p can be generated in the form (15) by a primitive root of $F(x)$. This gives the proof of Theorem 7:

Every p -modulus is simple.

An important special case is the case where the p -modulus is a finite field with p^n elements; the corresponding p -polynomial is then $x^{p^n} - x$. Theorem 11 shows that there exist $\Phi(x^n - 1)$ numbers μ such that every element can be represented in the form

$$\omega = a_0\mu + a_1\mu^p + \cdots + a_{n-1}\mu^{p^{n-1}}.$$

We have therefore proved

THEOREM 12. *In a finite field of degree n there exist $(1/n)\Phi(x^n - 1)$ different bases consisting of conjugate elements:*

$$\mu, \mu^p, \cdots, \mu^{p^{n-1}}.$$

Theorem 12 gives the answer to a problem proposed already by Eisenstein*, and partly solved by Schönemann.† The first complete solution was given by Hensel‡; it should also be observed that the existence of such a basis is a consequence of a much more general theorem by Noether and Deuring§, proving the existence of a basis consisting of conjugate elements for an arbitrary Galois field.

5. Symbolic multiplication. Let $F(x)$ be a p -polynomial with the exponent n , $f(x)$ the corresponding polynomial and M_p the p -modulus of the roots. All elements of M_p are then of the form

$$(22) \quad Q(\mu) = a_0\mu + \cdots + a_{n-1}\mu^{p^{n-1}}$$

where μ is a primitive root. The number $Q(\mu)$ belongs to some divisor $F_1(x)$ of $F(x)$ and this divisor can easily be found. If namely $F(x) = F_1(x) \times F_2(x)$ and $F_1(Q(\mu)) = 0$, then one must have $F_1(x) \times Q(x) \equiv 0 \pmod{F(x)}$ or $Q(x) \equiv 0 \pmod{F_2(x)}$, and one finds

THEOREM 13. When $F_1(x) \times F_2(x) = F(x)$, then an element (22) in M_p belongs to $F_1(x)$ if and only if

$$Q(x) = Q_1(x) \times F_2(x),$$

where $Q_1(x)$ is relatively prime to $F_1(x)$.

The primitive elements of M_p consequently consist of those $Q(\mu)$ for which $Q(x)$ is relatively prime to $F(x)$.

The existence of a primitive element also permits us to introduce a symbolic multiplication in a p -modulus and make the p -modulus a ring; and this can even be done in several ways. Let μ as formerly be a primitive element; to define the product of two elements

$$\alpha = A(\mu), \quad \beta = B(\mu),$$

we put

$$(23) \quad \alpha \times \beta = \beta \times \alpha = [A(x) \times B(x)]_{x=\mu}.$$

This product is associative, distributive and commutative; it should be observed that the definition (23) depends essentially upon the choice of the primitive element μ , because μ must be the unit element of the symbolic mul-

* G. Eisenstein, *Über irreduzible Kongruenzen*, Journal für Mathematik, vol. 39 (1850), p. 182.

† Schönemann, *Über einige von Herrn Dr. Eisenstein aufgestellte Lehrsätze etc.*, Journal für Mathematik, vol. 40 (1850), pp. 185-187.

‡ K. Hensel, *Über die Darstellung der Zahlen eines Gattungsbereiches für einen beliebigen Primdivisor*, Journal für Mathematik, vol. 103 (1888), pp. 230-237.

§ M. Deuring, *Galoissche Theorie und Darstellungstheorie*, Mathematische Annalen, vol. 107 (1932), pp. 140-144.

tiplication. The ring M_p defined by a particular μ is seen to be isomorphic to the ring of all residue-classes $(\text{mod } f(x))$, where $f(x)$ is the polynomial corresponding to $F(x)$; M_p is a field only when $F(x)$ is symbolically irreducible. When applied to a finite field, one obtains in particular

THEOREM 14. *Let μ be a primitive element in a finite field K_μ , such that the conjugates of μ form a basis. Each element in K_μ is then a p -polynomial in μ and the symbolic multiplication of these p -polynomials introduces a new definition of multiplication in K_μ . With regard to this multiplication K_μ is a ring isomorphic to the ring of residue-classes for the double modulus $(\text{modd } p, x^n - 1)$.*

Now let $F(x)$ be a p -polynomial with the symbolic prime polynomial decomposition

$$(24) \quad F(x) = \Phi_1(x)^{(e_1)} \times \cdots \times \Phi_r(x)^{(e_r)}$$

and let us put

$$A_i(x) = F(x) \times \Phi_i(x)^{(-e_i)} \quad (i = 1, 2, \dots, r).$$

The primitive roots of $\Phi_i(x)^{(e_i)} = 0$ are then $Q_i(\mu) \times A_i(\mu)$, where $Q_i(x)$ is not divisible by $\Phi_i(x)$, and where μ as before denotes a primitive root of $F(x) = 0$. Every root ω of $F(x)$ is representable uniquely in the form

$$\omega = \sum_{i=1}^r R_i(\mu) \times A_i(\mu),$$

where the degree of $R_i(x)$ is smaller than the degree of $\Phi_i(x)^{(e_i)}$. This shows that each root is uniquely representable in the form

$$\omega = \mu_1 + \mu_2 + \cdots + \mu_r,$$

where μ_i is a root of

$$(25) \quad \Phi_i(x)^{(e_i)} = 0.$$

The root ω is primitive if and only if all μ_i are primitive roots of their corresponding equations (25).

Now let

$$(26) \quad G(x) = \Phi_1(x)^{(f_1)} \times \cdots \times \Phi_r(x)^{(f_r)}$$

be a second p -polynomial and

$$v = v_1 + v_2 + \cdots + v_r, \quad \Phi_i(v_i)^{(f_i)} = 0,$$

the representation of one of its primitive roots. The number

$$\mu \pm v = (\mu_1 \pm v_1) + \cdots + (\mu_r \pm v_r)$$

is then a root of the union $[F(x), G(x)]$. When for an index i we have $e_i > f_i$, then the element $\lambda_i = \mu_i \pm \nu_i$ is a primitive root of $\Phi_i(x)^{(e_i)} = 0$ as one easily sees, and correspondingly for $f_i > e_i$. When $e_i = f_i$ it may happen, however, that λ_i is not a primitive root, but when $p \neq 2$ it is always possible even to a fixed μ_i to choose a ν_i such that λ_i is a primitive root, for instance $\nu_i = \pm \mu_i$. When $p = 2$ and $\Phi_i(x)$ has the exponent 1, one finds that no primitive root ν_i with the property indicated exists.

THEOREM 15. *Let $F(x)$ be two p -polynomials with the symbolic prime polynomial decompositions (24) and (26), and let μ and ν be two primitive roots. When for all i $e_i \neq f_i$, then $\lambda = \mu \pm \nu$ is a primitive root of the least common multiple $[F(x), G(x)]$. If $e_i = f_i$ for some i and $p \neq 2$, one can always to every primitive μ find a primitive ν such that λ is a primitive root of the union.*

6. p' -polynomials. We shall finally make a slight generalization of the former theory by considering p' -polynomials

$$F(x) = \alpha_0 x^{p^n} + \alpha_1 x^{p^{n-1}} + \cdots + \alpha_{n-1} x^{p'} + \alpha_n x,$$

where the coefficients α_i are elements of a finite field K_f of degree f . The polynomial corresponding to $F(x)$ is

$$F(x) = \alpha_0 x^n + \alpha_1 x^{n-1} + \cdots + \alpha_{n-1} x + \alpha_n.$$

One can define the symbolic multiplication by substitution as in §1, and one finds that the symbolic multiplication is commutative and that the polynomial corresponding to a symbolic product is equal to the product of the corresponding factors; Theorems 2 and 3 also hold without change.

The decomposition of

$$x^{p^n} - x$$

into p' -factors corresponds uniquely to the decomposition of $x^n - 1$ into irreducible factors in K_f .

The roots of a p' -polynomial form a p' -modulus, i.e., a modulus with the following properties:

1. When μ belongs to $M_{p'}$, then $\alpha\mu$ also belongs for all elements α of K_f .
2. When μ belongs to $M_{p'}$, then $\mu^{p'}$ also belongs. Every p' -modulus forms the set of roots of a p' -polynomial.

One finds that every polynomial with coefficients in K_f belongs to a p' -polynomial. The smallest exponent N such that $F(x)$ divides

$$x^{p^N} - x$$

is called the index of $F(x)$, and Theorem 10 will hold unchanged. One can then prove the existence of primitive roots for a p' -polynomial and obtain

similar formulas for their number. It follows that every p' -modulus is simple and can be represented in the form

$$M_{p'} = \alpha_0 \mu + \alpha_1 \mu^{p'} + \dots + \alpha_{n-1} \mu^{p^{f(n-1)}}.$$

When applied to a finite field of degree ff' this gives

THEOREM 16. *In a finite field $K_{ff'}$ of degree ff' one can always find bases with respect to K_f consisting of conjugate elements*

$$\mu, \mu^{p'}, \dots, \mu^{p^{f(f'-1)}}.$$

The analogue of Theorem 15 can easily be deduced.

CHAPTER II. DECOMPOSITION THEOREMS

1. Identities for $x^{p^n} - x$. Let $F(x)$ and $G(x)$ be two p -polynomials, and let α be an arbitrary root of $F(x) = 0$ and β an arbitrary root of $G(x) = 0$. From the definition of the symbolic multiplication it follows that the following identities must hold:

$$(1) \quad F(x) \times G(x) = \prod_{\beta} (F(x) - \beta) = \prod_{\alpha} (G(x) - \alpha).$$

This simple remark gives, when applied to $x^{p^n} - x$,

THEOREM 1. *Let $f(x)$ and $g(x)$ be two complementary divisors of $x^n - 1$ such that*

$$(2) \quad x^n - 1 = f(x)g(x),$$

and let $F(x)$ and $G(x)$ be the corresponding p -polynomials. Then

$$(3) \quad x^{p^n} - x = \prod_{\beta} (F(x) - \beta) = \prod_{\alpha} (G(x) - \alpha)$$

where α runs through all the roots of $F(x) = 0$ and β through all the roots of $G(x) = 0$.

Using p' -polynomials one obtains a similar theorem for the decomposition of $x^{p^f n} - x$. Since $x^n - 1$ always has the two factors

$$f(x) = x^{n-1} + \dots + x + 1, \quad g(x) = x - 1,$$

one obtains as a special case of the decomposition (3) the decompositions given by Mathieu*:

* E. Mathieu, *Mémoire sur l'étude de fonctions de plusieurs quantités etc.*, Journal de Mathématiques, (2), vol. 6 (1861), pp. 241-323.

$$\begin{aligned}
 (4) \quad x^{p^n} - x &= \prod_{a=0}^{p-1} (x^{p^{n-1}} + \cdots + x^p + x + a) \\
 &= \prod_{\beta} (x^p - x - \beta),
 \end{aligned}$$

where β runs through all solutions of

$$(5) \quad x^{p^{n-1}} + x^{p^{n-2}} + \cdots + x^p + x = 0.$$

When p' -polynomials are applied one obtains

$$\begin{aligned}
 (6) \quad x^{p'^n} - x &= \prod_{\alpha} (x^{p'^{n-1}} + \cdots + x^{p'} + x + \alpha) \\
 &= \prod_{\beta} (x^{p'} - x - \beta),
 \end{aligned}$$

where α runs through all elements of K_f , while β runs through the roots of

$$(7) \quad x^{p'^{n-1}} + x^{p'^{n-2}} + \cdots + x^{p'} + x = 0.$$

The significance of the conditions (5) and (7) is seen to be the following: When β is a root of (7) it is an element of the finite field K_{n_f} of relative degree n with respect to K_f , and it therefore satisfies an irreducible equation in K_f of degree n_{β} , where n_{β} divides n . When $\alpha_1^{(\beta)}$ denotes the coefficient of $x^{n_{\beta}-1}$ in this equation, one finds

$$\beta^{p'^{n-1}} + \cdots + \beta^{p'} + \beta = -\frac{n}{n_{\beta}} \alpha_1^{(\beta)},$$

and the condition (7) is equivalent to

$$(8) \quad \frac{n}{n_{\beta}} \alpha_1^{(\beta)} \equiv 0 \pmod{p},$$

or simply $\alpha_1^{(\beta)} \equiv 0 \pmod{p}$ when n is not divisible by p .

2. Decomposition theorems. The object of the following considerations is to give a method to determine the prime polynomials belonging to a product $F(x) = F_1(x) \times F_2(x)$ of two p' -polynomials, when the prime factors of $F_1(x)$ and $F_2(x)$ are known. According to (1) we have the decomposition

$$(9) \quad F(x) = \prod_{\alpha_1} (F_1(x) - \alpha_2) = \prod_{\alpha_1} (F_2(x) - \alpha_1),$$

where α_1 and α_2 run through the roots of $F_1(x) = 0$ and $F_2(x) = 0$ respectively. Each root of $F(x)$ then satisfies an equation

$$(10) \quad F_2(x) = \alpha_1.$$

We shall determine all equations (10) satisfied by primitive roots of $F(x)$;

first, it is obvious that a primitive root can only satisfy (10) when α_1 is a primitive root of $F_1(x)$. Next let μ be a primitive root of $F(x)$; then according to Theorem 16, α_1 must have the form $\alpha_1 = Q(\mu) \times F_2(\mu)$, where $Q(x)$ is relatively prime to $F_1(x)$; when $R(\mu)$ denotes an arbitrary root of (10), then one obtains

$$F_2(x) \times (R(x) - Q(x)) \equiv 0 \pmod{F(x)}$$

or

$$(11) \quad R(x) = Q(x) - K(x) \times F_1(x).$$

The relation (11) gives the general form of a root of (10), including also the case where α_1 is not a primitive root of $F_1(x)$.

Let us next write

$$(12) \quad F_1(x) = G_1(x) \times D_1(x), \quad F_2(x) = G_2(x) \times D_2(x),$$

where $G_1(x)$ and $G_2(x)$ are relatively prime and $D_1(x)$ and $D_2(x)$ contain only prime factors which are common to $F_1(x)$ and $F_2(x)$. When $Q(x)$ is relatively prime to $F_1(x)$ it follows from (11) that any common factor of $R(x)$ and $F(x)$ must be a divisor $\bar{G}_2(x)$ of $G_2(x)$, and this shows that every root of (10) belongs to a polynomial

$$(13) \quad \bar{G}_2(x) \times D_2(x) \times F_1(x),$$

where $G_2(x) = \bar{G}_2(x) \times \bar{G}_2(x)$.

In order to determine the exact number of roots of (10) belonging to a given polynomial (13), we observe that $R(x)$ must be of the form $R_1(x) \times \bar{G}_2(x)$, where $R_1(x)$ is relatively prime to (13); comparing this with (11) one finds

$$(14) \quad R_1(x) \times \bar{G}_2(x) + K(x) \times F_1(x) = Q(x)$$

and our problem is equivalent to the determination of the number of solutions $R_1(x)$ of degree less than the degree of (13) and relatively prime to this polynomial, i.e., relatively prime to $\bar{G}_2(x)$ since no solution of (14) can have a factor in common with $F_1(x)$. Since $\bar{G}_2(x)$ is relatively prime to $F_1(x)$, it follows that (14) has a special solution $R_1^{(0)}(x)$ such that the general solution is

$$(15) \quad R_1(x) = R_1^{(0)}(x) + M(x) \times F_1(x),$$

where $M(x)$ is an arbitrary polynomial whose degree is smaller than the degree of $\bar{G}_2(x) \times D_2(x)$. The total number of polynomials $M(x)$ is then p^{f^*} , where $f^* = f(\bar{g}_2 + d_2)$ and where \bar{g}_2 and d_2 are the exponents of $\bar{G}_2(x)$ and $D_2(x)$. One finds by the usual argument in number theory that the number of solutions of (15) which are relatively prime to $\bar{G}_2(x)$ will be

$$(16) \quad N = p^{f d_2} \Phi(\bar{g}_2(x)),$$

where $\bar{g}_2(x)$ is the polynomial corresponding to $\bar{G}_2(x)$ and Φ is the generalized Euler function introduced in §4, chapter 1. A well known property of the Φ -function shows that the sum of all numbers (16) taken over all divisors $\bar{g}_2(x)$ of $g_2(x)$ is equal to the degree of $F_2(x)$ as one should expect.

THEOREM 2. *Let $F(x) = F_1(x) \times F_2(x)$ be the product of two p^f -polynomials and*

$$F(x) = \prod_{\alpha_1} (F_2(x) - \alpha_1)$$

the corresponding decomposition, where α_1 runs through all roots of $F_1(x)$. The primitive roots of $F(x)$ are roots of the equations

$$(17) \quad F_2(x) = \alpha_1,$$

where α_1 runs through the primitive roots of $F_1(x)$. When

$$F_1(x) = G_1(x) \times D_1(x), \quad F_2(x) = G_2(x) \times D_2(x),$$

where $D_1(x)$ and $D_2(x)$ contain the prime factors which are common to $F_1(x)$ and $F_2(x)$, then every root of (17) belongs to a polynomial

$$(18) \quad \bar{D}(x) \times D_2(x) \times F_1(x),$$

where $\bar{D}(x)$ is a divisor of $G_2(x)$. The exact number of roots belonging to a given polynomial (18) is

$$(19) \quad N(\bar{D}) = p^{f d_2} \Phi(\bar{d}(x)),$$

where d_2 is the exponent of $D_2(x)$ and $\bar{d}(x)$ the polynomial corresponding to $\bar{D}(x)$.

This theorem shows, in particular, that the number of roots of the various categories of an equation (17) is the same for all primitive α_1 and the number of primitive roots is $p^{f d_2} \Phi(g_2(x))$, where $g_2(x)$ is the polynomial corresponding to $G_2(x)$.

Instead of considering (17) one could have determined the primitive roots of $F(x)$ as a root of an equation

$$(20) \quad F_1(x) = \alpha_2.$$

The common roots of two equations (20) and (17) can be obtained in the following manner: one can write α_2 in the symbolic form $\alpha_2 = Q_1(\mu) \times F_1(\mu)$ and one finds as in (11) that the general root of (20) has the form

$$(21) \quad R_1(x) = Q_1(x) - L(x) \times F_2(x).$$

The comparison of (21) with (11) shows that in case of a common root the polynomials $K(x)$ and $L(x)$ must satisfy the condition

$$(22) \quad Q(x) - Q_1(x) = K(x) \times F_1(x) - L(x) \times F_2(x).$$

This equation is solvable if and only if

$$(23) \quad Q(x) \equiv Q_1(x) \pmod{T(x)},$$

where $T(x)$ is the greatest common factor of $F_1(x)$ and $F_2(x)$; when the condition (23) is satisfied, one obtains exactly p^t common solutions from (22), where t denotes the exponent of $T(x)$. A special case of particular importance is the following:

THEOREM 3. *Let $F_1(x)$ and $F_2(x)$ be two p' -polynomials without common factor; the equations*

$$(24) \quad F_1(x) = \alpha_2, \quad F_2(x) = \alpha_1,$$

where α_1 is a root of $F_1(x)$ and α_2 is a root of $F_2(x)$, have then exactly one root in common.

The common root can be found from (22); when α_1 is a primitive root of $F_1(x)$ and α_2 is a primitive root of $F_2(x)$, then the common root μ in (24) is a primitive root of $F(x) = F_1(x) \times F_2(x)$, and this remark gives a simple method for determining all primitive roots of $F(x)$.

3. Applications. The theorems derived in §2 have a number of applications. Let us use the former notation and let $\phi_1(x)$ be an irreducible polynomial in K_f belonging to the p' -polynomial $F_1(x)$. When α_1 is an arbitrary root of $\phi_1(x)$, then

$$(25) \quad \phi_1(F_2(x)) = (F_2(x) - \alpha_1)(F_2(x) - \alpha_1^{p'}) \cdots (F_2(x) - \alpha_1^{p^{f(N_1-1)}}),$$

where N_1 is the degree of $\phi_1(x)$. We now join all factors in (9) in the form (25) and Theorem 2 gives the following result:

THEOREM 4. *Let $\phi_1(x)$ be an irreducible polynomial of degree N_1 belonging to the p' -polynomial $F_1(x)$; let $F_2(x)$ be a second p' -polynomial and*

$$F_1(x) = G_1(x) \times D_1(x), \quad F_2(x) = G_2(x) \times D_2(x),$$

where $D_1(x)$ and $D_2(x)$ contain the prime factors common to $F_1(x)$ and $F_2(x)$. The polynomial $\phi_1(F_2(x))$ is then equal to a product of prime polynomials belonging to p' -polynomials

$$(26) \quad \bar{D}(x) \times D_2(x) \times F_1(x),$$

where $\bar{D}(x)$ is a divisor of $G_2(x)$. The number of prime polynomials belonging to a given polynomial (26) is

$$\frac{N_1}{N} p^{d_2} \Phi(\bar{d}(x)),$$

where N is the index of (26), d_2 the exponent of $D_2(x)$ and $\bar{d}(x)$ the polynomial corresponding to $\bar{D}(x)$.

There are several cases of Theorem 4 which are of special interest. Since all prime factors of $\phi_1(F_2(x))$ belong to a multiple of $F_1(x)$, it is clear that the degrees of all prime polynomials are divisible by N_1 . In the case where $F_1(x)$ is relatively prime to $F_2(x)$, all prime factors of $\phi_1(F_2(x))$ belong to some $\bar{D}(x) \times F_1(x)$, where $\bar{D}(x)$ is a divisor of $F_2(x)$ and the number of prime factors belonging to such a given polynomial is simply

$$\frac{N_1}{N} \Phi(\bar{d}(x)),$$

where N is the index of $\bar{D}(x) \times F_1(x)$. There will be exactly

$$\frac{(N_1, N_2)}{N_2} \Phi(f_2(x))$$

irreducible factors belonging to $F_1(x) \times F_2(x)$, where N_2 is the index of $F_2(x)$, while there will be only one prime polynomial belonging to $F_1(x)$ and dividing $\phi_1(F_2(x))$. The roots of this prime polynomial can easily be obtained from (11).

Theorem 3 gives a surprisingly simple method for determining the prime polynomials belonging to a product of p' -polynomials when those of the factors are known:

THEOREM 5. Let $F_1(x)$ be relatively prime to $F_2(x)$ and let $\phi_1(x)$ be a prime polynomial belonging to $F_1(x)$ while $\phi_2(x)$ belongs to $F_2(x)$. The greatest common factor of the two polynomials

$$(27) \quad \phi_1(F_2(x)), \quad \phi_2(F_1(x))$$

is then a prime polynomial belonging to $F_1(x) \times F_2(x)$ and all prime polynomials belonging to the product can be determined in this way.

CHAPTER III. CONSTRUCTION OF IRREDUCIBLE POLYNOMIALS

1. A class of irreducible polynomials. One of the most interesting but also most difficult problems in the theory of higher congruences is the determination of irreducible polynomials of a given degree in explicit form. At the pres-

ent time this problem has only been solved for very special cases, but it is of interest to observe that almost all of the results obtained are closely related to the theory of p' -polynomials.

Before we illustrate this fact, we shall however use some of the former results to obtain a new class of irreducible polynomials. Let $f(x)$ be an ordinary irreducible polynomial of degree n with coefficients in a finite field K_f . We shall suppose in addition that $f(x)$ is a primitive polynomial, i.e. $p'^n - 1$ is the smallest exponent such that

$$x^{p'^n - 1} \equiv 1 \pmod{f(x)}.$$

For the p' -polynomial $F(x)$ corresponding to $f(x)$ one then has symbolically

$$x^{p^{f(p'^n - 1)}} - x \equiv 0 \pmod{F(x)}$$

and the index of $F(x)$ is $p'^n - 1$. Theorem 10 then shows that any ordinary prime polynomial $\neq x$ dividing $F(x)$ has the degree $p'^n - 1$. This gives

THEOREM 1. *When*

$$f(x) = x^n + \alpha_1 x^{n-1} + \dots + \alpha_n$$

is an irreducible primitive polynomial in K_f , then

$$\phi(x) = x^{p'^n - 1} + \alpha_1 x^{p^{f(n-1)}} + \dots + \alpha_{n-1} x^{p'^{n-1}} + \alpha_n$$

is an irreducible polynomial in the same field.

A consequence of Theorem 1 is obviously that the polynomial

$$\phi_1(x) = x^{(p'^n - 1)/(p' - 1)} + \alpha_1 x^{(p^{f(n-1)} - 1)/(p' - 1)} + \dots + \alpha_{n-1} x + \alpha_n$$

is irreducible.

As an illustration of Theorem 1 we may take $f(x) = x - \alpha$ and we obtain the well known result that

$$\phi(x) = x^{p'^n - 1} - \alpha$$

is irreducible when α belongs to the exponent $p' - 1$ and hence

$$\phi(x) = x^\delta - \alpha$$

is also irreducible when δ is any divisor of $p' - 1$.

2. Substitution of a prime polynomial. Our next considerations are based on Theorem 4, chapter 2, and this theorem shall be applied particularly for the case where $F_2(x)$ is an irreducible p -polynomial. We use the former notations, letting N_1 and N_2 be the indices of $F_1(x)$ and $F_2(x)$, while $f_1(x)$ and $f_2(x)$ denote the polynomials corresponding to $F_1(x)$ and $F_2(x)$.

Let us first deal with the case where $F_2(x)$ symbolically divides $F_1(x)$ and $F_1(x)$ contains $F_2(x)$ symbolically e times. Furthermore let $N_1 = p^A N'_1$, where N'_1 is not divisible by p . The exponent A is obviously the smallest number such that p^A is not surpassed by any of the exponents occurring in the symbolic prime function decomposition of $F_1(x)$. If now $e+1 \leq p^A$, then $F_2(x) \times F_1(x)$ still has the index N_1 , and when $\phi_1(x)$ is a polynomial belonging to $F_1(x)$ and hence of degree N_1 , then according to Theorem 4 $\phi_1(F_2(x))$ decomposes into irreducible factors of degree N_1 . If however $e+1 > p^A$, then e is the largest exponent occurring in $F_1(x)$ and the index of $F_1(x) \times F_2(x)$ must be pN_1 , and hence $\phi_1(F_2(x))$ decomposes into factors of degree pN_1 .

Next let $F_1(x)$ not be divisible by $F_2(x)$. The index of the product $F_1(x) \times F_2(x)$ is

$$[N_1, N_2] = \frac{N_1 N_2}{(N_1, N_2)}$$

and each irreducible factor of $\phi_1(F_2(x))$ will, according to Theorem 4, belong to $F_1(x) \times F_2(x)$ or to $F_1(x)$, and there will be one prime polynomial of degree N_1 belonging to $F_1(x)$ and

$$\frac{(N_1, N_2)}{N_2} \Phi(f_2(x)) = \frac{(N_1, N_2)}{N_2} (p^{f_{n_2}} - 1)$$

polynomials of degree $[N_1, N_2]$ belonging to $F_1(x) \times F_2(x)$, where n_2 denotes the degree of $f_2(x)$.

THEOREM 2. *Let $F_1(x)$ and $F_2(x)$ be two p' -polynomials with the indices N_1 and N_2 ; we shall suppose that $F_2(x)$ is symbolically irreducible and that $\phi_1(x)$ is a prime polynomial belonging to $F_1(x)$. When $F_2(x)$ divides $F_1(x)$, then $\phi_1(F_2(x))$ is the product of prime polynomials of degree N_1 except when N_1 is divisible exactly by p^A and $F_1(x)$ contains $F_2(x)$ to the same power p^A ; then $\phi_1(F_2(x))$ is the product of prime polynomials of degree pN_1 .*

When $F_2(x)$ does not divide $F_1(x)$, then $\phi_1(F_2(x))$ contains one prime factor of degree N_1 , while the remaining factors have the degree $[N_1, N_2]$.

3. Prime polynomials whose degrees are divisible by p . We shall now apply the first part of Theorem 2 to obtain various irreducible polynomials whose degrees are divisible by p . We shall suppose for the moment that all polynomials have rational coefficients, and we put

$$F_2(x) = x^p - ax, \quad a^d = 1,$$

where the exponent d of a divides $p-1$ and is identical with the index of $F_2(x)$. Since we shall suppose that $F_1(x)$ is divisible by $x^p - ax$, we must have

$x^{N_1} - 1$ divisible by $x - a$, which in turn shows that $N_1 \equiv 0 \pmod{d}$. To insure that the exceptional case of Theorem 2 occurs, we shall have to suppose furthermore that $F_1(x)$ divides $x^{p^{N_1}} - x$ but not

$$(x^{p^{N_1}} - x) \times (x^p - ax)^{-1} = x^{p^{N_1-1}} + ax^{p^{N_1-2}} + \dots + a^{N_1-1}x.$$

This shows

THEOREM 3. *Let a be a rational integer belonging to the exponent d and $\phi(x)$ a prime polynomial of degree N divisible by d . Then $\phi(x^p - ax)$ is a prime polynomial of degree pN , when $\phi(x)$ does not divide*

$$x^{p^{N-1}} + ax^{p^{N-2}} + \dots + a^{N-1}x.$$

When $a=1$, then $d=1$ and the last condition of Theorem 3 is equivalent to $a_1 \neq 0$, where a_1 is the coefficient of x^{N-1} in $\phi(x)$. This gives the following well known result:

When $\phi(x)$ is a prime polynomial of degree N in which the coefficient of x^{N-1} does not vanish, then $\phi(x^p - x)$ is a prime polynomial of degree pN .

When applied to $\phi(x) = x + a$, this shows that

$$x^p - x + a, \quad a \neq 0,$$

is irreducible. I observe without proof that Theorem 3 can be modified to hold in an arbitrary field K_f .

We shall also make an application of the first part of Theorem 2 to obtain in a simple way the results of Serret* and Dickson† on prime polynomials in a field K_f , whose degrees are powers of p . Let us denote by $\Pi_r(x)$ the product of r equal symbolic factors $x^{p^f} - x$

$$(1) \quad \Pi_r(x) = (x^{p^f} - x)^{(r)} = x^{p^f r} - \binom{r}{1} x^{p^f(r-1)} + \dots + (-1)^r x.$$

For $r = p^n$ one obtains simply

$$(2) \quad \Pi_{p^n}(x) = x^{p^f p^n} - x.$$

The polynomial corresponding to $\Pi_r(x)$ is $(x-1)^r$ and all symbolic divisors of $\Pi_{p^n}(x)$ are of the form $\Pi_r(x)$. Since a prime polynomial of degree p^n must divide (2) every prime polynomial having this degree must belong to a unique polynomial

$$\Pi_r(x) \quad (r = p^{n-1} + 1, p^{n-1} + 2, \dots, p^n).$$

* See Serret, *Cours d'Algèbre*.

† See Dickson, *Linear Groups*.

In this way one obtains a division of all prime polynomials of degree p^n into $p^n - p^{n-1}$ classes. The class corresponding to $r = p^{n-1} + 1$ shall be called the *first class* and the class corresponding to $r = p^n$ the *last class of degree p^n* . Since

$$\Pi_r(x) = \Pi_{r-1}(x) \times (x^{p^r} - x) = \Pi_{r-1}(x)^{p^r} - \Pi_{r-1}(x),$$

and since all polynomials dividing, but not belonging to, $\Pi_r(x)$ must divide $\Pi_{r-1}(x)$, it follows that

$$\Gamma_r(x) = \Pi_{r-1}(x)^{p^r-1} - 1 = \prod_{\alpha} \left(\Pi_{r-1}(x) - \alpha \right),$$

where $\alpha \neq 0$ runs through all of the elements of K_f , represents the product of all prime polynomials belonging to $\Pi_r(x)$. The first part of Theorem 2 gives immediately

THEOREM 4. *When $\phi(x)$ is a prime polynomial of degree p^n belonging to the class ρ , then $\phi(x^{p^f} - x)$ is the product of p^f different prime polynomials of the class $\rho + 1$ except in the case where $\phi(x)$ belongs to the last class of degree p^n , when $\phi(x^{p^f} - x)$ is the product of p^{f-1} prime polynomials of the first class of degree p^{n+1} .*

4. Further applications. The second part of Theorem 2 may also be used to obtain results on irreducible polynomials. We saw that, with the same notation as before, $\phi_1(F_2(x))$ contains one irreducible polynomial of degree N_1 belonging to $F_1(x)$ and

$$T = \frac{(N_1, N_2)}{N_2} (p^{f n_2} - 1)$$

irreducible polynomials of degree $[N_1, N_2]$ belonging to $F_1(x) \times F_2(x)$. We see that $T=1$ only when the indices N_1 and N_2 are relatively prime and $N_2 = p^{f n_2} - 1$. We can then write $\phi_1(F_2(x)) = \lambda(x) \cdot \mu(x)$ where $\lambda(x)$ has the degree $N_1 N_2$ and $\mu(x)$ divides $F_1(x)$. Hence we can write

$$\mu(x) = (\phi_1(F_2(x)), F_1(x))$$

and this gives the following: *Let $f_2(x)$ be an irreducible primitive polynomial of degree n_2 and let $f_1(x)$ be an arbitrary polynomial belonging to the exponent N_1 , where $(N_1, p^{f n_2} - 1) = 1$. When $F_1(x)$ and $F_2(x)$ are the corresponding p^f -polynomials and $\phi_1(x)$ a prime polynomial belonging to $F_1(x)$, then*

$$\lambda(x) = \frac{\phi_1(F_2(x))}{(\phi_1(F_2(x)), F_1(x))}$$

is a prime polynomial of degree $N_1(p^{f n_2} - 1)$. It may be observed that this

result contains Theorem 1 for $F_1(x) = x$. We shall give a further application to the case where

$$F_1(x) = x^p - x.$$

One then has $N_1 = 1$ and $\phi_1(x) = x - \alpha$. Let us suppose

$$F_2(x) = x^{p^n} + \beta_1 x^{p^{n-1}} + \cdots + \beta_n x,$$

and hence

$$\phi_1(F_2(x)) = x^{p^n} + \beta_1 x^{p^{n-1}} + \cdots + \beta_n x - \alpha.$$

According to the general result this polynomial must contain a linear factor $x - \gamma$ and we find

$$\gamma = \frac{\alpha}{1 + \beta_1 + \cdots + \beta_n}.$$

THEOREM 5. *Let $f(x)$ be an irreducible primitive polynomial of degree n and let $F(x)$ be the corresponding p^f -polynomial. Then*

$$\psi(x) = \frac{F(x) - \alpha}{x - \frac{\alpha}{F(1)}}$$

is an irreducible polynomial of degree $p^n - 1$.

This theorem may be considered as a restatement of Theorem 1.

5. Decompositions of p^f -polynomials. We shall now give the complete decomposition into prime factors of a few simple p^f -polynomials, thus also illustrating the general theorems.

1. In the simplest case

$$F(x) = x^p - \alpha x,$$

let δ be the smallest exponent such that $\alpha^\delta = 1$. The index of $F(x)$ is δ and one finds the prime polynomial decomposition

$$F(x) = x \prod_{\beta} (x^\delta - \beta),$$

where β runs through all of the roots of

$$\beta^{(p^f-1)/\delta} = \alpha.$$

2. When

$$F(x) = (x^{p^f} - x) \times (x^p - x)$$

the irreducible factors must have the degrees 1 and p , and since

$$F(x) = \prod_{\alpha} (x^{p^f} - x - \alpha),$$

where α runs through all of the elements of K_f , it is sufficient to decompose the factors of this product. One finds

$$x^{p^f} - x - \alpha = \prod_{\beta} (x^p - \alpha^{p^{-1}}x - \beta), \quad \alpha \neq 0,$$

where $\beta\alpha^{-p}$ runs through all solutions of

$$x^{p^{f-1}} + \dots + x^p + x = 1.$$

One can also show that

$$f(x) = x^p - \alpha^{p^{-1}}x - \beta$$

is reducible if and only if $\beta\alpha^{-p}$ satisfies

$$x^{p^{f-1}} + \dots + x^p + x = 0.$$

3. In the case

$$F(x) = (x^{p^f} - \alpha x) \times (x^{p^f} - x), \quad \alpha^{\delta} = 1,$$

it follows from the general theory that the irreducible factors are of degree 1 and δ . One obtains

$$F(x) = \prod_{\beta} (x^{p^f} - \alpha x - \beta),$$

and putting $\sigma = \beta/(1 - \alpha)$ one finds

$$x^{p^f} - \alpha x - \beta = (x - \sigma) \prod_{\gamma} ((x - \sigma)^{\delta} - \gamma),$$

where γ runs through all solutions of

$$\gamma^{(p^f-1)/\delta} = \alpha.$$

At this point it may be of interest to determine the decomposition of a polynomial

$$f(x) = x^p - \alpha x - \beta.$$

This problem occurs in connection with the determination of prime ideal decompositions in relative Kummer fields. The number

$$a = \alpha^{(p^f-1)/(p-1)}$$

is rational and we can suppose $a \neq 1$ since this case has been treated under 2. One finds that $f(x)$ has the root

$$\sigma = \frac{1}{1-\alpha} \sum_{i=1}^f \beta^{p^i-1} \alpha^{(p^i-p^{i-1}-1)/(p-1)}$$

in K_f and consequently

$$x^p - \alpha x - \beta = (x - \sigma) \prod_{\lambda} ((x - \sigma)^\Delta - \lambda),$$

where

$$\alpha^\delta = 1, \quad \Delta = \left(\frac{p^f - 1}{\delta}, p - 1 \right), \quad \lambda^{(p-1)/\Delta} = \alpha.$$

6. Irreducible polynomials and linear substitutions. Now let

$$(3) \quad F(x) = x^{p^f} + \alpha x^{p^f} + \beta x$$

be a p^f -polynomial whose corresponding polynomial

$$(4) \quad f(x) = x^2 + \alpha x + \beta$$

is irreducible in K_f . In order to study the prime polynomial decomposition of $F(x)$ we put $t = x^{p^f-1}$ and obtain

$$(5) \quad \Phi(t) = F(x) \cdot x^{-1} = x^{p^f-1} + \alpha x^{p^f-1} + \beta = t^{p^f+1} + \alpha t + \beta.$$

Any root of $\Phi(t)$ must satisfy the relation

$$t^{p^f} = -\alpha - \frac{\beta}{t},$$

and so we are naturally led to the study of irreducible polynomials whose roots are connected by linear substitutions

$$(6) \quad x^{p^f} = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

Such prime polynomials are obviously divisors of

$$(7) \quad \Psi(x) = \gamma x^{p^f+1} + \delta x^{p^f} - \alpha x - \beta.$$

In (6) and (7) we can always assume $\gamma \neq 0$ since the polynomials

$$(8) \quad x^{p^f} + \lambda x + \mu$$

have been completely decomposed in Nos. 2 and 3 of the preceding section. We are also mainly interested in the case where the linear substitution (6) leaves no element of K_f unchanged. We suppose then that the equation

$$x = \frac{\alpha x + \beta}{\gamma x + \delta}$$

has no solution in K_f and this is equivalent to the statement that the equation

$$(9) \quad \psi(x) = \gamma x^2 + (\delta - \alpha)x - \beta = 0$$

is irreducible in K_f .

If namely $\psi(x)$ in (9) has the root ρ in K_f , then $\Psi(x)$ in (7) also has the root ρ and one finds after putting $y = x - \rho$

$$\Psi(x) = \gamma y \cdot \left[y^{p'} + \left(\rho + \frac{\delta}{\gamma} \right) y^{p'-1} + \rho - \frac{\alpha}{\gamma} \right].$$

The second factor in this product is again of the type (8) when z is substituted for $1/y$.

We suppose, then, that (9) is irreducible and has the two roots ψ_1 and ψ_2 . This corresponds in our first special case to the assumption that the polynomial (4) is irreducible. In this case $\Psi(x)$ has no linear factors.

From (6) one obtains through iteration

$$(10) \quad x^{p^n} = \frac{\alpha_n x + \beta_n}{\gamma_n x + \delta_n}$$

and one verifies that the coefficients of this substitution are given by*

$$(11) \quad \begin{aligned} (\omega_1 - \omega_2)\alpha_n &= (\alpha - \omega_2)\omega_1^n - (\alpha - \omega_1)\omega_2^n, \\ (\omega_1 - \omega_2)\beta_n &= \beta(\omega_1^n - \omega_2^n), \\ (\omega_1 - \omega_2)\gamma_n &= \gamma(\omega_1^n - \omega_2^n), \\ (\omega_1 - \omega_2)\delta_n &= (\omega_1 - \alpha)\omega_1^n - (\omega_2 - \alpha)\omega_2^n, \end{aligned}$$

where ω_1 and ω_2 are the roots of

$$(12) \quad \omega(x) = x^2 - (\alpha + \delta)x + \alpha\delta - \beta\gamma = 0$$

and hence

$$\omega_1 = \gamma\psi_1 + \delta, \quad \omega_2 = \gamma\psi_2 + \delta.$$

Now let n be the degree of an irreducible factor of $\Psi(x)$. Then n is the smallest number such that the roots of the factor satisfy the equation

$$(13) \quad x^{p^n} = x = \frac{\alpha_n x + \beta_n}{\gamma_n x + \delta_n}.$$

If $\gamma_n \neq 0$ one finds that a solution ρ of (13) is also a solution of (9), and since

* These expressions were given by Serret, *Sur les fonctions rationnelles linéaires prises suivant un module premier* etc., Comptes Rendus, Paris, vol. 48 (1859), pp. 112-117.

such a root cannot be a root of $\Psi(x)$, we shall have to suppose $\gamma_n = 0$ and hence according to (11)

$$(14) \quad \omega_1^n = \omega_2^n.$$

In this case one obtains from (11) that the right-hand side of (10) reduces to x and it follows that the degree of any factor of $\Psi(x)$ is the smallest exponent such that (14) is satisfied. Since the number $\omega_1/\omega_2 = \phi_1$ is a root of the congruence

$$(15) \quad \phi(x) = x^2 + \left(\frac{(\alpha + \delta)^2}{\beta\gamma - \alpha\delta} + 2 \right)x + 1$$

and since the irreducibility of (9) follows from the irreducibility of (15), we conclude:

THEOREM 6. *Let*

$$\Psi(x) = \gamma x^{p'+1} + \delta x^{p'} - \alpha x - \beta, \quad \gamma \neq 0,$$

be a polynomial with coefficients in K_f chosen such that the polynomial

$$\phi(x) = x^2 + \left(\frac{(\alpha + \delta)^2}{\beta\gamma + \alpha\delta} + 2 \right)x + 1$$

is irreducible in K_f . When $\phi(x)$ belongs to the exponent n , then $\Psi(x)$ is the product of $(p'+1)/n$ irreducible factors of degree n , and when $\phi(x)$ belongs to the maximal exponent $p'+1$, $\Psi(x)$ is irreducible.

It is also possible to give the complete prime polynomial decomposition of $\Psi(x)$ and hence to exhibit explicitly irreducible polynomials having a degree equal to an arbitrary divisor of $p'+1$. One finds, namely, that $\Psi(x)$ may be brought into the form

$$(16) \quad \Psi(x) = \alpha_1(x - \psi_1)^{p'+1} + \alpha_2(x - \psi_2)^{p'+1},$$

where ψ_1 and ψ_2 as formerly denote the roots of (9), while $-\alpha_1/\alpha_2$ is a root ϕ_1 of (15). From (16) we obtain the decomposition

$$(17) \quad \Psi(x) = \prod_i (\rho_1^{(i)}(x - \psi_1)^n + \rho_2^{(i)}(x - \psi_2)^n)$$

where $\rho_1^{(i)}$ and $\rho_2^{(i)}$ are two conjugate elements in the field K_{2f} such that the quotient $-\rho_2^{(i)}/\rho_1^{(i)}$ runs through all $m = (p'+1)/n$ solutions of the equation

$$(18) \quad x^m = \phi_1.$$

The actual determination of the roots of (18) may be done in the following way. Any solution can be represented in the form

$$\kappa = (r + \omega_2)/(r + \omega_1)$$

and the equation (18) takes the symmetric form

$$(19) \quad \omega_1(r + \omega_1)^m = \omega_2(r + \omega_2)^m.$$

If we suppose $p \neq 2$ we can write

$$\begin{aligned} \omega_1 &= A + B^{1/2}, & \omega_2 &= A - B^{1/2}, \\ A &= \frac{\alpha + \delta}{2}, & B &= \frac{(\alpha - \delta)^2}{2} + \beta\gamma, \end{aligned}$$

and to satisfy (19) we must have

$$(20) \quad \begin{aligned} (r + A)^m + \binom{m}{2}(r + A)^{m-2} \cdot B + \binom{m}{4}(r + A)^{m-4} \cdot B^2 + \dots \\ + A \left(\binom{m}{1}(r + A)^{m-1} + \binom{m}{3}(r + A)^{m-3} \cdot B + \dots \right) = 0. \end{aligned}$$

This congruence must have m different solutions and each solution determines a factor of $\Psi(x)$ in (16).

One can, however, derive these irreducible factors in rational form in a different way, which more clearly shows their relation to the linear substitutions. Let the numbers $\alpha, \beta, \gamma, \delta$ satisfy the conditions of Theorem 6 and let us construct the expression

$$R(x) = x + \frac{\alpha x + \beta}{\gamma x + \delta} + \dots + \frac{\alpha_{n-1}x + \beta_{n-1}}{\gamma_{n-1}x + \delta_{n-1}}.$$

For a root λ of $\Psi(x)$ we have

$$R(\lambda) = \lambda + \lambda^{p'} + \dots + \lambda^{p^{f(n-1)}} = -a_1,$$

where a_1 is the coefficient of x^{n-1} in the corresponding irreducible factor in (16), hence

$$a_1 = n \frac{\rho_1 \psi_1 + \rho_2 \psi_2}{\rho_1 + \rho_2} = \frac{n}{\gamma} (r + \alpha).$$

Since the equation of n th degree

$$R(x) = -a_1$$

is satisfied by all roots of the irreducible factor of $\Psi(x)$ having the coefficient a_1 , we have

THEOREM 7. *Let $p \neq 2$, and let $\alpha, \beta, \gamma, \delta$ be chosen such that the polynomial (15) is irreducible in K_f , while the order of the linear substitution*

$$x' = \frac{\alpha x + \beta}{\gamma x + \delta}$$

is n , where $n \cdot m = p^f + 1$. The equation of n th degree

$$Q_i(x) = x + \frac{\alpha x + \beta}{\gamma x + \delta} + \cdots + \frac{\alpha_{n-1}x + \beta_{n-1}}{\gamma_{n-1}x + \delta_{n-1}} + \frac{n}{\gamma} (r^{(i)} + \alpha) = 0$$

is then irreducible for all $r^{(i)}$ satisfying (20).

One sees from the proof of this theorem that $\Psi(x)$ may be represented as the product of factors $P_i(x)$ where

$$P_i(x) = (\gamma x + \delta) \cdots (\gamma_{n-1}x + \delta_{n-1}) Q_i(x).$$

It should also be observed that one can obtain similar results through a consideration of the product of the linear transformations.

CHAPTER IV. MISCELLANEOUS THEOREMS ON HIGHER CONGRUENCES

1. Elements with unit norm. We shall now deduce a few results which may be considered as the rudiments of the class field theory in finite fields. We show first

THEOREM 1. *The necessary and sufficient condition that a number α in the field $K_{ff'}$ satisfy the relation*

$$(1) \quad \alpha^{(p^{ff'}-1)/(p^f-1)} = 1$$

is that α be representable in the form

$$(2) \quad \alpha = \beta^{p^f}/\beta.$$

It is obvious that every element of the form (2) satisfies (1). On the other hand, one finds that (1) represents the necessary and sufficient condition that $x^{p^{ff'}} - x$ be symbolically right-hand divisible by $x^{p^f} - \alpha x$, and hence when (1) is satisfied the equation

$$(3) \quad x^{p^f} - \alpha x = 0$$

has a solution $\beta \neq 0$ in $K_{ff'}$.

If one wishes to determine the form of the number β in the representation (2), we divide $x^{p^{ff'}} - x$ left-hand by $x^{p^f} - \alpha x$ and find

$$(4) \quad x^{p^{f'}} - x = (x^{p^f} - \alpha x) \times Q(x),$$

where

$$Q(x) = x^{p^{f'(f'-1)}} + \alpha^{p^{f'(f'-1)}} x^{p^{f'(f'-2)}} + \alpha^{p^{f'(f'-1)+p^{f'(f'-2)}}} x^{p^{f'(f'-3)}} + \dots$$

The relation (4) shows that

$$\beta = Q(\omega),$$

where ω is an arbitrary element in $K_{f'}$ such that $Q(\omega) \neq 0$.

The condition (1) may also be written

$$N_f(\alpha) = \alpha \cdot \alpha^{p^f} \cdot \dots \cdot \alpha^{p^{f(f'-1)}} = 1,$$

and Theorem 1 is seen to be the analogue of the well known theorem on cyclic fields, that *every element whose norm is unity may be represented as the quotient of two conjugate elements*. The ordinary proof for this theorem could not be applied in our case, since it requires that the field contain an infinite number of elements.

One may also state Theorem 1 in the following equivalent form:

THEOREM 2. *The necessary and sufficient condition that an irreducible polynomial $f(x)$ of degree n with coefficients in K_f belong to an exponent N dividing $(p^{f'n}-1)/(p^f-1)$ is that the last coefficient α_n in $f(x)$ be unity, and in this case every root ρ of $f(x)$ may be represented in the form*

$$\rho = \sigma^{p^f}/\sigma$$

where σ is an element of K_{nf} .

One may also express Theorem 1 in a somewhat more general form. Let namely

$$(5) \quad F(x) = x^{p^r} + \gamma_1 x^{p^{r-1}} + \dots + \gamma_{r-1} x^{p^f} + \gamma_r x$$

be a p^f -polynomial dividing $x^{p^{f'}} - x$, and let

$$(6) \quad x^{p^{f'}} - x = F(x) \times G(x).$$

Expressing the condition that $x^{p^{f'}} - \alpha x$ be a right-hand symbolic divisor of $F(x)$, we find

THEOREM 3. *Let $F(x)$ be a p^f -polynomial given by (5) and let $G(x)$ be its complementary polynomial such that (6) is satisfied. An element α in $K_{f'}$ satisfying the condition*

$$\alpha^{(p^{f'r}-1)/(p^f-1)} + \gamma_1 \alpha^{(p^{f'(r-1)}-1)/(p^f-1)} + \dots + \gamma_{r-1} \alpha + \gamma_r = 0$$

is then representable in the form

$$\alpha = \beta^{p^f}/\beta,$$

where β is a root of $F(x)=0$, hence $\beta=G(\omega)$ for a primitive element ω of $K_{f'}$.

2. The law of reciprocity. There exists for higher congruences a very simple and general law of reciprocity. This was first pointed out by F. K. Schmidt*, although special instances of it were already known to Dedekind.† Recently the theorem has been rediscovered by Carlitz‡, who seems to have overlooked the paper of Schmidt. Carlitz gives two different proofs mapped on the proofs for the quadratic law of reciprocity. In the following I give a new and very simple proof for the law of reciprocity in its most general form.

Let d be a divisor of $p^f - 1$ and let

$$d \cdot \delta = p^f - 1.$$

The equation

$$(7) \quad x^d = 1$$

is then solvable and has the d roots

$$(8) \quad 1 = \epsilon_1, \epsilon_2, \dots, \epsilon_d$$

in K_f . We define the field $K_{n,f}$ over K_f through a root ω of the irreducible equation

$$f(x) = \alpha_0 x^n + \dots + \alpha_{n-1} x + \alpha_n$$

where we do not, as usual, suppose that $\alpha_0 = 1$. Let then

$$g(\omega) = \beta_0 \omega^m + \dots + \beta_{m-1} \omega + \beta_m$$

be an arbitrary element in $K_{n,f}$, and hence

$$(9) \quad g(\omega)^{\delta(p^n-1)/(p^f-1)} = \epsilon,$$

where ϵ is one of the roots (8). One may obviously write (9) in the form of a congruence

$$g(x)^{\delta(p^n-1)/(p^f-1)} \equiv \epsilon \pmod{f(x)},$$

and when we introduce the d th power residue symbol

$$(10) \quad \left(\frac{g(x)}{f(x)} \right)_d = \epsilon \equiv g(x)^{\delta(p^n-1)/(p^f-1)} \pmod{f(x)},$$

we find that it has the property

* F. K. Schmidt, *Zur Zahlentheorie in Körpern von der Charakteristik p* , Erlangen Sitzungsberichte, vols. 58-59 (1928), pp. 159-172.

† R. Dedekind, *Abriss einer Theorie der höheren Kongruenzen in Bezug auf einen reellen Primzahlmodulus*, Journal für Mathematik, vol. 54 (1857), pp. 1-26; Werke, vol. 1, pp. 40-67.

‡ L. Carlitz, *The arithmetic of polynomials in a Galois field*, American Journal of Mathematics, vol. 54 (1932), pp. 39-50. See also *On a theorem of higher reciprocity*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 155-160.

$$\left(\frac{g(x) \cdot h(x)}{f(x)} \right)_d = \left(\frac{g(x)}{f(x)} \right)_d \left(\frac{h(x)}{f(x)} \right)_d$$

and

$$\left(\frac{g(x)}{f(x)} \right)_d = 1$$

is the necessary and sufficient condition that $g(x)$ be a d th power residue (mod $f(x)$).

This definition (10) gives the d th power residue symbol only for prime $f(x)$. In the general case, where $f(x)$ has the prime factor decomposition

$$f(x) = f_1(x) \cdots f_r(x),$$

we put

$$(11) \quad \left(\frac{g(x)}{f(x)} \right)_d = \left(\frac{g(x)}{f_1(x)} \right)_d \cdots \left(\frac{g(x)}{f_r(x)} \right)_d.$$

To prove the law of reciprocity, let us first consider the symbol for a prime $f(x)$. Then according to (10) we obtain

$$(12) \quad \left(\frac{g(x)}{f(x)} \right)_d = (g(\omega)g(\omega^{p^f}) \cdots g(\omega^{p^{f(n-1)}}))^{\delta} = \alpha_0^{-m\delta} R(f(x), g(x))^{\delta},$$

where $R(f, g)$ denotes the resultant of the two polynomials. The definition (11) then shows that the same formula (12) holds for an arbitrary $f(x)$. For the inverse symbol we obtain in the same way

$$\left(\frac{f(x)}{g(x)} \right)_d = \beta_0^{-n\delta} R(g(x), f(x))^{\delta} = (-1)^{mn} \beta_0^{-n\delta} R(f(x), g(x))^{\delta},$$

and hence

THEOREM 4. *For the d th power residue symbol one has the law of reciprocity*

$$\alpha_0^{m(p^f-1)/d} \left(\frac{f(x)}{g(x)} \right)_d = (-1)^{mn} \beta_0^{n(p^f-1)/d} \left(\frac{g(x)}{f(x)} \right)_d$$

where n and m are the degrees and α_0 and β_0 are the highest coefficients of the relatively prime polynomials $f(x)$ and $g(x)$.

This proof also suggests generalizations of the law of reciprocity using some other symmetric function than the resultant. Let

$$S_{n,m}(u_1, \dots, u_n; v_1, \dots, v_m)$$

denote a symmetric function in each of the sets u_i and v_j , and let us suppose in addition that

$$(13) \quad S_{n,m}(u, v) = S_{m,n}(v, u).$$

Various symmetric functions having these properties may be constructed. Now let $f(x)$ and $g(x)$ be two polynomials with the roots x_1, \dots, x_n and y_1, \dots, y_m , and let us define

$$\left\{ \frac{g(x)}{f(x)} \right\} = S_{n,m}(g(x_1), \dots, g(x_n), f(y_1), \dots, f(y_m)).$$

It is then obvious according to (13) that

$$\left\{ \frac{g(x)}{f(x)} \right\} = \left\{ \frac{f(x)}{g(x)} \right\}.$$

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ERRATA IN MY PAPER "ON A SPECIAL CLASS OF POLYNOMIALS"*

BY
OYSTEIN ORE

This paper contains a number of disturbing misprints: Equation (2) p. 560 should read

$$G_p(x) = a_0x^{pm} + a_1x^{p(m-1)} + \dots + a_{m-1}x^p + a_mx.$$

Line 17 p. 561 read $A_p(x) \times B_p(x)$ instead of $A_p(x)B_p(x)$.

The term *perfect* (vollkommen) in Theorem 1 is used in the sense of Steinitz, *Algebraische Theorie der Körper*, edited by Hasse and Baer, pp. 50-51.

Line 21 p. 562 should read

$$F_p(x) = Q_p(x) \times (x^p - \alpha x) + Ax.$$

Equation (9) p. 562 should read

$$A = a_0\alpha^{(p^m-1)/(p-1)} + a_1\alpha^{(p^{m-1}-1)/(p-1)} + \dots + a_{m-2}\alpha^{p+1} + a_{m-1}\alpha + a_m.$$

In the expression line 9 p. 563 the last term should be $A_n^{(n)}x$.

Equation (17) p. 564 should read

$$F_n(x) = F_{n-1}(x)^p - F_{n-1}(\omega_n)^{p-1}F_{n-1}(x).$$

Line 8 from below p. 574 should read

$$B_p^{(1)}(x) \times B_p(x) \equiv x \pmod{A_p(x)}.$$

In line 2 from below p. 575 the last term is

$$A_p^{(1)}B_p(x)A_p^{(1)^{-1}} \times A_p^{(1)}(x).$$

Line 18 p. 576 read $x^p - \omega^{p-1}x$.

Line 12 p. 580 read $F_p(x) = x^{p^r} \times G_p(x)$.

* These Transactions, vol. 35 (1933), pp. 559-584.

ALMOST PERIODIC TRANSFORMATIONS*

BY

R. H. CAMERON

1. INTRODUCTION

When one studies periodic transformations such as, for example, rotations, he often encounters transformations which are not periodic but which are, in a very real and non-technical sense, *almost* periodic. For instance, repeated rotation through an angle which is an irrational part of a revolution will never bring a point set back point-for-point into itself; yet this object may be as nearly attained as we please by repeating the process an appropriate number of times. Moreover, such "appropriate" numbers are relatively dense† among the integers. This example suggests the definition of an a.p. (almost periodic) transformation; it being only necessary to make precise the meaning of "as nearly as possible" when applied to bringing the points of a set back into themselves.

Consider, for example, a set \mathfrak{T} of uniformly continuous transformations which take each point of a complete metric space \mathfrak{C} into a point of \mathfrak{C} . Let \mathfrak{T} contain the identity and the product of any two of its elements. Then if ξ is a variable point of \mathfrak{C} and $\Phi(\xi)$ and $\Psi(\xi)$ are any two elements of \mathfrak{T} , let the smaller of the two quantities, unity and the least upper bound for all ξ in \mathfrak{C} of the distance between the points $\Phi(\xi)$ and $\Psi(\xi)$, be called the distance between the transformations $\Phi(\xi)$ and $\Psi(\xi)$, and let it be indicated by $\|\Phi(\xi), \Psi(\xi)\|$. Then $\|\Phi(\xi), \xi\|$ represents one way of telling how nearly the points $\Phi(\xi)$ approximate the points ξ . Moreover a transformation of \mathfrak{T} will be called a.p. if to each positive number ϵ there corresponds a positive integer L so great that among each L successive positive integers there is an integer N satisfying $\|\Phi^N(\xi), \xi\| \leq \epsilon$. This is merely an example of a definition of an a.p. transformation. A more general definition will be given in the next section. Transformations will be thought of as points in a new space, and a.p. points will be defined. Moreover, for simplicity in wording and notation, most of the theorems on a.p. transformations will be stated in terms of a.p. points. However, the reader may readily re-phrase them in terms of the more natural and significant a.p. transformations.

* Presented to the Society, December 29, 1932, and April 14, 1933; received by the editors November 28, 1932, and in revised form, September 6, 1933.

† A set of real numbers is called relatively dense if there exists a positive number L so great that every interval of length L contains at least one element of the set.

For the sake of generality the concepts of a.p. functions and sequences in a complete metric space have been introduced. They include ordinary a.p. functions and sequences as special cases; and may also be thought of as including a.p. transformations as a special case. However, this work is not merely a generalization of the standard theory of a.p. functions and sequences, for my most significant theorem—the climax of the whole theory—applies to a.p. transformations (or points) alone, and does not seem to be susceptible of generalization to space functions or sequences. The theorem to which I refer is Theorem V, §8, which shows that every a.p. transformation can be expressed as an infinite product of simpler transformations.

2. ALMOST PERIODIC POINTS, SPACE FUNCTIONS, AND SPACE SEQUENCES

DEFINITION. A space \mathfrak{T} will be called a \mathfrak{T} -space if it satisfies the postulates

a. \mathfrak{T} is metric and complete (Let $\|\phi, \psi\|$ denote the distance from ϕ to ψ .)

b. An operation called *multiplication* is defined so that to each ordered pair of points ϕ and ψ corresponds a unique point $\phi\psi$. The operation is associative, and the space contains an identity point I .

c. $\|\phi\theta, \psi\theta\| \leq \|\phi, \psi\|$ for any three points ϕ, ψ, θ .

d. The product $\theta\phi$ is a uniformly continuous function of the variable point ϕ for each point θ .

THEOREM I. In any \mathfrak{T} -space, $\|\phi\theta, \psi\theta\| = \|\phi, \psi\|$ if θ^{-1} exists.*

THEOREM II. In any \mathfrak{T} -space the product $\theta\phi$ is a continuous function of the points θ and ϕ .

DEFINITION. Let a positive number ϵ and a point ϕ of the \mathfrak{T} -space \mathfrak{T} be given. Then an integer N which satisfies $\|\phi^N, I\| \leq \epsilon$ is called an ϵ -iteration exponent of ϕ . Moreover a point ψ of \mathfrak{T} is called a.p. if to each positive number ϵ there corresponds a positive integer L so great that among every L successive positive integers there is an ϵ -iteration exponent of ψ .

DEFINITION. Let each point of a \mathfrak{T} -space \mathfrak{T} be a $(1, 1)$ transformation which takes a set or space \mathfrak{C} into a subset of itself. Then an a.p. point ϕ of \mathfrak{T} will be called an a.p. transformation.

It can readily be verified that this definition includes the special case of the definition given above. Moreover a.p. points are no more general than a.p. transformations, for if a \mathfrak{T} -space \mathfrak{T} is given, a space \mathfrak{T}' of transformations can always be set up isomorphic with \mathfrak{T} . Merely let $\xi' = \phi\xi$ correspond to the point ϕ .

* The symbol θ^{-1} denotes a point which satisfies the equations $\theta\theta^{-1} = \theta^{-1}\theta = I$.

As an example of a \mathcal{C} -space, consider the set of all complex numbers whose absolute value is unity or less. Take multiplication and distance in their ordinary sense. In this space, all the numbers whose absolute value is unity are a.p. points.

Another example of a \mathcal{C} -space is the set of all complex numbers; the product of two points being the sum of the numbers. In this case the unit point (the number zero) is the only a.p. point.

DEFINITION. Let \mathcal{S} be a complete metric space, and $\Phi(t)$ a function defined over a set of real numbers and having its set of values in \mathcal{S} . Let s be a real number such that $t+s$ is in the set of definition of $\Phi(t)$ for all values of t in that set of definition. Then s will be called an e -translation number of $\Phi(t)$ if the distance between $\Phi(t)$ and $\Phi(t+s)$ is never greater than the positive number e .

DEFINITION.* A continuous space function $\Phi(t)$ of the real variable t which has its set of values in a complete metric space \mathcal{S} is called a.p. if its e -translation numbers corresponding to each positive e are relatively dense. If each point of \mathcal{S} is a transformation of the points of a set \mathcal{C} into a subset of \mathcal{C} , $\Phi(t)$ is called an a.p. family of transformations.

DEFINITION.* Let $\{\Gamma_n\}$ be a two-way sequence of points in a complete metric space \mathcal{S} . Then if the e -translation numbers of Γ_n considered as a function of n are relatively dense for each positive e , Γ_n is called an a.p. sequence. If each point of \mathcal{S} is a transformation, Γ_n is called an a.p. sequence of transformations.

THEOREM III. An a.p. space function is uniformly continuous for all values of its argument.

THEOREM IV. An a.p. space function or sequence is bounded.

These two theorems can be proved in the same way as the special case of numerical a.p. functions or sequences.†

THEOREM V. In the complete metric space \mathcal{S} let $\{\Gamma_n\}$ be an a.p. sequence of points such that the distance between Γ_m and Γ_n equals the distance between Γ_{m+1} and Γ_{n+1} for every pair of integers m and n , and let \mathfrak{T} be the subspace consisting of the points Γ_n and their limit points. Then‡

* The author is indebted to Dr. I. J. Schoenberg for the suggestion which led to this generalization of a.p. transformations.

† H. Bohr, *Zur Theorie der fastperiodischen Funktionen*, Acta Mathematica, vol. 45, pp. 29-127, especially pp. 35 and 36.

‡ The notation

$$\lim_{\phi(\xi) \rightarrow \alpha} \theta(\xi) = \beta$$

means that to each $e > 0$ there corresponds a $d > 0$ such that every point ξ which satisfies $\|\phi(\xi), \alpha\| \leq d$ also satisfies $\|\theta(\xi), \beta\| \leq e$.

$$(1) \quad \phi\psi = \lim_{\Gamma_m \rightarrow \phi, \Gamma_n \rightarrow \psi} \Gamma_{m+n}$$

exists and is a point of \mathfrak{T} whenever ϕ and ψ are both points of \mathfrak{T} . Moreover if we let (1) define multiplication, the space \mathfrak{T} will be a \mathfrak{G} -space having Γ_1 as an a.p. point.

The existence of $\phi\psi$ follows from the inequality

$$\|\Gamma_{m+n}, \Gamma_{m'+n'}\| \leq \|\Gamma_m, \Gamma_{m'}\| + \|\Gamma_n, \Gamma_{n'}\|$$

and the completeness of \mathfrak{T} . It is easy to verify the fact that \mathfrak{T} is a \mathfrak{G} -space, and since $\Gamma_1^n = \Gamma_n$, the translation indices of the sequences are iteration exponents of Γ_1 .

3. ALMOST PERIODIC CONTINUATION

DEFINITION. Let \mathfrak{J} be the set $a \leq t < +\infty$ or the set $-\infty < t < +\infty$. Let \mathfrak{R} be an infinite set of real numbers such that the sum and difference of any two numbers in \mathfrak{R} is in \mathfrak{R} ; then a function $\Phi(t)$ defined on the intersection $\mathfrak{J}\mathfrak{R}$ and having its set of values in a complete metric space \mathfrak{S} will be called *asymptotically periodic* if there exists a sequence of positive numbers s_1, s_2, \dots in \mathfrak{R} such that $s_n \rightarrow \infty$ as $n \rightarrow \infty$ and such that uniformly in t on $\mathfrak{J}\mathfrak{R}$,

$$\lim_{n \rightarrow \infty} \Phi(t + s_n) = \Phi(t).$$

LEMMA 1. If $\Phi(t)$ is asymptotically periodic on $\mathfrak{J}\mathfrak{R}$, then there is one and only one asymptotically periodic function $\Psi(t)$ which equals $\Phi(t)$ on $\mathfrak{J}\mathfrak{R}$ but is defined on $\mathfrak{Y}\mathfrak{R}$, where \mathfrak{Y} is an interval containing \mathfrak{J} .

For if t is any number in \mathfrak{R} , for sufficiently great integers m and n the numbers $t+s_m, t+s_n, t+s_m+s_n$ are all in $\mathfrak{J}\mathfrak{R}$, and

$$\begin{aligned} \|\Phi(t+s_m), \Phi(t+s_n)\| &\leq \|\Phi(t+s_m+s_n), \Phi(t+s_m)\| \\ &\quad + \|\Phi(t+s_m+s_n), \Phi(t+s_n)\|. \end{aligned}$$

It follows from the hypothesis concerning Φ that the second member approaches zero as m and n approach infinity, and hence that

$$\Psi(t) = \lim_{n \rightarrow \infty} \Phi(t + s_n)$$

is defined for all t on \mathfrak{R} ; and we have $\Psi(t) = \Phi(t)$ on $\mathfrak{J}\mathfrak{R}$. Now corresponding to an arbitrary $\epsilon > 0$ there exists an integer N so great that for all t' on $\mathfrak{J}\mathfrak{R}$ and any $n > N$,

$$\|\Phi(t' + s_n), \Phi(t')\| \leq \epsilon.$$

Thus for any t on \mathfrak{R} and any $n > N$,

$$\|\Psi(t + s_n), \Psi(t)\| = \lim_{p \rightarrow \infty} \|\Phi(t + s_n + s_p), \Phi(t + s_p)\| \leq \epsilon,$$

and

$$\lim_{n \rightarrow \infty} \Psi(t + s_n) = \Psi(t)$$

uniformly over the whole set \mathfrak{R} . Finally suppose $\Theta(t)$ is equal to $\Phi(t)$ on $\mathfrak{J}\mathfrak{R}$ and satisfies uniformly on its set of definition $\mathfrak{Y}\mathfrak{R}$ the equation

$$\lim_{n \rightarrow \infty} \Theta(t + s'_n) = \Theta(t),$$

where $s'_n \rightarrow \infty$. Then on $\mathfrak{Y}\mathfrak{R}$, for sufficiently great m and n ,

$$\begin{aligned} \|\Psi(t), \Theta(t)\| &\leq \|\Psi(t), \Psi(t + s_n)\| + \|\Theta(t + s_n), \Theta(t + s_n + s'_n)\| \\ &\quad + \|\Psi(t + s_n + s'_n), \Psi(t + s'_n)\| + \|\Theta(t + s'_n), \Theta(t)\|, \end{aligned}$$

and since the second member approaches zero as n approaches infinity, $\Theta(t) = \Psi(t)$ on $\mathfrak{Y}\mathfrak{R}$, and the lemma is proved.

THEOREM I. *An a.p. space function or space sequence is completely determined by its values on any half infinite interval.*

THEOREM II. *An a.p. point or transformation ϕ possesses an inverse ϕ^{-1} (in the case of the transformation, a single-valued inverse over its whole set of definition), and its integer powers form an a.p. sequence or sequence of transformations.*

For if N_n is a $(1/n)$ -iteration exponent of ϕ and its value is greater than n for each positive n , then uniformly for non-negative k ,

$$\lim_{n \rightarrow \infty} \phi^{N_n + k} = \phi^k$$

and ϕ^n is an asymptotically periodic function ϕ_n of the non-negative integer n . But by the lemma, ϕ_n is defined for all integers n , and hence

$$\phi\phi_{-1} = \lim_{n \rightarrow \infty} \phi\phi_{-1+N_n} = I = \lim_{n \rightarrow \infty} \phi_{-1+N_n}\phi = \phi_{-1}\phi;$$

so that $\phi_n = \phi^n$ for all integers n . Moreover negative translation numbers for ϕ_n exist, since $\|\phi^{k-n}, \phi^k\| = \|\phi^k, \phi^{k+n}\|$.

4. FOURIER SEQUENCES

In the future it will often be convenient to state two or more theorems or definitions at once; and this will be done by the use of brackets. Where alternative words or sets of words are to be used, both alternatives will be inserted in the brackets and separated by a semi-colon. If no words are needed for

one of the alternatives, that will be indicated by a dash. In reading the theorem, read one set of words taken in the same relative position from each pair of brackets. Parenthetical expressions are indicated in the ordinary way and have nothing to do with the brackets.

DEFINITION. A finite or infinite sequence f_1, f_2, \dots of real numbers will be called an *upper Fourier sequence* of [a continuous space function; a two-way space sequence; a point in a \mathfrak{G} -space] Θ if to each $\epsilon > 0$ correspond a positive integer N and a positive number d such that any [number; integer; integer] t whose multiples tf_1, tf_2, \dots, tf_N all differ from integers by less than d is an [ϵ -translation number; ϵ -translation index; ϵ -iteration exponent] of Θ .

DEFINITION. A sequence will be called a *lower Fourier sequence* of Θ if to each positive number d and positive integer N (not greater than the number of elements f_i) corresponds a positive number ϵ such that all the multiples tf_1, tf_2, \dots, tf_N of any [ϵ -translation number; ϵ -translation index; ϵ -iteration exponent] t of Θ differ from integers by less than d .

The relationship between upper and lower Fourier sequences will be given in Theorem V.

DEFINITION. A sequence which is both an upper and a lower Fourier sequence is called a *Fourier sequence*.

THEOREM I. Every a.p. [function; sequence] in [Bohr's; Walther's] sense has at least one Fourier sequence.

In the case of the function, two of Bohr's* theorems show that a Fourier sequence can be obtained by dividing each Fourier exponent by 2π and arranging them in countable order. Moreover, Walther† has shown how to construct corresponding to a given a.p. sequence an a.p. function whose set of integer ϵ -translation numbers corresponding to each given $\epsilon > 0$ will be identical with the set of ϵ -translation indices of the sequence. Thus a Fourier sequence of the function will be a Fourier sequence of the sequence.

DEFINITION. If s and t are variable real [numbers; integers] and $\Theta(t)$ is an a.p. space [function; sequence], the real [function; sequence] $f(t) = \sup \Theta(s+t)$, $\Theta(s)$ is called the *Bochner translation* [function; sequence] of $\Theta(t)$.

THEOREM II. The Bochner translation [function; sequence] of a given a.p. space [function; sequence] is a.p., and its set of ϵ -translation [numbers; indices] for each given $\epsilon > 0$ is identical with the set of ϵ -translation [numbers; indices] of the given space [function; sequence].

* Zur Theorie der fastperiodischen Funktionen, Acta Mathematica, vol. 46 (1925), pp. 101-214, especially pp. 105 and 110.

† Fastperiodische Folgen und ihre Fouriersche Analyse, Atti del Congresso Internazionale dei Matematici, 1928 (VII), vol. 2, pp. 289-298, especially p. 290.

From Theorems I and II of this section and Theorem II, §3, we have immediately the following theorem which is fundamental in this work:

THEOREM III. *Every a.p. [space function; space sequence; point] has at least one Fourier sequence.*

THEOREM IV. *The necessary and sufficient condition that a [continuous space function; space sequence; point in a \mathcal{G} -space] be a.p. is that it have an upper Fourier sequence.*

Here sufficiency follows from Wennberg's* theorem on Diophantine approximation.

THEOREM V. *Each element f_p of a lower Fourier sequence of an a.p. [space function; space sequence; point] Θ is linearly dependent with integer coefficients on a finite number (dependent on p) of the elements of any upper Fourier sequence f'_1, f'_2, \dots of Θ [—; and unity; and unity].*

For let M be a positive integer greater than unity. Let e_M be a positive number such that tf_p differs from an integer by less than $1/(2M)$ whenever t is an e_M -[translation number; translation index; iteration exponent] of Θ . Let d_M be a positive number less than $1/(2M)$ and N_M a positive integer such that t is an e_M -[translation number; translation index; iteration exponent] of Θ whenever the numbers $tf'_1, tf'_2, \dots, tf'_{N_M}$ all differ from integers by less than d_M . Then there exists no number t at all which will make $tf_p + 1/M, tf'_1, tf'_2, \dots, tf'_{N_M}$ all differ from integers by less than d_M . Now according to a theorem of [Bohr†; Giraud‡; Giraud‡], a necessary and sufficient condition that there exist values of the variable [number; integer; integer] t which bring a given set of linear functions $a_i t + b_i$ ($i=0, 1, \dots, Q$) arbitrarily close to integers is that every set of integer multipliers g_0, g_1, \dots, g_Q which make the quantity $\sum_{i=0}^Q g_i a_i$ become [zero; an integer; an integer] should make the quantity $\sum_{i=0}^Q g_i b_i$ an integer. In the present case arbitrarily good approximating values of t do not exist if we put $a_0 = f_p, a_1 = f'_1, \dots, a_{N_M} = f'_{N_M}$ and $b_0 = 1/M, b_1 = 0, \dots, b_{N_M} = 0$; hence the condition is not satisfied, and there exist integers $g_M, g_{M,1}, g_{M,2}, \dots, g_{M,N_M}$ such that

$$g_M f_p + \sum_{i=1}^{N_M} g_{M,i} f'_i$$

* *Zur Theorie der Dirichlet'schen Reihen*, Dissertation, Upsala, 1920, p. 19.

† *Neuerer Beweis eines allgemeinen Kronecker'schen Approximationssatzes*, Det Kgl. Danske Videnskabernes Selskab, Mathematisk-Fysiske Meddelelser, vol. 6 (1924-25), Article 8.

‡ *Sur la résolution approchée en nombres entiers d'un système d'équations linéaires non homogènes*, Société Mathématique de France, Comptes Rendus des Séances, 1914, pp. 29-32.

is [zero; an integer; an integer] and such that g_M/M is not an integer. Thus each of the quantities $g_2 f_p, g_3 f_p, \dots$ can be expressed in a finite linear combination of [— —; unity and; unity and] the quantities f'_1, f'_2, \dots with integer coefficients. Now if k_1, k_2, \dots, k_n are the prime factors of g_2 , unity can be expressed as a finite linear combination of $g_2, g_{k_1}, g_{k_2}, \dots, g_{k_n}$, since g_M/M is not an integer; and hence f_p can be so expressed in terms of $g_2 f_p, g_{k_1} f_p, \dots, g_{k_n} f_p$.

THEOREM VI. *A sequence whose elements are linearly dependent with integer coefficients on [— —; unity and; unity and] a finite number of the elements of a lower Fourier sequence of an a.p. [space function; space sequence; point] Θ is itself a lower Fourier sequence of Θ . A sequence on a finite number of whose elements [— —; and unity; and unity] each element of an upper Fourier sequence of Θ is linearly dependent with integer coefficients is itself an upper Fourier sequence of Θ .*

For a linear combination of numbers with integer coefficients can be brought as close to an integer as we please by bringing the numbers sufficiently close to integers. The last two theorems lead immediately to the

THEOREM VII. *Let f_1, f_2, \dots be a Fourier sequence of an a.p. [space function; space sequence; point] Θ . Then a necessary and sufficient condition that a sequence f'_1, f'_2, \dots should also be a Fourier sequence of Θ is that each f_p be linearly dependent with integer coefficients on a finite number of f'_1, f'_2, \dots [— —; and unity; and unity], and vice versa.*

DEFINITION. A number module is a set of real numbers which forms a group under the operation of addition. It is called *complete* if it contains the number unity; otherwise *incomplete*. A denumerable number module which when arranged as a sequence constitutes a Fourier sequence is called a *Fourier module*.

Obviously any countable arrangement of a Fourier module is a Fourier sequence.

THEOREM VIII. *Each a.p. [space function; space sequence; point] has one and only one [— —; complete; complete] Fourier module.*

For the [function; sequence; point] has a Fourier sequence f_1, f_2, \dots . Let ϕ be the set of all numbers which are linearly dependent with integer coefficients on a finite number of the f_i [— —; and unity; and unity]. By Theorem VII, a sequence obtained by ordering ϕ is a Fourier sequence; hence ϕ is a Fourier module. Now if ϕ' is any [— —; complete; complete]

Fourier module, it is linearly dependent on ϕ [—; and unity; and unity] and vice versa, and must therefore be ϕ .

5. SCALARS

DEFINITION. A finite or infinite sequence of real numbers will be called a *scalar*, and the number of elements in it (which may and usually will be the symbol ∞) will be called its *length*. The [sum; product] of two scalars of the same length or *product of one scalar by a number* is the scalar obtained by [adding; multiplying] corresponding elements or multiplying each element by the number. The *scalar identity* ι is the sequence 1, 1, 1, \dots ; and the *scalar zero* (indicated by an ordinary zero) is the sequence 0, 0, 0, \dots ; for both ι and 0, the length of the scalar will be indicated by the context. Scalars will be indicated by small Greek letters and their elements by corresponding italics, thus: α ; a_1, a_2, \dots .

DEFINITION. The *absolute value* $|\alpha|$ of an infinite scalar α ; a_1, a_2, \dots will be the greatest lower bound for all positive integers n and k of

$$\frac{1}{n} + \frac{1}{k} + \max_{0 < j \leq n} \min_{-\infty < i < +\infty} |a_j + k!i|.$$

The *absolute value* $|\alpha|$ of a finite scalar α ; a_1, \dots, a_p will be the greatest lower bound for all positive integers k of

$$\frac{1}{k} + \max_{0 < j \leq p} \min_{-\infty < i < +\infty} |a_j + k!i|.$$

Using $|\alpha - \beta|$ as the distance between α and β , one can verify that the set of all scalars of the same given length is a metric space.

DEFINITION. A *reduced upper Fourier sequence* of a [space function; space sequence; point in a \mathfrak{G} -space] is a sequence on a finite number of whose elements each element of an upper Fourier sequence is rationally linearly dependent. A *base* is a reduced upper Fourier sequence every finite subset of which is rationally linearly independent. A base is called *minimal* if each of its elements is rationally linearly dependent on a finite number of the elements of an incomplete Fourier module, or in case none exists, the complete Fourier module. A base for a space sequence or a point in a \mathfrak{G} -space is called *proper* either if it contains a rational element or if unity is not rationally linearly dependent on any finite subset of its elements.

It follows from Theorem IV, §4, that the statements that a [continuous space function; space sequence; point in a \mathfrak{G} -space] has a reduced upper Fourier sequence, has a base, has a proper minimal base, or is a.p. are all equivalent.

THEOREM I. *If s and t are variable real [numbers; integers], a necessary and sufficient condition that the scalar γ be a reduced upper Fourier sequence of the [continuous space function $\Theta(t)$; space sequence $\Theta_i = \Theta(t)$] is that uniformly in s*

$$\lim_{t\gamma \rightarrow 0} \Theta(s+t) = \Theta(s).$$

To prove sufficiency, let f_1, f_2, \dots be an upper Fourier sequence of Θ whose elements are rationally linearly dependent on $\gamma: c_1, c_2, \dots$. By making $|t\gamma|$ sufficiently small, an arbitrarily large number of the quantities tc_1, tc_2, \dots can be brought arbitrarily close to multiples of an arbitrarily large $k!$. There exist integers p_i such that each $p_i f_i$ is a finite linear combination of the c_j with integer coefficients. Thus an arbitrarily large number of the quantities $tp_i f_i$ can be brought arbitrarily close to multiples of $k!$, and by choosing k large enough so that $k!$ contains all of the corresponding p_i as factors, arbitrarily many of the tf_i may be brought arbitrarily close to integers. Hence t will be an arbitrarily good translation number. To prove necessity we need only notice that the integer sub-multiples of the elements of γ when arranged in countable order form an upper Fourier sequence.

From the above theorem and Theorem II, §3, we obtain

THEOREM II. *If n is an integer, a necessary and sufficient condition that a scalar γ be a reduced upper Fourier sequence of the point Λ in a \mathcal{G} -space having the identity point I is that*

$$\lim_{n\gamma \rightarrow 0} \Lambda^n = I.$$

6. ALMOST PERIODIC PROPERTIES INVARIANT UNDER MULTIPLICATION

THEOREM I. *If $[\Phi(t); \Gamma_n; \Lambda]$ is an a.p. [function; sequence; point] in a \mathcal{G} -space, then $[\Phi(t)\Theta; \Gamma_n\Theta; \Lambda^n\Theta]$ is a uniformly continuous function of the point Θ uniformly with respect to $[t; n; n]$.*

For any finite set of values of t or n , the theorem is obvious. Since $\Phi(t)$ is uniformly continuous, for an arbitrary $\epsilon > 0$ we can divide any finite interval \mathfrak{J} up into a sufficiently large number of equal intervals so that on any such interval $\Phi(t)$ varies by less than $\epsilon/3$. After choosing a point t_i from each interval, we can bring all the points $\Phi(t_i)\Theta'$ within a distance of $\epsilon/3$ from the corresponding points $\Phi(t_i)\Theta''$ by bringing Θ' sufficiently close to Θ'' . Thus we can bring $\Phi(t)\Theta'$ within ϵ of $\Phi(t)\Theta''$ for all t on \mathfrak{J} by bringing Θ' sufficiently close to Θ'' . Thus the theorem is true for functions, sequences or points on any finite interval. That it is also true for the infinite interval can be seen by choosing a length L corresponding to $\epsilon > 0$ so great that on any interval of this length there is always an $(\epsilon/3)$ -translation number or

index or iteration exponent. Then $[\Phi(t); \Gamma_n; \Lambda^*]$ for any value of $[t; n; n]$ can be replaced by $[\Phi(t_0); \Gamma_{n_0}; \Lambda^{*0}]$ with an error not greater than $\epsilon/3$, where $[t_0; n_0; n_0]$ lies between 0 and L . But by bringing Θ' and Θ'' sufficiently close, $[\Phi(t_0)\Theta'; \Gamma_{n_0}\Theta'; \Lambda^{*0}\Theta']$ can be brought within a distance of $\epsilon/3$ from $[\Phi(t_0)\Theta''; \Gamma_{n_0}\Theta''; \Lambda^{*0}\Theta'']$ for all $[t_0; n_0; n_0]$ between zero and L at once. Thus $[\Phi(t)\Theta'; \Gamma_n\Theta'; \Lambda^*\Theta']$ would be at a distance not greater than ϵ for all $[t; n; n]$ at once from $[\Phi(t)\Theta''; \Gamma_n\Theta''; \Lambda^*\Theta'']$.

THEOREM II. *The product of any two [—; —; permutable] a. p. [functions; sequences; points] in a \mathcal{C} -space is a p.*

For let s and t be real [numbers; integers; integers] and let $\Theta_1(t)$ and $\Theta_2(t)$ be the [a.p. functions; a.p. sequences; t th powers of the a.p. points] having the reduced upper Fourier sequences γ_1 and γ_2 . Let γ^* be a sequence whose elements comprise all the elements of γ_1 and γ_2 . Then both $t\gamma_1$ and $t\gamma_2$ can be brought arbitrarily close to zero by bringing $t\gamma^*$ sufficiently close to zero. Thus, uniformly in s ,

$$\lim_{t\gamma^* \rightarrow 0} \Theta_1(s+t) = \Theta_1(s) \quad \text{and} \quad \lim_{t\gamma^* \rightarrow 0} \Theta_2(s+t) = \Theta_2(s).$$

Then since $\Theta_1(t)\Gamma$ is uniformly continuous in Γ uniformly with respect to t , it follows that uniformly in s

$$\lim_{t\gamma^* \rightarrow 0} \|\Theta_1(s+t)\Theta_2(s+t); \Theta_1(s+t)\Theta_2(s)\| = 0$$

and hence that uniformly in s ,

$$\lim_{t\gamma^* \rightarrow 0} \|\Theta_1(s+t)\Theta_2(s+t); \Theta_1(s)\Theta_2(s)\| = 0.$$

Now we note that in case $\Theta_1(t)$ and $\Theta_2(t)$ represent the t th powers of permutable points, $\Theta_1(t)\Theta_2(t)$ represents the t th power of the product of the points; and hence in all three cases our theorem is proved.

COROLLARY. *A sequence whose elements comprise all the elements of γ_1 and γ_2 which are reduced upper Fourier sequences of two a.p. [functions; sequences; permutable points] in a \mathcal{C} -space is a reduced upper Fourier sequence for the product of the [functions; sequences; points].*

DEFINITION. A sequence of points $\Lambda_1, \Lambda_2, \dots$ will be said to converge exponentially uniformly if $\Lambda_1^n, \Lambda_2^n, \dots$ converges uniformly with respect to n for all integers n .

THEOREM III. *If a sequence of a.p. [space functions; space sequences; points] converges [—; —; exponentially] uniformly, its limit is a.p.*

For, using the natural extension of the notation of the last theorem, it follows from the fact that $\lim_{n \rightarrow \infty} \Theta_n(t)$ is uniform in t that

$$\lim_{t\gamma^s \rightarrow 0} \lim_{n \rightarrow \infty} \Theta_n(s+t) = \lim_{n \rightarrow \infty} \lim_{t\gamma^s \rightarrow 0} \Theta_n(s+t) = \lim_{n \rightarrow \infty} \Theta_n(s),$$

where the limits with respect to t are uniform in s .

COROLLARY. If $\gamma_1, \gamma_2, \dots$ are reduced upper Fourier sequences for an [—; —; exponentially] uniformly convergent sequence of a.p. [space functions; space sequences; points] and γ^* is a sequence which has each of the γ_i as subsequences, then γ^* is a reduced upper Fourier sequence for the limit [function; sequence; point].

THEOREM IV. The [—; —; exponentially] uniform limit of an infinite product of [—; —; permutable] a.p. [space functions; space sequences; points] in a \mathcal{G} -space is a.p.

7. PSEUDO-ARGUMENTS, INDICES, AND EXPONENTS

DEFINITION. If $[\Theta(t); \Gamma_n; \Lambda]$ is an a.p. [space function; space sequence; point] having the base γ ; and if α is any scalar of the same length as γ , then the symbol $[\Theta(\alpha)_\gamma; (\Gamma_\alpha)_\gamma; \Lambda_\gamma^a]$ will denote

$$[\lim_{t\gamma \rightarrow \alpha\gamma} \Theta(t); \lim_{n\gamma \rightarrow \alpha\gamma} \Gamma_n; \lim_{n\gamma \rightarrow \alpha\gamma} \Lambda^n]$$

and will be called the *pseudo- [value of the function; element of the sequence; power of the point] corresponding to the pseudo- [argument; index; exponent] α with respect to the base γ* . The base γ will be omitted from the notation when the context makes clear what base is to be used. If the points of the space are transformations, the same nomenclature will be used except that for a family of transformations the terms *pseudo-value* or *argument* will be replaced by *pseudo-member* or *parameter*. A pseudo- [element; power] of an a.p. [space sequence; point] with respect to a base γ will be called *proper* if γ is a proper base and no non-integer element of the pseudo- [index; exponent] corresponds to a rational element of γ .

THEOREM I. All [—; proper; proper] pseudo- [values; elements; powers] of an a.p. [space function; space sequence; point] exist.

For if t is a variable real [number; integer; integer] and α is any scalar [argument; index; exponent] which satisfies the hypothesis, then $\alpha\gamma$ can be approached arbitrarily closely by the scalar $t\gamma$; and if $\Theta(t)$ is the [function; sequence; t th power of the point], then uniformly in t

$$\lim_{s\gamma, s'\gamma \rightarrow \alpha\gamma} \Theta(s - s' + t) = \Theta(t);$$

so that

$$\lim_{s\gamma, s'\gamma \rightarrow \alpha\gamma} \|\Theta(s); \Theta(s')\| = 0$$

and the theorem follows.

THEOREM II. *If t is a real [number; integer; integer] and ι is the identity scalar of the same length as the base γ of the a.p. [space function $\Theta(t)$; space sequence Γ_t ; point Λ], then $[\Theta(t)\gamma; (\Gamma_t)_\gamma; \Lambda_t^\iota]$ is the same point as $[\Theta(t); \Gamma_t; \Lambda^\iota]$.*

THEOREM III. *The [—; proper; proper] pseudo- [value $\Theta(\alpha)_\gamma$; element $(\Gamma_\alpha)_\gamma$; power Λ_γ^α] of the a.p. [space function $\Theta(t)$; space sequence Γ_n ; point Λ] is a uniformly continuous function of the scalar $\alpha\gamma$ for all [—; admissible; admissible] values of α .*

For in the case of the function having the base γ , to a given $\epsilon > 0$ corresponds $d > 0$ so small that for all t and t' satisfying $|(t-t')\gamma| \leq d$,

$$(1) \quad \|\Theta(t); \Theta(t')\| \leq \epsilon.$$

Now let α and α' be any scalar satisfying $|\alpha - \alpha'|\gamma| < d$. Then when $t\gamma$ and $t'\gamma$ are sufficiently close to $\alpha\gamma$ and $\alpha'\gamma$ respectively, the equation (1) is satisfied; and hence

$$\|\Theta(\alpha); \Theta(\alpha')\| = \lim_{t\gamma \rightarrow \alpha\gamma, t'\gamma \rightarrow \alpha'\gamma} \|\Theta(t); \Theta(t')\| \leq \epsilon.$$

Similar arguments show that the theorem holds for sequences and points also.

THEOREM IV. *If Λ^α and Λ^β are proper pseudo-powers of the a.p. point Λ taken with respect to the same base, then*

$$\Lambda^\alpha \Lambda^\beta = \Lambda^{\alpha+\beta}.$$

For

$$\lim_{m\gamma \rightarrow (\alpha+\beta)\gamma} \Lambda^n = \lim_{m\gamma \rightarrow \alpha\gamma, n\gamma \rightarrow \beta\gamma} \Lambda^{m+n} = \lim_{m\gamma \rightarrow \alpha\gamma} \Lambda^m \lim_{n\gamma \rightarrow \beta\gamma} \Lambda^n.$$

COROLLARY 1. *Any two proper pseudo-powers of an a.p. point are permutable.*

COROLLARY 2. *If Λ^α is a proper pseudo-power of the a.p. point Λ , then $(\Lambda^\alpha)^n = \Lambda^{n\alpha}$.*

THEOREM V. *If γ is a base for the a.p. [space function $\Theta(t)$; space sequence Γ_n ; point Λ] and $[\alpha$ and β are any scalars; Γ_α and Γ_β are proper pseudo-elements; Λ^α is a proper pseudo-power], then $[\Theta(t\alpha+\beta); \Gamma_{n\alpha+\beta}; \Lambda^\alpha]$ is a.p. and has the reduced upper Fourier sequence $\alpha\gamma$.*

For in the case of the function, as $t\alpha\gamma \rightarrow 0$, $[(s+t)\alpha + \beta]\gamma \rightarrow [s\alpha + \beta]\gamma$ uniformly in s ; and since $\Theta(\alpha)$ is uniformly continuous in $\alpha\gamma$ it follows that uniformly in s

$$\lim_{t\alpha\gamma \rightarrow 0} \Theta[(s+t)\alpha + \beta] = \Theta[s\alpha + \beta].$$

The proof is essentially the same for the functions and sequences.

8. MONO-BASAL FUNCTIONS, SEQUENCES AND POINTS

Notation. Let ω_p denote the scalar which has its p th element equal to unity and all other elements zero. Its length will be indicated by the context.

DEFINITION. A [space function; space sequence; point] is called *mono-basal* if it is a.p. and has a base consisting of but one element.

THEOREM I. Any pure periodic [space function; space sequence; point] is mono-basal, having unity as a base.

THEOREM II. If the a.p. [space function $\Theta(t)$; space sequence Γ_n ; point Λ] has the [—; proper; proper] base $\gamma: c_1, c_2, \dots$ and $[\beta$ is any scalar; $(\Gamma_\beta)_\gamma$ is any proper pseudo-element; —], then $[\Theta(t\omega_p + \beta)_\gamma; (\Gamma_{n\omega_p + \beta})_\gamma; \Lambda_\gamma \omega_p]$ is mono-basal and has c_p as a minimal base.

For it has $\omega_p\gamma$ as a reduced upper Fourier sequence.

DEFINITION. A space [function; multiple sequence] of any countable set of [variables; indices] will be called *mono-basal* in any one of its [variables t ; indices n] if it is a mono-basal [function of t ; sequence in n] for each set of constant values of the other [variables; indices]. The diagonal [function; sequence] of the [function $\Theta(t_1, t_2, \dots)$; multiple sequence $\Gamma_{n_1, n_2, \dots}$] is the [function $\Theta(t, t, \dots)$; sequence $\Gamma_{n, n, \dots}$].

THEOREM III. An a.p. space [function; sequence] can be expressed as the diagonal [function; sequence] on a mono-basal space [function; multiple sequence] of a countable set of variables.

For let $\Theta(t)$ be the a.p. [function; sequence] of the real [number; integer] t . Then $\Theta(t) = \Theta(t) = \Theta(t_1\omega_1 + t_2\omega_2 + \dots)$ is the diagonal of $\Theta(t_1\omega_1 + t_2\omega_2 + \dots)$, which is mono-basal in each of its arguments.

THEOREM IV. If Λ is an a.p. point, then with respect to any proper base the infinite product $\Lambda^{\omega_1}\Lambda^{\omega_2}\dots$ or $\dots \Lambda^{\omega_2}\Lambda^{\omega_1}$ converges absolutely and exponentially uniformly to the value Λ .

For $n\omega_1 + n\omega_2 + \dots$ converges uniformly in n to n , and hence $\Lambda^{n\omega_1 + n\omega_2 + \dots}$ converges uniformly in n to Λ^n . Moreover changing the order of the $\omega_1, \omega_2, \dots$ would not destroy the convergence.

Because of its importance in this work, the following theorem will be stated in terms of both points and transformations.

THEOREM V. *A necessary and sufficient condition that a [transformation; point] be a.p. is that it be the exponentially uniformly convergent infinite product of permutable mono-basal [transformations; points].*

9. EXAMPLES

I will bring this paper to a close by giving two examples of a.p. transformations.

I. Let A_k be a two-way sequence of non-negative real numbers such that $\sum_{k=-\infty}^{\infty} A_k$ converges. Let the space \mathfrak{C} have as its points the complex functions $f(x)$ of a real variable x having the Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx},$$

where the a_k are numbers satisfying $|a_k| \leq A_k$. Let the distance between any two transformations $F(x) = \Theta_1[f(x)]$ and $F(x) = \Theta_2[f(x)]$ be

$$\max_{x, f} |\Theta_1[f(x)] - \Theta_2[f(x)]|.$$

Let c_k be any two-way sequence of complex numbers each having the absolute value 1, and let

$$g(y, t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} c_k e^{iky} t^{|k|}$$

be a function of the real variables y and t . Then the transformation

$$(1) \quad \Phi[f(x)] = F(x) = \lim_{t \rightarrow 1^-} \int_0^{2\pi} f(x-y) g(y, t) dy$$

is a.p.

For the set of transformations of the form (1) which is obtained by using all possible sets of values for the coefficients c_k of the function $g(y, t)$ is a \mathfrak{C} -space. Moreover it can be shown that

$$\Phi[f(x)] = \sum_{k=-\infty}^{+\infty} c_k a_k e^{ikx}$$

and hence that

$$\frac{\log c_0}{2\pi i}, \quad \frac{\log c_1}{2\pi i}, \quad \frac{\log c_{-1}}{2\pi i}, \quad \frac{\log c_2}{2\pi i}, \quad \frac{\log c_{-2}}{2\pi i}, \quad \dots$$

is a reduced upper Fourier sequence of Φ .

II. Let c_1, c_2, \dots be an infinite sequence of distinct complex numbers each having the absolute value 1, and let $\sum A_k$ be a convergent series of non-negative real numbers. Let \mathfrak{C} have as its points all functions $f(z)$ of the complex variable z of the form

$$(1) \quad f(z) = \sum_{k=1}^{\infty} a_k e^{c_k z},$$

where the a_k are any complex numbers satisfying $|a_k| \leq A_k$.

Let the distance between two transformations

$$F(z) = \Theta_1[f(z)] \quad \text{and} \quad F(z) = \Theta_2[f(z)]$$

be the least upper bound for all functions $f(z)$ in \mathfrak{C} of $\sum_{k=1}^{\infty} |a'_k - a''_k|$, where a'_k and a''_k are the coefficients of the series of the form (1) for $\Theta_1[f(z)]$ and $\Theta_2[f(z)]$. Then the transformation

$$\Phi[f(z)] = F(z) = \frac{d}{dz} f(z)$$

is a.p.

For it can be shown that each function of \mathfrak{C} has a unique representation of the form (1). Let us associate with each sequence of numbers c'_1, c'_2, \dots , each of whose absolute values is unity the transformation which takes each function of the form (1) into the corresponding function

$$F(z) = \sum_{k=1}^{\infty} c'_k a_k e^{c_k z}.$$

This set of transformations is a \mathfrak{G} -space, and it contains the transformation Φ which corresponds to c_1, c_2, \dots and has the reduced upper Fourier sequence

$$\frac{\log c_1}{2\pi i}, \quad \frac{\log c_2}{2\pi i}, \quad \dots$$

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THE BERTINI TRANSFORMATION IN SPACE*

BY

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1. **Introduction.** Examples are known of involutorial transformations I having an invariant pencil of planes through a line l , such that in each plane of the pencil there is a transformation of the Geiser or the de Jonquières type. We have shown in this paper that involutorial transformations exist which have in each plane through l a Bertini transformation with 6 of the fundamental points lying on a C_6 , $p=3$, the other two being on l , and either fixed or variable. There is another type in which 6 of the 8 fundamental points lie on a C_6 , $p=4$, and in every plane through l there is a degenerate Bertini transformation. A third type is discussed in which there is a net of invariant quartic surfaces through a C_{11} , $p=14$. The method of obtaining this last transformation leads also to an involutorial transformation with a net of invariant surfaces of order $n+1$ through a C_{5n-3} of genus $12n-19$. This type has on each plane through l a Geiser transformation having the 7 fundamental points on C_{5n-3} .

2. **The involutorial Bertini transformation I_B on a cubic surface F_3 .** The conics tangent to a cubic surface F_3 at two fixed points O_1, O_2 meet F_3 in two residual points P, P' which are conjugate points of an involutorial Bertini transformation I_B on F_3 . The web of quadrics tangent to F_3 at O_1, O_2 meet F_3 in a web of sextic curves of genus 2 which is invariant under I_B as is also the pencil of plane sections through the line $l:O_1+O_2$. If the space (y) of F_3 is transformed into a space (z) by means of the web of cubic surfaces through a fixed C_6 , $p=3$, on F_3 , then F_3 is transformed into a plane meeting the fundamental sextic of the transformation in 6 points Q_3, \dots, Q_8 . If Q_1, Q_2 are the transforms of O_1, O_2 , then I_B becomes a plane transformation of order 17, a line going into a $C_{17}:8Q^6$. The image of each six-fold point Q_i is a $C_6:Q_i^3+7Q_j^3$ ($j \neq i$). The line Q_1Q_2 is the transform of a cubic curve on F_3 through O_1, O_2 .

Analytically, if $y_2=0, y_1=0$ are the planes tangent to F_3 at $O_1 \equiv (1, 0, 0, 0)$, $O_2 \equiv (0, 1, 0, 0)$, the equation of F_3 may be written

$$(1) \quad Ay_1 + By_2 + C \equiv y_1(a^2y_1y_2 + \alpha) + y_2(b^2y_1y_2 + \beta) + \gamma y_1y_2 + \delta = 0,$$

where $\alpha, \beta, \gamma, \delta$ are binary forms in y_3, y_4 . The transformation I_B is defined by

$$(2) \quad Ay'_1 = By_2, By'_2 = Ay_1, y'_3 = y_3, y'_4 = y_4.$$

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The images of O_1, O_2 are the sextics in which the quadrics $A=0, B=0$ meet F_3 . Since the second polars of O_1, O_2 differ from A, B by terms containing y_3, y_1 respectively, the quadrics have second-order contact with F_3 at O_1, O_2 respectively and meet F_3 in sextics having triple points at O_1, O_2 respectively.

3. The transformation I_B for a pencil of cubic surfaces. Since a point P determines an F_3 of the pencil (parameter λ) on which P' can be found by the method of §2, we can define involutorial space transformations by either taking O_1, O_2 as fixed points lying on each F_3 of the pencil, or by taking one or both of them variable (the coordinates being functions of λ), on a rational curve lying on each F_3 .

4. Case I, O_1, O_2 are fixed. For this case we take a pencil of cubic surfaces F_3 having in common a $C_9, p=10$, through $O_1 \equiv (1, 0, 0, 0), O_2 \equiv (0, 1, 0, 0)$, and write the equation of the F_3 in the form

$$(3) \quad \begin{aligned} F_3 &\equiv F'_3 - \lambda F''_3 \\ &\equiv ax_1^2x_2 + bx_1x_2^2 + cx_1^3 + dx_1x_2 + ex_2^3 + fx_1 + gx_2 + h = 0, \end{aligned}$$

where $a \equiv a' - \lambda a''$, etc., and c', c'' , etc., are binary forms in x_3, x_4 . A change of coordinate system given by

$$(4) \quad y_1 = bx_1 + e, \quad y_2 = ax_2 + c, \quad y_3 = x_3, \quad y_4 = x_4$$

will express F_3 in the form (1). The transformation (2) in terms of x_i is

$$(5) \quad \begin{aligned} x'_1 &= (ax_2B + cB - eA)Ba, \\ x'_2 &= (bx_1A - cB + eA)Ab, \\ x'_3 &= x_3ABab, \quad x'_4 = x_4ABab. \end{aligned}$$

The surface of invariant points $K \equiv y_1A - y_2B = 0$ contains ab as a factor as does the transformation (5). The forms A, B are of degree 4 in λ and of degree 2 in x_i , so that the x'_i are of degree 5 in x_i and of degree 8 in λ . If λ is replaced by F'_3/F''_3 we have an I_{29} in which the image of O_1 is $A=0$, the image of O_2 is $B=0$, and the image of C_9 can be obtained by applying the transformation to an S_{29} . The table of characteristics of the I_{29} is

$$\begin{aligned} O_1 &\sim F_{14}: O_1^7 + O_2^6 + C_9^4, \\ O_2 &\sim F_{14}: O_1^6 + O_2^7 + C_9^4, \\ C_9 &\sim F_{66}: O_1^{28} + O_2^{28} + C_9^{15}, \\ S_1 &\sim S_{29}: O_1^{14} + O_2^{14} + C_9^8, \\ K_{12} &: O_1^6 + O_2^6 + C_9^3. \end{aligned}$$

Every plane through the line $l: O_1 + O_2$ cuts from the F_{56} a composite curve of order 56, the 7 components of which are the images of the 7 residual intersections of C_9 with the plane. If O_i ($i=3, \dots, 9$) is any one of these 7 points and $6O_j$ ($j=3, \dots, 9$) are the others, then

$$O_i \sim C_9: O_1^4 + O_2^4 + O_i^3 + 6O_j^2 \quad (i, j = 3, \dots, 9; i \neq j).$$

In each of these planes there is a transformation of the Bertini type of order 29. If it is transformed by a quadratic transformation having O_1, O_2, O_i for fundamental points it becomes the usual Bertini transformation of order 17 with 8 six-fold points at $O_1, O_2, 6O_j$.

Since C_9 is of genus 10 there are 11 trisecants of the C_9 which pass through O_1 or O_2 . Any one of these 22 lines meets an S_{29} in $14 + 2 \cdot 8 = 30$ points and therefore lies on the S_{29} . These lines are the fundamental lines of the second species in the I_{29} . The surface R_{42} of trisecants of C_9 contains C_9 as an 11-fold curve. The line l meets R_{42} in 20 points not on C_9 from which trisecants of C_9 may be drawn. In any one of the 20 planes determined by one of these trisecants and l , the 6 residual intersections of C_9 lie on a conic. Each of these 20 conics meets an S_{29} in $2 \cdot 14 + 4 \cdot 8 = 60$ points and therefore lies doubly on the S_{29} . They are the fundamental conics of the second species in the I_{29} . The tangent planes to the pencil of cubic surfaces at O_1 form a pencil of planes through the tangent line to C_9 at O_1 . The plane of the pencil which passes through O_2 cuts from the corresponding F_3 a cubic curve with a double point at O_1 and through O_2 and the 6 residual intersections of C_9 with the plane. There is another such cubic curve with the roles of O_1, O_2 interchanged. Each of these cubics meets an S_{29} in $28 + 14 + 6 \cdot 8 = 90$ points and hence lies triply on the S_{29} . They are the fundamental cubics of the second species in the I_{29} . There exist then 22 lines, 20 conics, and 2 cubics which are parasitic curves in the involutorial transformation I_{29} .

If the C_9 is composed of a space cubic C_3 through O_1, O_2 and a C_6 , $p=3$, $[C_3, C_6]=8$, the surface F_{56} breaks up into an $F_8: O_1^4 + O_2^4 + C_3^3 + C_6^2$, the image of C_3 , and an $F_{48}: O_1^{24} + O_2^{24} + C_3^{12} + C_6^{12}$, the image of C_6 . If we transform the space (x) into a space (z) by means of the cubic transformation $T_{3,3}: C_6$ the pencil of F_3 's becomes a pencil of planes through the line l' which is the transform of C_3 . In each plane through l' there is a Bertini transformation of order 17. The transform of the surface F_8 of trisecants of C_6 is C_6' , and to O_1, O_2 correspond the points Q_1, Q_2 . The characteristics of the I_{29} in the (z) space are

$$\begin{aligned} Q_1 &\sim F_{10}: Q_1^7 + Q_2^6 + l'^4 + C_6'^2, \\ Q_2 &\sim F_{10}: Q_1^6 + Q_2^7 + l'^4 + C_6'^2, \end{aligned}$$

$$\begin{aligned}
 l' &\sim F_8 : Q_1^4 + Q_2^4 + l'^3 + C_6'^2, \\
 C_6' &\sim F_{64} : Q_1^{40} + Q_2^{40} + l'^{28} + C_6'^{13}, \\
 S_1 &\sim S_{29} : Q_1^{18} + Q_2^{18} + l'^{12} + C_6'^6, \\
 K_{12} &: Q_1^6 + Q_2^6 + l'^3 + C_6'^3.
 \end{aligned}$$

In the (x) space in place of the surface $R_{42}:C_3^{11}$ of trisecants of C_3 we have the surface $R_3:C_6^3$ of trisecants of C_6 , the surface $R_3':C_3^4+C_6$ of bisecants of C_3 which meet C_6 , and the surface $R_{26}:C_3^7+C_6^7$ of bisecants of C_6 which meet C_3 . The 22 parasitic lines in the (x) space are (a) the 4 bisecants of C_3 through O_1 which meet C_6 , (b) the 4 bisecants of C_3 through O_2 which meet C_6 , (c) the 7 bisecants of C_6 through O_1 , (d) the 7 bisecants of C_6 through O_2 . The lines of types (a), (b) correspond to parasitic conics in the (z) space through Q_1 or Q_2 and meeting C_6' in 5 points. The lines of types (c), (d) correspond to parasitic lines which are bisecants of C_6' from Q_1 or Q_2 . The 8 trisecants of C_6' meeting l' are parasitic and correspond in the (x) space to the 8 points $[C_3, C_6]$.

The line l meets the surface $R_3:C_6^3$ in 8 points, hence 8 trisecants of C_6 meet l . Each of the planes determined by l and one of these trisecants meets the F_3 containing the trisecant in a residual conic which is parasitic. The 8 conics go into parasitic cubics in the (z) space which have double points on C_6' and pass through Q_1, Q_2 , and 5 points on C_6' . The surface $R_{26}:C_3^7+C_6^7$ is met by l in 12 points, hence 12 bisecants of C_6 meet C_3 and l . In each of the 12 planes determined by these lines and l there is a parasitic conic which corresponds to a parasitic conic in the (z) space through Q_1, Q_2 , and 4 points of C_6' . The two parasitic cubics with double points at O_1 or O_2 and through O_2 or O_1 and 6 points of C_6 correspond to similar cubics in the (z) space. The I_{29} in the (z) space has 22 lines, 20 conics, and 10 cubics which are fundamental curves of the second species.

5. **Case II, O_1 is variable on a space cubic curve C_3 .** We take a pencil of cubic surfaces (parameter λ) through a space cubic curve C_3 containing the points $O_1=(1, \lambda^3, \lambda^2, \lambda)$, $O_2=(0, 1, 0, 0)$, and having the equation

$$(6) \quad F_3 = \begin{vmatrix} (px) & x_1 & x_4 \\ (qx) & x_4 & x_3 \\ (rx) & x_3 & x_2 \end{vmatrix} \equiv F_3' - \lambda F_3'' \equiv (px)H_p + (qx)H_q + (rx)H_r = 0,$$

where $(px) = p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4$, $p_i = p_i' - \lambda p_i''$, etc., and H_p, H_q, H_r are quadrics through C_3 . A point $P(x)$ determines an F_3 of the pencil and a definite point O_1 so that by the construction of §2 we can determine a point $P'(x')$. A change of variables is made by

$$y_1 = p_2x_4 - q_2x_1,$$

$$y_2 = p(x_2 - 2\lambda x_3 + \lambda^2x_4) - q(x_3 - 2\lambda x_4 + \lambda^2x_1),$$

$$y_3 = x_3 - \lambda x_4,$$

$$y_4 = x_4 - \lambda x_1,$$

where $p \equiv \lambda P - Q$, $q \equiv \lambda Q - R$, $P \equiv p_1 + p_2\lambda^3 + p_3\lambda^2 + p_4\lambda$, etc. The planes $y_1=0$, $y_2=0$ are the tangent planes at O_2 , O_1 respectively, and the planes $y_3=0$, $y_4=0$ are a pair of planes through the line $l:O_1+O_2$. The pencil of cubic surfaces is now of the form (1) and the transformation (2) gives I_B .

The equations of the surfaces $A=0$, $B=0$, $K=0$ may be written in terms of x_i and y_i as follows:

$$A \equiv mp[p(PH_p + QH_q + RH_r) + (y_3 - \lambda y_4)\{\lambda q(px) - M(qx) + p(rx)\}] \\ - y_2[m(y_3 - \lambda y_4)(\lambda q p_2 - Mq_2 + pr_2) + py_4(Qp_2 - Pq_2)] = 0,$$

$$B \equiv mp[m(p_2H_p + q_2H_q + r_2H_r) - y_4\{q_2(px) - p_2(qx)\}] \\ - y_1[m(y_3 - \lambda y_4)(\lambda q p_2 - Mq_2 + pr_2) + py_4(Qp_2 - Pq_2)] = 0,$$

$$K \equiv y_1[p(PH_p + QH_q + RH_r) + (y_3 - \lambda y_4)\{\lambda q(px) - M(qx) + p(rx)\}] \\ - y_2[m(p_2H_p + q_2H_q + r_2H_r) - y_4\{q_2(px) - p_2(rx)\}] = 0,$$

where $m \equiv \lambda p_2 - q_2$, $M \equiv \lambda p + q$. The surfaces $A=0$, $B=0$ are of the second degree in x_i and of degrees 16, 10 in λ respectively. The surface $K=0$ is of the third degree in x_i and of degree 9 in λ .

We transform the space (x) into a space (z) as was done in the latter part of §4. A surface of the web in the (x) space goes into a surface of the web in the (z) space such that in any plane through l' there is a Bertini transformation of order 17. By this I_{17} the image of l' is a $C_5:Q_1+Q_2+6Q^2$ which with l' makes up a $C_6:8Q^2$ that is the plane section of a sextic surface having Q_1 , Q_2 , C'_6 as double elements. This sextic surface is the transform of a quadric surface through C_3 and tangent to F_3 at O_1 , O_2 . This quadric which is the image of C_3 and is of the sixth degree in λ has the equation

$$p p_2 H_p + p q_2 H_q + (\lambda p q_2 - \lambda q p_2 + q q_2) H_r = 0.$$

Any plane through l' is invariant under I_B , hence a pencil of surfaces of the web in the (z) space is made up of the pencil of planes through l' together with the image surfaces of l' , Q_1 , Q_2 . Since the image of Q_1 by the I_{17} in a plane through l' is a $C_6:Q_1^3+Q_2^2+6Q^2$ and since $A=0$ is of order 16 in λ , the image of Q_1 by the I_B in the (z) space is a surface of order $6+16=22$ on which l' is a 16-fold line with 3 sheets of the surface having contact along l' . The point Q_1 is $16+3=19$ -fold; Q_2 is a $16+2=18$ -fold point; C'_6 is a double curve. In the same way we obtain the surfaces corresponding to Q_2 and l' , and

the invariant surface K . The table of characteristics of the I_{51} is

$$\begin{aligned} l' &\sim F_{12} : l'^{7+t} + C_6'^2 + Q_1^8 + Q_2^8, \\ Q_1 &\sim F_{22} : l'^{16+3t} + C_6'^2 + Q_1^{19} + Q_2^{18}, \\ Q_2 &\sim F_{16} : l'^{10+2t} + C_6'^2 + Q_1^{12} + Q_2^{13}, \\ C_6' &\sim F_{112} : l'^{76+12t} + C_6'^{13} + Q_1^{88} + Q_2^{88}, \\ S_1 &\sim S_{51} : l'^{34+6t} + C_6'^6 + Q_1^{40} + Q_2^{40}, \\ K_{18} &: l'^{9+3t} + C_6'^3 + Q_1^{12} + Q_2^{12}, \end{aligned}$$

where the coefficient of t indicates the number of fixed tangent planes at a point of l' .

In determining the number of parasitic lines, conics, and cubics the methods in the previous section have to be changed when the variable point Q_1 is involved. There are 7 bisecants of C_6' through Q_2 and 8 trisecants of C_6' meeting l' which are parasitic. In any plane λ through l' the 15 bisecants of C_6' meet l' in 15 points μ ; through any point μ on l' the 7 bisecants of C_6' determine 7 planes λ through l' . The number of coincidences in the (λ, μ) correspondence is $15+7=22$, and hence in 22 positions of Q_1 a bisecant of C_6' can be drawn from Q_1 in the plane through l' associated with Q_1 . These bisecants are parasitic lines.

There are 4 conics through Q_2 and 5 points of C_6' in planes through l' which are parasitic. In any plane λ through l' the 6 conics through 5 points of C_6' meet l' in 12 points μ ; through any point μ on l' the 4 conics through 5 points of C_6' lie in 4 planes λ through l' . There are $12+4=16$ coincidences in this (λ, μ) correspondence and therefore 16 positions of Q_1 such that Q_1 and 5 points of C_6' lie on a conic in the plane through l' associated with Q_1 . In any plane λ through l' the 15 conics through Q_2 and 4 points of C_6' meet l' in 15 points μ ; through any point μ on l' the 12 conics through Q_2 and 4 points of C_6' lie in 12 planes λ through l' . The $15+12=27$ coincidences of this (λ, μ) correspondence determine 27 positions of Q_1 such that Q_1 , Q_2 , and 4 points of C_6' lie on a conic in the plane through l' associated with Q_1 . These 47 conics are all parasitic.

There are 2 values of λ given by $m=0$ for which the tangent plane to F_3 at O_2 contains O_1 , and there are 5 values of λ given by $p=0$ for which the tangent plane to F_3 at O_1 contains O_2 . In each of these 7 planes there is a cubic with a double point at O_2 or O_1 and passing through O_1 or O_2 and 6 points of C_6 . These 7 cubics correspond to 7 similar cubics in the (z) space. In any plane λ through l' there are 6 cubics with a double point on C_6' and through Q_2 and the 5 remaining points of C_6' . These 6 cubics meet l' in 12 points μ .

Through any point μ on l' there are 8 cubics with a double point on C'_6 and through Q_2 and the other 5 points of C'_6 . These cubics lie in 8 planes λ through l' so that there are $12+8=20$ coincidences in the (λ, μ) correspondence and 20 positions of Q_1 such that Q_1, Q_2 , and the 6 points of C'_6 lie on a cubic with a double point at one of these latter points. These cubics lie in planes through l' associated with the positions of Q_1 . There are then 37 lines, 47 conics, and 27 cubics which are fundamental curves of the second species in the I_{51} .

6. Case III, O_1, O_2 are both variable on a space cubic C_3 . To illustrate the case where the points O_1, O_2 are variable on a rational curve which is part of the basis curve of a pencil of cubic surfaces we again utilize a rational space cubic. Other rational curves might be considered and other arrangements of the points O_1, O_2 might be used, but the transformations obtained resemble the I_{51} in Case II and the I_{51} derived in the following case.

The points $O_1 \equiv (1, \mu^3, \mu^2, \mu), O_2 \equiv (1, -\mu^3, \mu^2, -\mu)$, where $\lambda = \mu^2$, lie on the C_3 which with C_6 makes up the basis of the pencil of cubic surfaces F_3 given by (6). A change of coordinate system is made by

$$y_1 = \bar{p}(x_2 + 2\mu x_3 + \mu^2 x_4) - \bar{q}(x_2 + 2\mu x_4 + \mu^2 x_1),$$

$$y_2 = p(x_2 - 2\mu x_3 + \mu^2 x_4) - q(x_3 - 2\mu x_4 + \mu^2 x_1),$$

$$y_3 = x_2 - \mu^2 x_4,$$

$$y_4 = x_3 - \mu^2 x_1,$$

where $\bar{p} = \mu P - Q, \bar{q} = \mu Q - R, P = p_1 + p_2 \mu^3 + p_3 \mu^2 + p_4 \mu$, etc., and the dashed letters indicate a change of sign in μ . The surface F_3 is now in the form (1) and the involutorial transformation (2) is determined. We have the following expressions for A, B, K written in terms of x_i and y_i for the sake of conciseness:

$$\begin{aligned} A \equiv & 4\mu^2 \bar{M} M [M(PH_p + QH_q + RH_r) - (y_3 - \mu y_4) \{ -\mu q(p x) + M(q x) - \bar{p}(r x) \}] \\ & + y_2 [y_3 \{ \bar{M}(\mu \bar{P} q - \bar{Q} M + \bar{R} p) + M(-\mu P \bar{q} - Q \bar{M} + R \bar{p}) \} \\ & + \mu y_4 \{ -\bar{M}(\mu \bar{P} q - \bar{Q} M + \bar{R} p) + M(-\mu P \bar{q} - Q \bar{M} + R \bar{p}) \}], \end{aligned}$$

$$\begin{aligned} B \equiv & 4\mu^2 M \bar{M} [\bar{M}(\bar{P} H_p + \bar{Q} H_q + \bar{R} H_r) - (y_3 + \mu y_4) \{ \mu \bar{q}(p x) + \bar{M}(q x) - \bar{p}(r x) \}] \\ & + y_1 [y_3 \{ \bar{M}(\mu \bar{P} q - \bar{Q} M + \bar{R} p) + M(-\mu P \bar{q} - Q \bar{M} + R \bar{p}) \} \\ & + \mu y_4 \{ -\bar{M}(\mu \bar{P} q - \bar{Q} M + \bar{R} p) + M(-\mu P \bar{q} - Q \bar{M} + R \bar{p}) \}], \end{aligned}$$

$$\begin{aligned} K \equiv & y_1 [M(PH_p + QH_q + RH_r) - (y_3 - \mu y_4) \{ -\mu q(p x) + M(q x) - \bar{p}(r x) \}] \\ & - y_2 [\bar{M}(\bar{P} H_p + \bar{Q} H_q + \bar{R} H_r) - (y_3 + \mu y_4) \{ \mu \bar{q}(p x) + \bar{M}(q x) - \bar{p}(r x) \}], \end{aligned}$$

where $M = \mu p + q$.

The surfaces $A=0$, $B=0$ are of order 2 in x_i and of order 13 in μ^2 after the removal of a factor μ^2 . The invariant surface $K=0$ is of order 9 in μ^2 and of order 3 in x_i . The image of C_3 which is the quadric which contains C_3 and is tangent to F_3 at O_1 , O_2 is of order 6 in μ^2 and has the equation

$$(\bar{p}M - p\bar{M})H_p + \mu(p\bar{q} + q\bar{p})H_q + \mu(\bar{q}M + q\bar{M})H_r = 0.$$

The table of characteristics of the I_B in the (z) space may be obtained as in Case II and with the same results except that the images of the points Q_1 , Q_2 combine and the joint image is

$$(Q_1, Q_2) \sim F_{33}: l'^{26+51} + C_6'^4 + (Q_1, Q_2)^{31}.$$

In any plane λ through l' the 15 bisecants of C_6' meet l' in 15 points μ ; through any point μ on l' the 7 bisecants of C_6' determine 7 planes λ through l' . In the correspondence (λ, μ) there are $15+7+7=29$ coincidences since $\lambda=\mu^2$, and hence in 29 positions of the pair of points Q_1 , Q_2 a bisecant of C_6' can be drawn from one of them in the plane through l' associated with the pair. There are 8 trisecants of C_6' which meet l' . These 37 lines are parasitic in I_{51} .

In any plane λ through l' the 6 conics through 5 points of C_6' meet l' in 12 points μ ; through any point μ on l' the 4 conics through 5 points of C_6' lie in 4 planes λ through l' . The number of coincidences in the (λ, μ) correspondence is $12+4+4=20$ and hence in 20 positions of the pair of points Q_1 , Q_2 , one of the pair and 5 points of C_6' lie on a conic in the plane through l' associated with the pair. In any plane λ there is a pencil of conics through each of the 15 sets of 4 of the 6 points of C_6' . Each pencil determines an involution on l' which has one pair in common with the involution of points μ , hence 15 pairs of points μ^2 are determined. Given any pair of points μ^2 on l' there are 12 planes λ through l' in which there are conics through the pair μ^2 and 4 points of C_6' . In the correspondence (λ, μ^2) the $15+12=27$ coincidences fix 27 positions of the pair Q_1 , Q_2 such that conics in the associated planes pass through them and 4 of the points of C_6' .

The 7 values of λ given by $M\bar{M}=0$ determine 7 planes tangent to F_3 at O_1 or O_2 which pass through O_2 or O_1 . From the associated F_3 each of these planes cuts a cubic with a double point at O_1 or O_2 and passing through O_2 or O_1 and 6 points of C_6 . These 7 cubics correspond to similar cubics in the (z) space which are parasitic in the I_{51} . In any plane λ through l' there are 6 pencils of cubics through the 6 points of C_6' and with a double point at one of them. Each pencil determines an involution of the third order on l' which has 2 pairs in common with the involution of points μ , hence to a λ correspond 12 pairs of points μ^2 . Given any pair of points μ^2 on l' there are 8 planes λ

through l' in which there are cubics through the pair μ^2 and 6 points of C'_6 and which have a double point at one of the points of C'_6 . The correspondence (λ, μ^2) has $12+8=20$ coincidences which determine 20 positions of the pair Q_1, Q_2 such that in the associated plane there will be a cubic through Q_1, Q_2 and the 6 points of C'_6 and having a double point at one of the points of C'_6 . Hence as in Case II we have 37 lines, 47 conics, and 27 cubics which are fundamental curves of the second species in the I_{51} .

7. A Bertini transformation on a cubic variety in S_4 . In a space of four dimensions we take a cubic variety V_3 with a double point at $O_6 \equiv (0, 0, 0, 0, 1)$ and through the points $O_1 \equiv (1, 0, 0, 0, 0)$, $O_2 \equiv (0, 1, 0, 0, 0)$. The equation of the variety is

$$V_3 \equiv \phi_2 x_5 + \phi_3 = 0,$$

where ϕ_2, ϕ_3 are quaternary forms in x_1, x_2, x_3, x_4 with the x_1^3, x_2^3 terms missing in ϕ_3 . The conics tangent to V_3 at the points O_1, O_2 meet V_3 in two residual points P, P' which are conjugate points in a Bertini involution J_B on V_3 . This involution can be mapped on the 3-space $x_5=0$, and a Bertini involution I_B in 3-space is thus determined. The hyperplane $x_5=0$ meets V_3 in the cubic surface $\phi_3=0$, and meets the tangent hypercone to V_3 at O_6 in the quadric $\phi_2=0$. The surfaces $\phi_2=0, \phi_3=0$ meet in a sextic curve C_6 of genus 4. Any plane π through the line O_1O_2 meets C_6 in 6 points R which lie on a conic. The hyperplanes through O_1, O_2 are invariant under J_B , and the planes π are invariant under I_B . Since the 6 points R lie on a conic in each plane π , the Bertini involution in such a plane is degenerate and of the form $I_{13}: O_1^6 + O_2^6 + 6R^4$, with an invariant curve $k_7: O_1^3 + O_2^3 + 6R^2$. The I_B in the space $x_5=0$ has the characteristics

$$\begin{aligned} O_1 &\sim F_6: O_1^3 + O_2^3 + C_6^2, \\ O_2 &\sim F_6: O_1^3 + O_2^3 + C_6^2, \\ C_6 &\sim F_{24}: O_1^{12} + O_2^{12} + C_6^7, \\ S_1 &\sim S_{13}: O_1^6 + O_2^6 + C_6^4, \\ K_7 &: O_1^3 + O_2^3 + C_6^2. \end{aligned}$$

The 6 bisecants of C_6 from O_1 and the 6 from O_2 are parasitic lines in I_B and correspond to lines on the V_3 through O_1 or O_2 . To determine the number of parasitic conics we must find the number of conics which lie on V_3 and pass through O_1 and O_2 , since in any such conic the construction used to determine J_B will fail in the sense that to a point on the conic corresponds the whole conic. By a proper choice of coordinate system we can write the equation of any cubic variety in the form

$$(7) \quad x_1^2 x_2 + x_1 x_2^2 + a x_1 + b x_2 + c x_1 x_2 + d = 0$$

where a, b, c, d are ternary forms in x_3, x_4, x_5 . The left hand member of (7) can be factored as follows:

$$(x_1 x_2 + b)(x_1 + x_2 + c),$$

if

$$(8) \quad a - b = 0 \text{ and } ac - d = 0.$$

Equations (8) represent two hypercones of the second and third orders respectively whose rulings are planes. The 6 planes common to the two hypercones cut conics from the cubic variety through the points O_1, O_2 . Hence there are 6 fundamental conics of the second species in the I_{13} besides the 12 lines of the second species.

8. A family of space Bertini transformations. A net of planes $\pi \equiv \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$ through the point $(0, 0, 0, 1)$ and a net of cubic surfaces

$$(9) \quad F_3 \equiv x_4(ax)^2 + x_1 x_2(bx) \equiv \lambda_1 F_3' + \lambda_2 F_3'' + \lambda_3 F_3''' = 0,$$

where $(ax)^2$ and (bx) are quaternary forms in x_1, x_2, x_3, x_4 , and

$$a_{ij} \equiv \lambda_1 a'_{ij} + \lambda_2 a''_{ij} + \lambda_3 a'''_{ij}, \text{ and } b_i \equiv \lambda_1 b'_i + \lambda_2 b''_i + \lambda_3 b'''_i,$$

through the lines $l_1 \equiv x_2 = x_4 = 0$ and $l_2 \equiv x_1 = x_4 = 0$ may be used to determine a transformation of the Bertini type. A point $P(x)$ determines a set of λ_i and hence a plane π and a surface F_3 . The plane π cuts the lines l_1, l_2 in a pair of points $O_1(\lambda_3, 0, -\lambda_1, 0), O_2(0, \lambda_3, -\lambda_2, 0)$. The conic through $P(x)$ and tangent to F_3 at O_1 and O_2 will meet F_3 in a residual point $P'(x')$ which is the conjugate of $P(x)$ in an involutorial transformation I .

If we make the linear transformation

$$\begin{aligned} y_1 &= \lambda_3 B_2 x_1 + A_2 x_4, \\ y_2 &= \lambda_3 B_1 x_2 + A_1 x_4, \\ y_3 &= \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3, \\ y_4 &= x_4, \end{aligned}$$

where

$$\begin{aligned} B_1 &\equiv b_1 \lambda_3 - b_3 \lambda_1, \\ B_2 &\equiv b_2 \lambda_3 - b_3 \lambda_2, \\ A_1 &\equiv a_{11} \lambda_3^2 - 2a_{13} \lambda_1 \lambda_3 + a_{33} \lambda_1^2, \\ A_2 &\equiv a_{22} \lambda_3^2 - 2a_{23} \lambda_2 \lambda_3 + a_{33} \lambda_2^2, \end{aligned}$$

then equation (9) is in the form of (1), and the transformation (2) may be used to obtain

$$\begin{aligned}
x'_1 &= B_1B(By_2 - A_2Ay_4), \\
x'_2 &= B_2A(Ay_1 - A_1By_4), \\
x'_3 &= -B_1B(By_2 - A_2Ay_4) - B_2A(Ay_1 - A_1By_4), \\
x'_4 &= \lambda_3B_1B_2AB_4y_4,
\end{aligned}$$

where

$$\begin{aligned}
A &\equiv B_1^2y_1y_2 + B_2y_4^2[A_1^2B_2 + 2\lambda_3^2B_1^2(a_{14}\lambda_3 - a_{34}\lambda_1) \\
&\quad + A_1B_1(2a_{13}\lambda_2\lambda_3 + 2a_{23}\lambda_1\lambda_3 - 2a_{12}\lambda_3^2 - 2a_{33}\lambda_1\lambda_2 - b_4\lambda_3^2)], \\
B &\equiv B_2^2y_1y_2 + B_1y_4^2[A_2^2B_1 + 2\lambda_3^2B_2^2(a_{24}\lambda_3 - a_{34}\lambda_2) \\
&\quad + A_2B_2(2a_{13}\lambda_2\lambda_3 + 2a_{23}\lambda_1\lambda_3 - 2a_{12}\lambda_3^2 - 2a_{33}\lambda_1\lambda_2 - b_4\lambda_3^2)].
\end{aligned}$$

The λ_i are now replaced by

$$\begin{aligned}
\lambda_1 &\equiv \phi_1 \equiv x_2F_3''' - x_3F_3'', \\
\lambda_2 &\equiv \phi_2 \equiv x_3F_3' - x_1F_3''', \\
\lambda_3 &\equiv \phi_3 \equiv x_1F_3'' - x_2F_3'.
\end{aligned}$$

The quartic surfaces $\phi_i=0$ have in common the lines l_1, l_2 , and a residual curve C_{11} of order 11 and genus 14. The surfaces $A=0, B=0$ which are the images of the lines l_1, l_2 are of order 8 in ϕ_i and 2 in x_i after a factor ϕ_3^2 is removed. The factor $\phi_3^2B_1B_2$ can be removed from the transformation, and the invariant surface $K \equiv y_1A - y_2B=0$ has the factor $\phi_3^2B_1B_2$. The characteristics of the transformation are

$$\begin{aligned}
l_1 &\sim F_{34} : l_1^{11} + l_2^{10} + C_{11}^8, \\
l_2 &\sim F_{34} : l_1^{10} + l_2^{11} + C_{11}^8, \\
C_{11} &\sim F_{204} : l_1^{66} + l_2^{66} + C_{11}^{47}, \\
S_1 &\sim S_{69} : l_1^{22} + l_2^{22} + C_{11}^{16}, \\
K_{27} &: l_1^9 + l_2^9 + C_{11}^6.
\end{aligned}$$

The x parasitic lines of I are trisecants of C_{11} which meet either l_1 or l_2 . Since C_{11} meets l_1 in 4 points there are 7 residual intersections R_i in a plane through l_1 . In any such plane a line R_iR_j meets l_1 in a point P , and through each of R_i and R_j pass 5 other bisecants of C_{11} meeting l_1 in 10 points Q . If h' is the number of bisecants of C_{11} through any point of l_1 , the points P, Q are in $(10h', 10h')$ correspondence. The $20h'$ coincidences are determined by the x trisecants of C_{11} meeting l_1 , the r' tangents of C_{11} meeting l_1 , and the 4 tangents to C_{11} where it meets l_1 . Hence

$$20h' = 6x + 5r' + 30 \cdot 4.$$

Since the C_{11} is of class $r=48$ and has $h=31$ apparent double points, then

$h' = h - 4 \cdot 3/2 = 25$, and $r' = r - 2 \cdot 4 = 40$. These values make $x = 30$, but among the 30 trisecants the line l_2 , which is a quadrisecant, is counted 4 times. Hence there are 26 trisecants of C_{11} meeting l_1 and 26 more which meet l_2 . These 52 lines are the parasitic lines of the transformation I .

Let y be the number of parasitic conics and z be the number of parasitic cubics of I . The complete intersection of two surfaces of the web of S_{69} is made up of

$$69^2 = 69 + 22^2 + 22^2 + 11 \cdot 16^2 + 52 + 8y + 27z,$$

and the complete intersection of an S_{69} and the K_{27} is made up of

$$69 \cdot 27 = 27 + 9 \cdot 22 + 9 \cdot 22 + 6 \cdot 16 \cdot 11 + 52 + 4y + 9z.$$

The solution of these equations is $y = 45$, $z = 18$, whence we can conclude that the fundamental curves of the second species in I consist of 52 lines, 45 conics, and 18 cubics.

9. A family of space Geiser transformations. If $F_n = 0$ is a surface of order n with an $(n-2)$ -fold line $l \equiv x_3 = x_4 = 0$, the equations

$$(10) \quad \pi \equiv \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0,$$

$$(11) \quad F_n \equiv ax_1^2 + bx_2^2 + 2hx_1x_2 + 2fx_2 + 2gx_1 + c \equiv \lambda_1 F'_n + \lambda_2 F''_n + \lambda_3 F'''_n = 0,$$

where $a \equiv \lambda_1 a' + \lambda_2 a'' + \lambda_3 a'''$, etc., and a' , a'' , a''' , etc., are binary forms in x_3 , x_4 , define a net of plane curves C_n of order n with an $(n-2)$ -fold point $Q \equiv (\lambda_2, -\lambda_1, 0, 0)$. A line through Q and a point $P(x)$ on C_n meets it in a residual point $P'(x')$, thus defining an involutorial transformation I having the invariant net of surfaces

$$\begin{aligned} & k_1 \phi_1 + k_2 \phi_2 + k_3 \phi_3 \\ & \equiv k_1(x_2 F'''_n - x_3 F''_n) + k_2(x_3 F'_n - x_1 F'''_n) + k_3(x_1 F''_n - x_2 F'_n) = 0. \end{aligned}$$

The pencil of planes $p \equiv x_4 - \mu x_3 = 0$ through l are invariant under I and in any such plane F_n takes the form

$$(12) \quad ax_1^2 + bx_2^2 + cx_3^2 + 2hx_1x_2 + 2fx_2x_3 + 2gx_1x_3 = 0,$$

where the coefficients are polynomials in μ . This net of conics enables us to map I on a double space $S(\lambda_1 : \lambda_2 : \lambda_3, \mu)$. A plane

$$(13) \quad m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4 = 0$$

is mapped on S by eliminating x_i between (10) and (13) and using $x_4 = \mu x_3$. The values of x_i thus obtained are substituted in (12) giving

$$\begin{aligned}
 & a(m_2\lambda_3 - \bar{m}_3\lambda_2)^2 + b(\bar{m}_3\lambda_1 - m_1\lambda_3)^2 + c(m_1\lambda_2 - m_2\lambda_1)^2 \\
 & + 2h(m_2\lambda_3 - \bar{m}_3\lambda_2)(\bar{m}_3\lambda_1 - m_1\lambda_3) + 2f(\bar{m}_3\lambda_1 - m_1\lambda_3)(m_1\lambda_2 - m_2\lambda_1) \\
 & + 2g(m_2\lambda_3 - \bar{m}_3\lambda_2)(m_1\lambda_2 - m_2\lambda_1) = 0, \text{ where } \bar{m}_3 \equiv m_3 + \mu m_4,
 \end{aligned}$$

which must be identical with

$$(m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4)(m_1x'_1 + m_2x'_2 + m_3x'_3 + m_4x'_4) = 0.$$

From this identity we have

$$\begin{aligned}
 x_1x'_1 &= b\lambda_3^2 - 2f\lambda_2\lambda_3 + c\lambda_2^2, \\
 x_2x'_2 &= c\lambda_1^2 - 2g\lambda_1\lambda_3 + a\lambda_3^2, \\
 x_3x'_3 &= a\lambda_2^2 - 2h\lambda_1\lambda_2 + b\lambda_2^2, \\
 x'_4 &= \mu x'_3.
 \end{aligned}$$

If we replace μ by x_4/x_3 and λ_i by ϕ_i we have the transformation I in the form

$$\begin{aligned}
 x_1x'_1 &= b\phi_3^2 - 2f\phi_2\phi_3 + c\phi_2^2, \\
 x_2x'_2 &= c\phi_1^2 - 2g\phi_1\phi_3 + a\phi_3^2, \\
 x'_3 &= x_3(a\phi_2^2 - 2h\phi_1\phi_2 + b\phi_2^2), \\
 x'_4 &= x_4(a\phi_2^2 - 2h\phi_1\phi_2 + b\phi_2^2),
 \end{aligned}$$

where x_1, x_2 are factors of the first two equations respectively. The surfaces $\phi_i = 0$ are of order $n+1$ and have l as an $(n-2)$ -fold line. The residual basis curve of the net of ϕ_i is a C_{5n-3} of order $5n-3$ and genus $12n-19$ through the point $(0, 0, 0, 1)$. The image of l in I is the surface $L \equiv a\phi_2^2 - 2h\phi_1\phi_2 + b\phi_2^2 = 0$, which is of order 3 in ϕ_i and of order $n-2$ in x_3, x_4 . The image in S of the invariant surface K has the equation

$$\begin{vmatrix} a & h & g & \lambda_1 \\ h & b & f & \lambda_2 \\ g & f & c & \lambda_3 \\ \lambda_1 & \lambda_2 & \lambda_3 & 0 \end{vmatrix} = 0,$$

which corresponds to K^2 in the space (x) . Hence K is of order 2 in ϕ_i and of order $n-1$ in x_3, x_4 . The table of characteristics of I is

$$\begin{aligned}
 l &\sim L_{4n+1} : l^{4n-7} + C_{5n-3}^3, \\
 C_{5n-3} &\sim F_{12n+3} : l^{12n-18} + C_{5n-3}^8, \\
 S_1 &\sim S_{4n+2} : l^{4n-6} + C_{5n-3}^3, \\
 K_{2n+1} &: l^{2n-5} + C_{5n-3}^2.
 \end{aligned}$$

In any plane p through l there is an ordinary Geiser transformation, therefore the C_{5n-3} meets such a plane in the 7 fundamental points R_i of the Geiser transformation and in $5n-10$ points on l . The section of C_{5n-3} by the plane $x_3=0$ is the point $(0, 0, 0, 1)$ and 6 points lying on the conic $x_3=0$, $F_n'''=0$. Hence on this plane the Geiser transformation degenerates and the conic is parasitic for I .

The x parasitic lines of I are trisecants of C_{5n-3} meeting l . Since C_{5n-3} meets any plane p in 7 points not on l the method of §8 may be used in determining the number of trisecants of C_{5n-3} which meet l . The number $x=15n-15$ is obtained from the equation

$$20h' = 6x + 5r' + 30(5n - 10),$$

where $r'=24n-26$, and $h'=18n-26$. Therefore the fundamental curves of the second species for I consist of $15n-15$ lines and one conic.

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ON THE PROBLEM OF n BODIES*

BY

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Introduction. For the problem of three bodies, Sundman† established together with other results that if the angular momentum of the three bodies is not zero about every axis through the center of gravity of the system, the greatest of the three mutual distances will always exceed a specifiable constant depending upon the initial configuration of the bodies, and hence that triple collision is impossible. The problem was then considered from a different point of view by Birkhoff‡ in his Chicago Colloquium lectures of 1920. He considered the case for which the angular momentum of the three bodies about every axis through the center of gravity of the system is not zero and for which the constant K appearing in the energy integral: $T = U - K$, is (1) equal to or less than zero, and (2) greater than zero. Here T denotes the kinetic energy and $-U$ denotes the potential energy of the system. He showed for the first case that at least two if not all three of the mutual distances increase indefinitely as the time increases and decreases. For the second case, he showed if the motion of the three bodies is such that for some instant all three bodies approach sufficiently near to one another, that two of the mutual distances become infinite with the time while the third mutual distance remains less than a definite constant depending only upon the energy constant and the total mass of the system. After stating and proving various other results, he concluded by stating without formal proof that the results described above may be extended to the case of n bodies attracting one another according to the Newtonian law of force as well as to the case of n bodies attracting one another according to a more general law of force. The present paper has as its object the investigation of the conditions under which these extensions apply.

The equations of motion and other fundamental relationships. We shall denote the n bodies (assumed to be particles) by P_i ($i = 1, 2, \dots, n$), and suppose them to have positive finite masses m_i and real coordinates (x_i, y_i, z_i) . The distance from P_i to P_j will be denoted by r_{ij} . We shall suppose that the bodies attract one another in such a way that there exists a potential function

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† Sundman, *Mémoire sur le problème des trois corps*, Acta Mathematica, vol. 36 (1913), p. 105.

‡ Birkhoff, *Dynamical Systems*, 1927, p. 260. This book is volume IX of the American Mathematical Society Colloquium Publications.

$$U = \sum_{i,j=1}^n m_i m_j / (r_{ij})^d, \quad i \neq j, \quad 0 < d < 2.$$

If $d=1$, this function reduces to that for the Newtonian law of attraction. Inasmuch as the probability of collision among particles moving according to this law is zero in the general case, we shall assume that none of the n bodies ever collide. Then all of the r_{ij} will always be positive.

If t denotes the time, the equations of motion will be

$$m_i \frac{d^2 x_i}{dt^2} = \frac{\partial U}{\partial x_i}, \quad m_i \frac{d^2 y_i}{dt^2} = \frac{\partial U}{\partial y_i}, \quad m_i \frac{d^2 z_i}{dt^2} = \frac{\partial U}{\partial z_i}.$$

The ordinary existence theorems for a system of differential equations may be applied to yield the result that for assigned values of the coordinates and velocity components for $t=\bar{t}$ where $t_0 < \bar{t} < t_1$, there exists a unique set of analytic functions $x_i(t)$, $y_i(t)$, $z_i(t)$, $x'_i(t)$, $y'_i(t)$, $z'_i(t)$ defined and satisfying the system of equations for $t_0 < t < t_1$ and taking on the assigned values for $t=\bar{t}$. Furthermore, since we assume that the distances r_{ij} are always positive, the interval of definition may be extended to the interval $-\infty < t < \infty$.

The equations of motion admit the following ten integrals:

$$\begin{aligned} \sum m_i (x_i'^2 + y_i'^2 + z_i'^2) &= 2(U - K), \\ \sum m_i x_i &= \sum m_i y_i = \sum m_i z_i = 0, \\ \sum m_i x_i' &= \sum m_i y_i' = \sum m_i z_i' = 0, \\ \sum m_i (y_i z_i' - z_i y_i') &= c_1, \\ \sum m_i (z_i x_i' - x_i z_i') &= c_2, \\ \sum m_i (x_i y_i' - y_i x_i') &= c_3, \end{aligned}$$

where the summations for i are to be taken from 1 to n . Here K , c_1 , c_2 , c_3 are constants of integration and the primes denote derivatives with respect to t . The coordinate system has been so chosen that the center of gravity of the system is fixed at the origin.

If we define

$$R^2(t) = \left(\sum_{i,j=1}^n m_i m_j r_{ij}^2 \right) / (2M),$$

where M represents the total mass of the system, it is not difficult to obtain the analogue of Lagrange's Identity*:

$$(R^2)'' = 2(2-d)U - 4K.$$

We shall suppose $0 < d < 2$ in order that the coefficient of U may be positive.

* Lagrange, *Essai sur le problème des trois corps*, Oeuvres, vol. 6, p. 240.

We shall now proceed to derive the analogue of Sundman's Identity* for the problem of n bodies. Let us choose the coordinate axes in such a way that the products of inertia of the n bodies vanish, and if the moments of inertia about the x, y, z axes are A, B, C respectively, that $A \geq B \geq C$. We propose to find a minimum for the kinetic energy of the system

$$T = \frac{1}{2} \sum m_i (x_i'^2 + y_i'^2 + z_i'^2),$$

when the $3n$ space coordinates are fixed and the $3n$ velocity components are allowed to vary except for being required to satisfy the integrals of angular momentum and

$$RR' \equiv \sum m_i (x_i x_i' + y_i y_i' + z_i z_i') = c_4.$$

By the Lagrange method of multipliers† the minimum value of T under these conditions is found to be

$$\frac{1}{2} \left(\frac{c_1^2}{A} + \frac{c_2^2}{B} + \frac{c_3^2}{C} + \frac{c_4^2}{R^2} \right) \geq \frac{1}{2} \left(\frac{f^2}{A} + R'^2 \right),$$

where $f^2 = c_1^2 + c_2^2 + c_3^2$. On applying the energy integral, we may express this result by writing

$$R'^2 + P = 2(U - K) \quad \text{where} \quad P \geq f^2/R^2.$$

Let us now eliminate U between the above two fundamental identities. If we define

$$F \equiv 2RR'' + dR'^2 + 2dK - (2 - d)f^2/R^2,$$

the relationship obtained will show that $F \geq 0$. Let us define

$$H \equiv R^d(R'^2 + 2K + f^2/R^2),$$

and differentiate with respect to t . In terms of F , we obtain $H' = FR^{d-1} \cdot R'$, from which we have the following result: *If R increases, H cannot decrease, and if R decreases, H cannot increase.* We furthermore note if $f > 0$, R cannot approach zero, since then, by its definition, H would become infinite.

By means of the six integrals of linear momentum, the system of equations of motion may be reduced to a system of order $6n - 6$. We shall carry out this reduction in the following manner. For any instant, consider first all possible ways of dividing the n bodies into two groups G_1, G_2 , and choose that one for which at the given instant the distance from the center of gravity of one group to that of the complementary group is greatest. There may be

* Sundman, loc. cit., p. 148.

† See for example Goursat, *Cours d'Analyse Mathématique*, 1923, vol. 1, p. 119.

more than one such method of subdivision giving this maximum distance, in which case we shall divide the n bodies into two groups in any one of the several possible ways. Let the coordinates of the center of gravity of one group, G_2 , with respect to the center of gravity of the complementary group, G_1 , as origin be (ξ_1, η_1, ζ_1) and define $\rho_1^2 = \xi_1^2 + \eta_1^2 + \zeta_1^2$.

If either G_1 or G_2 contains more than one body, consider the various possible ways of subdividing G_1 and G_2 into subgroups, and choose a method of subdividing one group to give the greatest possible distance from the center of gravity of one subgroup to that of the complementary subgroup. Let the coordinates of the center of gravity of one subgroup with respect to the center of gravity of the complementary subgroup as origin be (ξ_2, η_2, ζ_2) and define $\rho_2^2 = \xi_2^2 + \eta_2^2 + \zeta_2^2$. This process of subdivision may be repeated until each of the final groups contains only one body. When this stage has been reached, $n-1$ sets of coordinates (ξ_i, η_i, ζ_i) will have been introduced together with $n-1$ distances defined by $\rho_i^2 = \xi_i^2 + \eta_i^2 + \zeta_i^2$.

The equations of transformation from (x_i, y_i, z_i) , $i=1, 2, \dots, n$, to (ξ_j, η_j, ζ_j) , $j=1, 2, \dots, n-1$, will depend upon the distribution of the n bodies with respect to one another and hence will in general depend upon t . If the position of each of the bodies at a given instant is known, there will always exist at least one way of separating the n bodies into groups in the manner described above, and then the equations of transformation together with their inverse formulas may be written down. If the system of n bodies is divided into groups in a proper manner, the same equations will apply throughout some interval of time containing the given instant. As t increases or decreases, the intervals of time throughout which a particular grouping obtains may become smaller and smaller and approach zero as a limit. Since we exclude the possibility of collision, the velocities of the bodies are bounded and in a sufficiently small interval of time the position of the bodies can change by only a small amount. To make it possible to continue the transformation beyond a limit point of grouping intervals, we shall modify the above method of dividing the bodies into groups in a sufficiently restricted neighborhood of the limit point so as to preserve a constant grouping there. Then by setting up a finite number of sets of equations of transformation, we may for any given finite interval of time express the equations of motion together with the energy integral and the integrals of angular momentum in terms of the new variables (ξ_j, η_j, ζ_j) .

Throughout any interval of time $t' < t < t''$ for which $\rho_i(t)$ represents the distance between the centers of gravity of the same two fixed groups of bodies, $\rho_i(t)$ will be analytic. For an instant at which the grouping changes, some distance ρ_i must change to the distance between two new centers of gravity.

In this case all of the following ρ_j will in general also change to distances between new centers of gravity. Except for intervals of time containing limit points of grouping intervals, the ρ_j with the smallest subscript which changes will be continuous but will in general have a discontinuous first derivative. For such intervals of time as contain a limit point of grouping intervals the ρ_j with the smallest subscript which changes will itself have a break whose magnitude may be made arbitrarily small by taking the interval about the limit point small enough. The ρ_j with larger subscripts will in general be discontinuous in either case.

Suppose for $t' < t < t''$, the group of bodies $\sum P_h$ has its center of gravity at the (ξ_j, η_j, ζ_j) -origin and the group $\sum P_k$ has its center of gravity at the point whose coordinates are (ξ_j, η_j, ζ_j) . If we define

$$\mu_j = \frac{(\sum m_h)(\sum m_k)}{(\sum m_h) + (\sum m_k)} \quad (j = 1, 2, \dots, n-1),$$

the reduced system of equations of order $6n-6$ will assume the simple form

$$\mu_j \frac{d^2 \xi_j}{dt^2} = \frac{\partial U}{\partial \xi_j}, \quad \mu_j \frac{d^2 \eta_j}{dt^2} = \frac{\partial U}{\partial \eta_j}, \quad \mu_j \frac{d^2 \zeta_j}{dt^2} = \frac{\partial U}{\partial \zeta_j}.$$

If we denote the derivatives of ξ_j, η_j, ζ_j with respect to t by $\xi'_j, \eta'_j, \zeta'_j$, the energy integral becomes

$$\sum \mu_j (\xi_j'^2 + \eta_j'^2 + \zeta_j'^2) = 2(U - K),$$

while the integrals of angular momentum become

$$\sum \mu_j (\eta_j \zeta'_j - \zeta_j \eta'_j) = c_1,$$

$$\sum \mu_j (\zeta_j \xi'_j - \xi_j \zeta'_j) = c_2,$$

$$\sum \mu_j (\xi_j \eta'_j - \eta_j \xi'_j) = c_3.$$

Finally, if the values of x_i, y_i, z_i in terms of ξ_j, η_j, ζ_j are substituted in the expression for R^2 , we obtain

$$R^2(t) = \frac{1}{2} \sum \mu_j \rho_j^2.$$

Some properties of the motions. We shall now proceed to consider those properties of the motions of the n bodies acting under the above law of force which correspond to the properties considered by Sundman* and Birkhoff† for the problem of three bodies under the Newtonian law of force. With the analogues of the fundamental identities of Lagrange and Sundman together with the $(\xi_j, \eta_j, \zeta_j), j=1, 2, \dots, n-1$, coordinates available, the proofs of

* Sundman, loc. cit., p. 105.

† Birkhoff, loc. cit., p. 275.

these theorems will be found similar to those for the classical problem. For this reason we shall merely state certain results and outline the proofs of others.

Directly from the analogue of Sundman's Identity we have the following: For the case $K \leq 0$, $0 < d < 2$, at least $n-1$ of the mutual distances r_{ij} increase indefinitely as the time increases or decreases. We shall now restrict ourselves to the case of $K > 0$.

If $K > 0$, the least of the mutual distances r_{ij} cannot exceed $[M^2/(2K)]^{1/d}$. This result follows immediately when the definition of U is applied to the energy integral.

For the case $K > 0$, the largest r_{ij} will necessarily exceed k times the smallest r_{ij} provided

$$R \leq \frac{m}{(2M)^{1/2}} \left(\frac{4f^2}{M^2 k^2} \right)^{1/(2-d)} \quad \text{or} \quad R \geq \frac{kM^{1/2}}{2} \left(\frac{M^2}{2K} \right)^{1/d},$$

where m denotes the least of the masses m_i . Here we must apply the analogue of Sundman's Identity together with inequalities obtained from R .

For the case $K > 0$, $0 < d < 2$, any part of the curve $R = R(t)$ (t , R rectangular coordinates) for which $R < f[(2-d)/(2dK)]^{1/2}$ consists of a finite arc concave upwards and with a single minimum. If $R = R_0$ gives this minimum, the curve rises on either side until R satisfies the inequality

$$(R^d - R_0^d)/[1 - (R_0/R)^{2-d}] \geq f^2/(2KR_0^{2-d}),$$

with a corresponding slope R' at least as great as is demanded by the inequality

$$R'^2 \geq E^2 \equiv f^2 \left(\frac{1}{R^d R_0^{2-d}} - \frac{1}{R^2} \right) + 2K \left(\frac{R_0^d - R^d}{R^d} \right),$$

at every intermediate stage. This result follows from a combination of the analogue of Lagrange's and the analogue of Sundman's Identity.

Since we are considering motions for which there are no collisions, f must be positive before this theorem may be applied. The n bodies are all near together at some instant $t = t_0$, the amount of separation being measured by R . The bodies separate in such a way that R increases and very rapidly as long as R is not too small or large until R has become very large. Since the least of the mutual distances is not greater than $[M^2/(2k)]^{1/d}$ for all values of the time, at least two bodies must remain relatively near together throughout the entire motion.

We shall now turn to consider the function $\rho_1(t)$. We shall prove the following theorem:

In the case $K > 0$, throughout any interval of time $\rho_1'' > -3dM^{d+2}/(2m\rho_1)^{d+1}$. If for any instant $\rho_1' > [3M^{d+2}/(2^d m^{d+1} \rho_1^d)]^{1/2}$, then ρ_1 will continue to increase indefinitely with t .

Let us first consider t in an interval $t' < t < t''$ sufficiently restricted so that the n bodies preserve one and the same grouping throughout. If one group consists of the k bodies $P_i (i=1, 2, \dots, k; k=1, 2, \dots, n-1)$, while the complementary group consists of the $n-k$ bodies $P_j, j=k+1, \dots, n$, then we can show that there exists a positive lower bound for the distances r_{ij} in terms of ρ_1 , namely $r_{ij} \geq 2m\rho_1/M$ for $i=1, 2, \dots, k; j=k+1, \dots, n$.

The distance ρ_1 may be written $\rho_1 = MW/\mu^2$, where

$$\mu^2 = \sum_{i=k+1}^n \sum_{j=1}^k m_i m_j \text{ and } W^2 = \left(\sum_{i=1}^k m_i x_i \right)^2 + \left(\sum_{i=1}^k m_i y_i \right)^2 + \left(\sum_{i=1}^k m_i z_i \right)^2.$$

Upon differentiating twice with respect to t and dropping three non-negative terms from the second member, we obtain

$$\rho_1'' \geq [M/(\mu^2 W)] [(\sum m_i x_i)(\sum m_i x_i'') + (\sum m_i y_i)(\sum m_i y_i'') + (\sum m_i z_i)(\sum m_i z_i'')],$$

where the summations are to be taken from $i=1$ to $i=k$. If furthermore we use the equations of motion to eliminate the second-order derivatives and simplify by applying inequalities of the type

$$x_i - x_j \leq r_{ij}, \quad \sum_{i=1}^k m_i x_i \leq W$$

we obtain the desired inequality concerning ρ_1'' . By integrating both sides of this inequality, we find if for any t in $t' < t < t''$ the inequality involving ρ_1' is satisfied, that ρ_1 will continue to increase indefinitely if the grouping of the n bodies does not change. For any instant that the grouping does change, either ρ_1' will be continuous or it will be increased and hence if this inequality is satisfied for any instant, ρ_1 will continue to increase indefinitely with t .

We proceed to combine these results in order to show that a motion having its minimum R sufficiently small is one for which R and ρ_1 increase indefinitely as t increases or decreases. According to what has been proved, for R^* and $R^{*'}$ arbitrarily large and for any fixed $d, 0 < d < 2$, a positive R_0 can be chosen so small that all motions for which the minimum R is not more than R_0 correspond to an R which increases from the minimum to R^* and has for $R=R^*$ a derivative R' which is at least as great as $R^{*'}$.

The function $\rho_1(t)$ is defined throughout any finite interval of time and will satisfy the inequality

$$2R/[(n-1)M^{1/2}] < \rho_1 < (2M)^{1/2}R/m,$$

from which it is evident that if R increases indefinitely so also must ρ_1 , and conversely if ρ_1 increases indefinitely so also must R .

Let us consider a fixed value of R_0 satisfying the inequality

$$(a) \quad 0 < R_0 < (2-d)^{1/2}f/(2dK)^{1/2}.$$

Then R must increase until

$$R^d/[1 - (R_0/R)^{2-d}] \geq f^2/(2KR_0^{2-d}).$$

Given any value R^* , we can choose R_0 so small that R becomes greater than R^* . We shall suppose therefore that R_0 has been chosen so small that in addition to satisfying (a) the motion is such that R increases until $R \geq 2^{1/(2-d)}R_0$. In this case R increases from R_0 until $2R^d \geq f^2/(2KR_0^{2-d})$ or until $R \geq R^* \equiv f^{2/d}/(2^2KR_0^{2-d})^{1/d}$. The above inequality will be satisfied if R_0 is chosen so small that $f^{2/d}/(2^2KR_0^{2-d})^{1/d} \geq 2^{1/(2-d)}R_0$, or if

$$(b) \quad R_0 \leq f/(2^{(4-d)/(2-d)}K)^{1/2}.$$

Now let us define $R^{**} = mR^*/[2^{3/2}(n-1)M]$. If we choose R_0 so small that

$$(c) \quad R_0 < m^{d/2}f/[2^{(5d+4)/2}(n-1)^dM^dK]^{1/2},$$

we shall have $R_0 < R^{**}/2$. If we define $R^{***} = R^*/2$, it is obvious that $R^{**} < R^{***}$. If t_0 denotes the first value of t for which $R(t) = R_0$, and t^* denotes the first value of t greater than t_0 for which $R(t) = R^*$, there will exist a unique pair of values t^{**}, t^{***} in the interval $t_0 < t < t^*$ such that $R(t^{**}) = R^{**}$ and $R(t^{***}) = R^{***}$. Since the function $R(t)$ is continuous and has a continuous derivative for t in any closed interval $t^{**} \leq t \leq t^{***}$, we may apply the law of the mean for derivatives which states that there exists at least one point \bar{t} in $t^{**} \leq t \leq t^{***}$ such that $(R^{***} - R^{**})/(t^{***} - t^{**}) = \bar{R}'$ where $\bar{R}' = R'(\bar{t})$. Since $\bar{R} < R^*$, \bar{R}' must satisfy our previous inequality and $(R^{***} - R^{**})/(t^{***} - t^{**}) \geq \bar{E}$ where \bar{E} denotes E with R replaced by \bar{R} .

Consider now the average rate of change of ρ_1 throughout the interval $t^{**} \leq t \leq t^{***}$. We find

$$[\rho_1(t^{***}) - \rho_1(t^{**})]/[t^{***} - t^{**}] > [R^{***} - R^{**}]/[(n-1)M^{1/2}(t^{***} - t^{**})].$$

There must exist a value of t , say \bar{t}_1 , satisfying $t^{**} < \bar{t}_1 < t^{***}$, such that $\rho_1'(\bar{t}_1) \geq \bar{E}/[(n-1)M^{1/2}]$.

We wish to show that a motion having its minimum R denoted by R_0 small enough is one for which R and ρ_1 become infinite with t . This result will follow if $\bar{E}/[(n-1)M^{1/2}] > [(2^{2-d}M^{2+d})/(m^{d+1}\rho_1^d(\bar{t}_1))]^{1/2}$, or on eliminating ρ_1 if

$$(m^{2d+1}f^2\bar{E}^2)/[2^{(8-d)/2}(n-1)^{2(d+1)}M^{(5d+6)/2}KR_0^{2-d}] > 1.$$

It is obvious by the choice of R^{**} and R^{***} , that $[R_0^d - \bar{R}^d]/\bar{R}^d > -1$. Since $\bar{R} < R^*/2$, we have $1/(\bar{R}^d R_0^{2-d}) > 2^{2+d}K/f^2$. Also since $\bar{R} > R^{**}$ we have

$$-1/\bar{R}^2 > -2^{(3d+4)/d}(n-1)^2 M^2 K^{2/d} R_0^{2(3-d)/d} / [m^2 f^{4/d}].$$

If furthermore we suppose

$$(d) \quad R_0 < \frac{m^{d/(2-d)} f}{2^{(4+2d-d^2)/[2(2-d)]} (n-1)^{d/(2-d)} M^{d/(2-d)} K^{1/2}},$$

then

$$f^2 \left(\frac{1}{\bar{R}^d R_0^{2-d}} - \frac{1}{\bar{R}^2} \right) > 2^{d+1} K,$$

and the desired inequality will be satisfied if

$$(e) \quad R_0 < \frac{(2^d - 1)^{1/(2-d)} m^{(3d+1)/(2-d)} f^{2/(2-d)}}{2^{(6-d)/[2(2-d)]} (n-1)^{2(d+1)/(2-d)} M^{(5d+6)/[2(2-d)]}}.$$

We have the following result: *In the case $K > 0$, $0 < d < 2$, if the motion is such that the n bodies approach so closely that the minimum R denoted by R_0 satisfies the inequalities (a), (b), (c), (d) and (e), then at least $n-1$ mutual distances become infinite with t while at least one such distance remains less than $[M^2/(2K)]^{1/d}$.*

We may also state one further property of motions of the above kind. *Any motion for which $f > 0$, $K > 0$, $0 < d < 2$ and the bodies are all near together at some instant $t = t_0$ is characterized by the property that one r_{ij} remains relatively large compared to the smallest r_{ij} throughout the entire motion.* This result follows from an earlier result, the definition of R , the energy integral and the analogue of Sundman's Identity.

The results of this paper may be extended to motions embracing instants of collision if any kind of continuation after multiple collision were possible in which the constants of linear and angular momentum as well as of energy are the same after as before collision and if also R' may be regarded as continuous at collision. In this case none of the analytic work would be affected even though for certain instants there did occur multiple collisions among the bodies.

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THE RULED V_4^1 IN S_5 ASSOCIATED WITH A SCHLÄFLI HEXAD*

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1. Introduction. In S_4 we know that all the lines which meet four generic planes also meet a fifth associated plane. These ∞^2 lines generate a ruled V_3^3 , i.e., the variety of Segre with 10 double points. This property cannot be generalized in space of more than four dimensions; that is: All the lines which meet n generic $(n-2)$ -flats will not in general meet an additional $(n-2)$ -flat when $n > 4$; $n+1$ generic flats will determine $n+1$ V_{n-1}^{n-1} 's.

If however the $n+1$ S_{n-2} 's form a Schläfli set, a single V_{n-1}^{n-1} is determined. An equation of this spread has been given by C. R. Rupp.† Let the Schläfli set be given as follows‡:

$$(1) \quad x_i = 0, \quad \sum_0^n b_{ik} x_k = 0 \quad (i = 0, 1, \dots, n),$$

where $b_{ii} = 0$, $b_{ik} = b_{ki}$. The equation of the V_{n-1}^{n-1} determined by the first n flats is then

$$(2) \quad \begin{vmatrix} b_{01} & b_{02} & b_{03} & \dots & b_{0,n-1} & b_{0n} \\ -\sum_a^n b_{1i} x_i & b_{12} x_1 & b_{13} x_1 & \dots & b_{1,n-1} x_1 & b_{1n} x_1 \\ b_{21} x_2 & -\sum_0^n b_{2i} x_i & b_{23} x_2 & \dots & b_{2,n-1} x_2 & b_{2n} x_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n-1,1} x_n & b_{n-1,2} x_n & b_{n-1,3} x_n & \dots & -\sum_0^n b_{n-1,i} x_i & b_{n-1,n} x_n \end{vmatrix} = 0.$$

It may easily be verified that the same V_{n-1}^{n-1} will be obtained by taking any other set of n flats from (1). All the $n+1$ flats lie on the spread and the fundamental (or regular) singular loci are $(n-2r)$ -flats of multiplicity $r = 2, 3, \dots, n/2$, when n is even, and of multiplicity $r = 2, 3, \dots, (n-1)/2$, when n is odd; there are $\binom{n+1}{r}$ such loci. The study of the remaining accessory singular loci for the case $n = 5$ will be the object of the present paper.

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† C. R. Rupp, *An extension of Pascal's theorem*, these Transactions, vol. 31, p. 578.

‡ Luigi Berzolari, *Sui sistemi di $n+1$ rette dello spazio ad n dimensioni, situate in posizione di Schläfli*, Rendiconti del Circolo Matematico di Palermo, vol. 20, pp. 229-247.

That this V_{n-1}^{n-1*} is not generic may be shown thus. By a projective transformation

$$x_0 = x'_0, \quad x_i = x'_i/b_{0i} \quad (i = 1, 2, \dots, n),$$

a Schläfli set may be carried into the slightly modified form

$$x_0 = 0, \quad \sum_1^n x_k = 0; \quad x_i = 0, \quad x_0 + \sum b_{ik} x_k = 0 \quad (i = 1, 2, \dots, n),$$

where $b_{ii}=0$, $b_{i0}=1$, $b_{ik}=b_{ki}$, $k \neq i$. A general Schläfli set depends therefore on $n(n-1)/2$ parameters. Let there now be given $n+1$ generic S_{n-2} 's in S_n :

$$\sum_0^n a_i^{(1)} x_i = \sum_0^n b_i^{(1)} x_i = 0; \quad \sum_0^n a_i^{(2)} x_i = \sum_0^n b_i^{(2)} x_i = 0, \dots,$$

$$\sum_0^n a_i^{(n+1)} x_i = \sum_0^n b_i^{(n+1)} x_i = 0,$$

which depend on $2(n-1)(n+1)$ parameters. A projective transformation can therefore be found which will reduce this number to

$$2(n-1)(n+1) - n(n+2) = n^2 - 2n - 2.$$

But this number is greater than $n(n-1)/2$ when $n > 4$, as we wished to prove.

2. The equation of V_4^4 in Grassmann-Plücker coordinates. The equation (2), being a determinant of the n th order, is rather unwieldy for the purpose of investigating the accessory singularities of a V_{n-1}^{n-1} ; even in the case for $n=5$ the analytical work becomes formidable. We shall therefore use the Grassmann-Plücker coordinates and start with the equation of the generic V_4^4 which has been derived in a former paper (R, pp. 341-342): we shall then find the conditions which must be satisfied in order that it shall be associated with a Schläfli hexad†, and thus incidentally obtain the invariants of the spread. We shall suppose that V_4^4 has no triple point, that is, no three of the fundamental flats intersect. The equation of V_4^4 is‡

$$(3) \quad V_4^4 = \begin{vmatrix} y_1 \sum_1^6 \alpha_{i1} y_i + y_2 \sum_1^6 \alpha_{i2} y_i & y_3 \sum_1^6 \alpha_{i3} y_i + y_6 \sum_1^6 \alpha_{i6} y_i \\ y_1 \sum_1^6 \beta_{i1} y_i + y_2 \sum_1^6 \beta_{i2} y_i & y_3 \sum_1^6 \beta_{i3} y_i + y_6 \sum_1^6 \beta_{i6} y_i \end{vmatrix} = 0,$$

* John Eiesland, *On a class of ruled $(n-1)$ -spreads in S_n* , Rendiconti del Circolo Matematico di Palermo, vol. 54, pp. 335-365. By a "generic" V_{n-1}^{n-1} is meant here the V_{n-1}^{n-1} whose equation is given on p. 337. In what follows this paper will be referred to as "R." In this paper, the V_4^4 , here denoted as the "generic V_4 ," is the generalization for $n=5$ of Segre's variety in S_4 .

† By a Schläfli hexad in S_4 we mean here six 3-flats in Schläfli position.

‡ R, pp. 341-342.

or, if we add the elements of the first column to those of the second,

$$(3') \quad V_4^4 = \begin{vmatrix} y_1 \sum_1^6 \alpha_{i1} y_i + y_2 \sum_1^6 \alpha_{i2} y_i & y_4 \sum_1^6 \alpha_{i4} y_i + y_6 \sum_1^6 \alpha_{i6} y_i \\ y_1 \sum_1^6 \beta_{i1} y_i + y_2 \sum_1^6 \beta_{i2} y_i & y_4 \sum_1^6 \beta_{i4} y_i + y_6 \sum_1^6 \beta_{i6} y_i \end{vmatrix} = 0.$$

The five fundamental flats are

$$y_1 = y_2 = 0; \quad y_3 = y_5 = 0; \quad y_4 = y_6 = 0; \quad \sum_1^6 a_i y_i = \sum_1^6 b_i y_i = 0;$$

$$\sum_1^6 c_i y_i = \sum_1^6 d_i y_i = 0, \quad \alpha_{ik} = a_i b_k - a_k b_i, \quad \beta_{ik} = c_i d_k - c_k d_i.$$

If now the V_4^4 belongs to a Schläfli hexad, all the lines which meet these five fundamental flats must also meet a sixth flat. In order to find such a flat we write (3) and (3') as follows:

$$V_4^4 = \begin{vmatrix} y_1 M_1 + y_2 M_2 & y_3 M_3 + y_5 M_5 \\ y_1(L_1 + M_1) + y_2(L_2 + M_2) & y_3(L_3 + M_3) + y_5(L_5 + M_5) \end{vmatrix}$$

$$= \begin{vmatrix} y_1 M_1 + y_2 M_2 & y_4 M_4 + y_6 M_6 \\ y_1(L_1 + M_1) + y_2(L_2 + M_2) & y_4(L_4 + M_4) + y_6(L_6 + M_6) \end{vmatrix} = 0,$$

where $M_k = \sum \alpha_{ik} y_i$, $L_k = \sum \beta_{ik} y_i$, $k = 1, 2, \dots, 6$. Consider the 3-flat

$$(4) \quad y_1 + M_2 + L_2 = 0, \quad y_2 - M_1 - L_1 = 0.$$

If it is to be the required sixth flat it must be identical with the two flats

$$y_3 + M_3 + L_3 = 0, \quad y_3 - M_3 - L_3 = 0; \quad y_4 + M_6 + L_6 = 0, \quad y_6 - M_4 - L_4 = 0.$$

If we set $P_{12} = 1 + \alpha_{12} + \beta_{12}$, $P_{35} = 1 + \alpha_{35} + \beta_{35}$, $P_{46} = 1 + \alpha_{46} + \beta_{46}$, and $(ik) = \alpha_{ik} + \beta_{ik}$, this means that the determinant

$$\begin{vmatrix} 0 & P_{12} & (13) & (15) & (14) & (16) \\ -P_{12} & 0 & (23) & (25) & (24) & (26) \\ (31) & (32) & 0 & P_{35} & (34) & (36) \\ (51) & (52) & -P_{35} & 0 & (54) & (56) \\ (41) & (42) & (43) & (45) & 0 & P_{46} \\ (61) & (62) & (63) & (65) & -P_{46} & 0 \end{vmatrix}$$

must be of rank 2. We thus obtain the following conditions:

$$P_{12}P_{35} = (13)(25) + (15)(32),$$

$$(5,a) \quad \begin{aligned} (36)P_{12} &= (13)(26) + (16)(32), & (56)P_{12} &= (15)(26) + (16)(52), \\ (34)P_{12} &= (13)(24) + (14)(32), & (54)P_{12} &= (15)(24) + (14)(52); \end{aligned}$$

$$P_{35}P_{46} = (45)(36) + (34)(56),$$

$$(5,b) \quad \begin{aligned} (24)P_{35} &= (23)(45) + (25)(34), & (14)P_{35} &= (13)(45) + (15)(34), \\ (26)P_{35} &= (23)(65) + (25)(36), & (16)P_{35} &= (13)(65) + (15)(36); \end{aligned}$$

$$P_{12}P_{46} = (16)(42) + (14)(26),$$

$$(5,c) \quad \begin{aligned} (23)P_{46} &= (24)(36) + (26)(43), & (13)P_{46} &= (14)(36) + (16)(43), \\ (25)P_{46} &= (24)(56) + (26)(45), & (15)P_{46} &= (14)(56) + (16)(45). \end{aligned}$$

These relations are not independent; from any six of them the remaining ones may easily be derived. We also obtain the following important relations:

$$(6) \quad \begin{aligned} \frac{\alpha_{14}\beta_{16} - \alpha_{16}\beta_{14}}{\alpha_{16}\beta_{13} - \alpha_{13}\beta_{15}} &= \frac{\alpha_{14}\beta_{26} - \alpha_{26}\beta_{14} + \alpha_{24}\beta_{16} - \alpha_{16}\beta_{24}}{\alpha_{25}\beta_{13} - \alpha_{13}\beta_{25} + \alpha_{15}\beta_{23} - \alpha_{23}\beta_{15}} = \frac{\alpha_{24}\beta_{26} - \alpha_{26}\beta_{24}}{\alpha_{25}\beta_{23} - \alpha_{23}\beta_{25}} = 1, \\ \frac{\alpha_{14}\beta_{24} - \alpha_{24}\beta_{14}}{\alpha_{54}\beta_{34} - \alpha_{34}\beta_{54}} &= \frac{\alpha_{14}\beta_{26} - \alpha_{26}\beta_{14} + \alpha_{16}\beta_{24} - \alpha_{24}\beta_{16}}{\alpha_{36}\beta_{45} - \alpha_{45}\beta_{36} + \alpha_{56}\beta_{31} - \alpha_{34}\beta_{56}} = \frac{\alpha_{16}\beta_{26} - \alpha_{26}\beta_{16}}{\alpha_{56}\beta_{36} - \alpha_{36}\beta_{56}} = 1, \\ \frac{\alpha_{34}\beta_{36} - \alpha_{36}\beta_{34}}{\alpha_{23}\beta_{13} - \alpha_{13}\beta_{23}} &= \frac{\alpha_{34}\beta_{56} - \alpha_{56}\beta_{34} + \alpha_{36}\beta_{45} - \alpha_{45}\beta_{36}}{\alpha_{26}\beta_{13} - \alpha_{13}\beta_{26} + \alpha_{23}\beta_{15} - \alpha_{15}\beta_{23}} = \frac{\alpha_{45}\beta_{65} - \alpha_{65}\beta_{45}}{\alpha_{26}\beta_{15} - \alpha_{16}\beta_{25}} = 1; \end{aligned}$$

$$(7) \quad 1 + \Sigma\alpha = 1 + \alpha_{12} + \alpha_{35} + \alpha_{46} = 0, \quad 1 + \Sigma\beta = 1 + \beta_{12} + \beta_{35} + \beta_{46} = 0.$$

The last two relations are obtained by using (5,b) and (6). Since there are six independent non-homogeneous relations between the 16 parameters of a generic V_4^1 , the V_4^1 belonging to a Schläfli hexad has 10 essential parameters, as was shown before (p. 316) by a different method.

3. The singular loci on the V_4^1 associated with a Schläfli hexad. We know that the generic V_4^1 in S_5 has 5 fundamental 3-flats and that the singular loci lie in each of these flats.* In any one flat we have four fundamental double lines which are the intersections of the flat with the remaining four flats; moreover, two accessory double lines which intersect these in 8 points and, finally, a cubic curve which has the four fundamental lines as bisecants. Through each of the 40 points pass two double lines of which one is accessory, and one cubic; but it is to be noted that this cubic does not belong to the fundamental flat in which the accessory line is immersed. We have thus 20 lines and 5 cubics as the complete set of double loci on a generic V_4^1 .

* R, pp. 344-352.

In the case at hand the V_4^4 has six fundamental flats in a Schläfli position. In each flat are five fundamental double lines and two accessory double lines which intersect these in 10 points. The cubic is composite, consisting of three lines, namely the fifth double line which is added to the four of the generic case, and two lines which remain. We shall prove that these two lines coincide with the two accessory lines. It will be sufficient to prove this for any one flat, since what happens in one must happen in all the other five flats.

The singular loci in the 3-flat $y_3 = y_5 = 0$ are the complete intersection of the two cubic surfaces

$$y_3 = y_5 = 0, \quad \frac{\partial V_4^4}{\partial y_3} = 0, \quad \frac{\partial V_4^4}{\partial y_5} = 0.$$

We have then, from (3),

$$(8) \quad \begin{aligned} \phi_1 &= \frac{\partial V_4^4}{\partial y_3} = P \sum_{1,2}^{4,6} \beta_{i3} y_i - Q \sum_{1,2}^{4,6} \alpha_{i3} y_i = 0, \\ \phi_2 &= \frac{\partial V_4^4}{\partial y_5} = P \sum_{1,2}^{4,6} \beta_{i5} y_i - Q \sum_{1,2}^{4,6} \alpha_{i5} y_i = 0. \end{aligned}$$

The four fundamental double lines are

$$(9) \quad \begin{aligned} y_3 = y_5 = y_1 = y_2 = 0, \quad y_3 = y_5 = y_4 = y_6 = 0, \\ y_3 = y_5 = M_3 = M_5 = 0, \quad y_3 = y_5 = L_3 = L_5 = 0, \end{aligned}$$

to which must be added the fifth double line

$$(10) \quad y_3 = y_5 = L_3 + M_3 = L_5 + M_5 = 0,$$

which is the intersection of the flat $y_3 = y_5 = 0$ with the fifth fundamental flat $y_3 + M_3 + L_3 = 0$, $y_5 - M_5 - L_5 = 0$. The two accessory double lines are

$$(11) \quad y_1 = \kappa y_2, \quad y_4 = \mu y_6,$$

where κ and μ are roots of the two quadratic equations

$$(12) \quad \begin{aligned} (\alpha_{14}\beta_{16} - \alpha_{10}\beta_{14})\kappa^2 + (\alpha_{10}\beta_{42} - \alpha_{42}\beta_{16} + \alpha_{26}\beta_{41} - \alpha_{41}\beta_{26})\kappa \\ + (\alpha_{26}\beta_{42} - \alpha_{42}\beta_{26}) = 0, \\ (\alpha_{14}\beta_{24} - \alpha_{24}\beta_{14})\mu^2 + (\alpha_{24}\beta_{61} - \alpha_{61}\beta_{24} + \alpha_{26}\beta_{41} - \alpha_{41}\beta_{26})\mu \\ + (\alpha_{24}\beta_{61} - \alpha_{61}\beta_{24}) = 0. \end{aligned}$$

These equations may also be written

$$(13) \quad \kappa = \frac{\alpha_{24}\mu + \alpha_{26}}{\alpha_{41}\mu + \alpha_{61}} = \frac{\beta_{24}\mu + \beta_{26}}{\beta_{41}\mu + \beta_{61}}, \quad \mu = \frac{\alpha_{10}\kappa + \alpha_{26}}{\alpha_{41}\kappa + \alpha_{42}} = \frac{\beta_{10}\kappa + \beta_{26}}{\beta_{41}\kappa + \beta_{42}}.$$

The cubic curve being composite, consisting of the fifth double line and two additional lines, it is to be proved that these latter coincide with the two accessory lines (11); in other words, these lines are tac-loci on the two surfaces $\phi_1 = \phi_2 = 0$. Take any point on the line (11), say $(\kappa\rho, \rho, \mu, 1)$. The tangent planes to the cubics at the point are

$$\left(\frac{\partial\phi_i}{\partial y_1}\right)y_1 + \left(\frac{\partial\phi_i}{\partial y_2}\right)y_2 + \left(\frac{\partial\phi_i}{\partial y_4}\right)y_4 + \left(\frac{\partial\phi_i}{\partial y_6}\right)y_6 = 0 \quad (i = 1, 2).$$

Noting that the accessory lines are generators of the two quadrics

$$P = y_1(\alpha_{41}y_4 + \alpha_{61}y_6) + y_2(\alpha_{42}y_4 + \alpha_{62}y_6) = 0,$$

$$Q = y_1(\beta_{41}y_4 + \beta_{61}y_6) + y_2(\beta_{42}y_4 + \beta_{62}y_6) = 0,$$

we have

$$\begin{aligned} \left(\frac{\partial\phi_1}{\partial y_1}\right) &= (\alpha_{41}\mu + \alpha_{61})L_3 - (\beta_{41}\mu + \beta_{61})M_3, & \left(\frac{\partial\phi_1}{\partial y_2}\right) &= (\alpha_{42}\mu + \alpha_{62})L_3 - (\beta_{42}\mu + \beta_{62})M_3, \\ \frac{1}{\rho}\left(\frac{\partial\phi_1}{\partial y_4}\right) &= (\alpha_{41}\kappa + \alpha_{42})L_3 - (\beta_{41}\kappa + \beta_{42})M_3, & \frac{1}{\rho}\left(\frac{\partial\phi_1}{\partial y_6}\right) &= (\alpha_{61}\kappa + \alpha_{62})L_3 - (\beta_{61}\kappa + \beta_{62})M_3, \\ \left(\frac{\partial\phi_2}{\partial y_1}\right) &= (\alpha_{41}\mu + \alpha_{61})L_5 - (\beta_{41}\mu + \beta_{61})M_5, & \left(\frac{\partial\phi_2}{\partial y_2}\right) &= (\alpha_{42}\mu + \alpha_{62})L_5 - (\beta_{42}\mu + \beta_{62})M_5, \\ \frac{1}{\rho}\left(\frac{\partial\phi_2}{\partial y_4}\right) &= (\alpha_{41}\kappa + \alpha_{42})L_5 - (\beta_{41}\kappa + \beta_{42})M_5, & \frac{1}{\rho}\left(\frac{\partial\phi_2}{\partial y_6}\right) &= (\alpha_{61}\kappa + \alpha_{62})L_5 - (\beta_{61}\kappa + \beta_{62})M_5, \end{aligned}$$

where we have set

$$L_3 = (\beta_{13}\kappa + \beta_{23})\rho + \beta_{43}\mu + \beta_{63}, \quad M_3 = (\alpha_{13}\kappa + \alpha_{23})\rho + \alpha_{43}\mu + \alpha_{63},$$

$$L_5 = (\beta_{15}\kappa + \beta_{25})\rho + \beta_{45}\mu + \beta_{65}, \quad M_5 = (\alpha_{15}\kappa + \alpha_{25})\rho + \alpha_{45}\mu + \alpha_{65}.$$

If now the line (11) is to be a tac-locus on $\phi_1 = \phi_2 = 0$ we must have

$$\begin{aligned} \frac{\left(\frac{\partial\phi_1}{\partial y_1}\right)}{\left(\frac{\partial\phi_2}{\partial y_1}\right)} &= \frac{\left(\frac{\partial\phi_1}{\partial y_2}\right)}{\left(\frac{\partial\phi_2}{\partial y_2}\right)} = \frac{\left(\frac{\partial\phi_1}{\partial y_4}\right)}{\left(\frac{\partial\phi_2}{\partial y_4}\right)} = \frac{\left(\frac{\partial\phi_1}{\partial y_6}\right)}{\left(\frac{\partial\phi_2}{\partial y_6}\right)} \\ &= \frac{L_3 - p_1M_3}{L_5 - p_1M_5} = \frac{L_3 - p_2M_3}{L_5 - p_2M_5} = \frac{L_3 - p_3M_3}{L_5 - p_3M_5} = \frac{L_3 - p_4M_3}{L_5 - p_4M_5}, \end{aligned}$$

where

$$p_1 = \frac{\beta_{41}\mu + \beta_{61}}{\alpha_{41}\mu + \alpha_{61}}, \quad p_2 = \frac{\beta_{42}\mu + \beta_{62}}{\alpha_{42}\mu + \alpha_{62}}, \quad p_3 = \frac{\beta_{41}\kappa + \beta_{42}}{\alpha_{41}\kappa + \alpha_{42}}, \quad p_4 = \frac{\beta_{61}\kappa + \beta_{62}}{\alpha_{61}\kappa + \alpha_{62}}.$$

But the equations (13) show that $p_1 = p_2$ and $p_3 = p_4$, hence we get the single condition

$$\frac{L_3 - p_1 M_3}{L_5 - p_1 M_5} = \frac{L_3 - p_3 M_3}{L_5 - p_3 M_5}, \quad p_1 - p_3 \neq 0,$$

which is equivalent to the condition $L_3 M_5 - L_5 M_3 = 0$, true for all values of ρ . We thus obtain the following three relations:

$$(14) \quad \begin{aligned} &(\beta_{13}\kappa + \beta_{23})(\alpha_{15}\kappa + \alpha_{25}) - (\beta_{15}\kappa + \beta_{25})(\alpha_{13}\kappa + \alpha_{23}) = 0, \\ &(\beta_{43}\mu + \beta_{63})(\alpha_{45}\mu + \alpha_{65}) - (\beta_{45}\mu + \beta_{65})(\alpha_{43}\mu + \alpha_{63}) = 0, \\ &(\beta_{13}\kappa + \beta_{23})(\alpha_{45}\mu + \alpha_{65}) - (\alpha_{15}\kappa + \alpha_{25})(\beta_{43}\mu + \beta_{63}) \\ &- (\beta_{15}\kappa + \beta_{25})(\alpha_{43}\mu + \alpha_{63}) - (\alpha_{13}\kappa + \alpha_{23})(\beta_{45}\mu + \beta_{65}) = 0. \end{aligned}$$

The third relation is satisfied by virtue of the first two, hence κ and μ must be roots of the two quadratic equations

$$(15) \quad \begin{aligned} &(\alpha_{15}\beta_{13} - \beta_{15}\alpha_{13})\kappa^2 + (\alpha_{15}\beta_{23} - \alpha_{23}\beta_{15} + \alpha_{25}\beta_{13} - \alpha_{13}\beta_{25})\kappa \\ &+ \alpha_{25}\beta_{23} - \beta_{25}\alpha_{23} = 0, \\ &(\alpha_{45}\beta_{43} - \alpha_{43}\beta_{45})\mu^2 + (\alpha_{45}\beta_{63} - \alpha_{63}\beta_{45} + \alpha_{65}\beta_{43} - \alpha_{43}\beta_{65})\mu \\ &+ \alpha_{65}\beta_{63} - \alpha_{63}\beta_{65} = 0. \end{aligned}$$

But κ and μ are roots of the quadratic equation (12), hence we must have

$$\begin{aligned} \frac{\alpha_{14}\beta_{16} - \alpha_{16}\beta_{14}}{\alpha_{15}\beta_{13} - \alpha_{13}\beta_{15}} &= \frac{\alpha_{16}\beta_{42} - \alpha_{42}\beta_{16} + \alpha_{26}\beta_{41} - \alpha_{41}\beta_{26}}{\alpha_{15}\beta_{23} - \alpha_{23}\beta_{15} + \alpha_{25}\beta_{13} - \alpha_{13}\beta_{25}} = \frac{\alpha_{26}\beta_{42} - \alpha_{42}\beta_{26}}{\alpha_{25}\beta_{23} - \alpha_{23}\beta_{25}}, \\ \frac{\alpha_{14}\beta_{24} - \alpha_{24}\beta_{14}}{\alpha_{45}\beta_{43} - \alpha_{43}\beta_{45}} &= \frac{\alpha_{24}\beta_{61} - \alpha_{61}\beta_{24} + \alpha_{26}\beta_{41} - \alpha_{41}\beta_{26}}{\alpha_{45}\beta_{63} - \alpha_{63}\beta_{45} + \alpha_{65}\beta_{43} - \beta_{65}\alpha_{43}} = \frac{\alpha_{26}\beta_{61} - \alpha_{61}\beta_{26}}{\alpha_{65}\beta_{63} - \alpha_{63}\beta_{65}}, \end{aligned}$$

which are true according to equations (6), as we wished to prove.

If we set $\lambda = (\alpha_{15}\kappa + \alpha_{25})/(\alpha_{31}\kappa + \alpha_{32})$, we have from (13) and (14),

$$(16) \quad \lambda = \frac{\alpha_{15}\kappa + \alpha_{25}}{\alpha_{31}\kappa + \alpha_{32}} = \frac{\beta_{15}\kappa + \beta_{25}}{\beta_{31}\kappa + \beta_{32}} = \frac{\alpha_{45}\mu + \alpha_{65}}{\alpha_{34}\mu + \alpha_{36}} = \frac{\beta_{45}\mu + \beta_{65}}{\beta_{34}\mu + \beta_{36}},$$

so that λ must be a root of the two equations

$$(17) \quad \begin{aligned} &(\alpha_{23}\beta_{31} - \alpha_{31}\beta_{23})\lambda^2 + (\alpha_{23}\beta_{51} - \alpha_{51}\beta_{23} + \alpha_{25}\beta_{31} - \alpha_{31}\beta_{25})\lambda \\ &+ \alpha_{25}\beta_{51} - \alpha_{51}\beta_{25} = 0, \\ &(\alpha_{63}\beta_{34} - \alpha_{34}\beta_{63})\lambda^2 + (\alpha_{63}\beta_{54} - \alpha_{54}\beta_{63} + \alpha_{65}\beta_{34} - \alpha_{34}\beta_{65})\lambda \\ &+ \alpha_{65}\beta_{54} - \alpha_{54}\beta_{65} = 0, \end{aligned}$$

which have identical roots, equation (6).

We have thus $6 \cdot 5/2 = 15$ fundamental double lines and 12 accessory double lines; these lines intersect in 30 points, three lines through each point. No point is a triple point. We shall prove the following important

THEOREM. *A V_4 in S_5 , associated with a Schläfli hexad, has two double planes. The 12 accessory lines form a double-six, each set of six lines forming a complete hexagon in each plane. The 15 fundamental double lines are outside of these planes and join the 15 pairs of corresponding vertices of the hexagon. The equations of the planes are*

$$(18) \quad y_1 = \kappa y_2, \quad y_3 = \lambda y_5, \quad y_4 = \mu y_6,$$

where κ , λ and μ are the roots of the three quadratic equations (12) and (17).

The proof of the first part of this theorem is immediate. If we substitute the values of y_1 , y_2 , and y_3 from (18) in the equation (3), the determinant reduces to one of rank zero, since all the elements vanish, account being taken of (13) and (16); to prove the second part we need only show that any one of the planes contains the six accessory lines

$$(19,a) \quad \begin{array}{llll} y_1 = 0, & y_2 = 0, & y_3 = \lambda y_5, & y_4 = \mu y_6, \\ y_3 = 0, & y_5 = 0, & y_1 = \kappa y_2, & y_4 = \mu y_6, \\ y_4 = 0, & y_6 = 0, & y_1 = \kappa y_2, & y_3 = \lambda y_5; \end{array}$$

$$(19,b) \quad \begin{array}{llll} y_1 = \kappa y_2, & y_3 = \lambda y_5, & \sum \alpha_{23} y_i = 0, & \sum \alpha_{45} y_i = 0, \\ y_1 = \kappa y_2, & y_3 = \lambda y_5, & \sum \beta_{13} y_i = 0, & \sum \beta_{45} y_i = 0, \\ y_1 = \kappa y_2, & y_3 = \lambda y_5, & y_1 + M_2 + L_2 = 0, & y_2 - M_1 - L_1 = 0. \end{array}$$

The second set (19, b) may also be written

$$\begin{aligned} y_1 = \kappa y_2, \quad y_3 = \lambda y_5, \quad y_4 = \mu y_6, \quad y_2 &= \frac{\alpha_{35} y_5}{\alpha_{13} \kappa + \alpha_{23}} + \frac{\alpha_{46} y_6}{\alpha_{14} \kappa + \alpha_{24}}, \\ y_1 = \kappa y_2, \quad y_3 = \lambda y_5, \quad y_4 = \mu y_6, \quad y_2 &= \frac{\beta_{35} y_5}{\beta_{13} \kappa + \beta_{23}} + \frac{\beta_{46} y_6}{\beta_{14} \kappa + \beta_{24}}, \\ y_1 = \kappa y_2, \quad y_3 = \lambda y_5, \quad y_4 = \mu y_6, \quad y_2 &= \frac{P_{35} y_5}{(13)\kappa + (23)} + \frac{P_{46} y_6}{(14)\kappa + (24)}. \end{aligned}$$

From these equations it is at once evident that on each of the planes (18) the six accessory lines form a complete hexagon, and that any one of the 15 fundamental double lines joins a pair of corresponding vertices of the two hexagons.

THEOREM. *The 12 sides of the two hexagons are bispatial.*

To prove this we transform the origin $(0, 0, 0, 0, 0, 1)$ to the point $(\rho\kappa, \rho, \sigma\lambda, \mu, \sigma, 1)$ on one of the double planes. The tangent cone, the equation of which being rather long, we shall not give here, is seen to be reducible for the following values of ρ and σ :

$$\rho = 0, \quad \sigma = 0, \quad \rho_1 = \frac{\alpha_{35}\sigma}{\alpha_{13}\kappa + \alpha_{23}} + \frac{\alpha_{46}}{\alpha_{14}\kappa + \alpha_{24}}, \quad \rho_2 = \frac{\beta_{35}\sigma}{\beta_{13}\kappa + \beta_{23}} + \frac{\alpha_{46}}{\beta_{14}\kappa + \beta_{24}},$$

$$\rho_3 = \frac{P_{35}\sigma}{(13)\kappa + (23)} + \frac{P_{46}}{(14)\kappa + (24)}.$$

These five values correspond to five sides of the hexagon. That the sixth side, $y_1 = \kappa y_2, y_3 = \lambda y_5, y_4 = y_6 = 0$, is also bispatial is proved by transforming the origin to the point $(\rho\kappa, \rho, \lambda, \mu\tau, 1, \tau)$. The cone is reducible when $\tau = 0$.

From these two theorems we derive the following

COROLLARY. *Given in S_5 a hexad of 3-flats in a Schläfli position. There exist two planes which intersect these flats in 12 lines forming a complete hexagon in each plane. The V_4^4 associated with the hexad has the two planes for double planes and the 15 lines joining the corresponding vertices of the hexagons are double lines on the V_4^4 . The 12 lines of intersection are bispatial.*

4. Transformation of the V_4^4 . In order to carry the V_4^4 into the form given by equation (2) for $n=5$ we set up the following transformation:

$$(20) \quad x_0 = y_1, \quad \sum_0^5 b_{0i}x_i = y_2, \quad \sum_0^5 b_{1i}x_i = y_5, \quad x_2 = y_4, \quad \sum_1^5 b_{2i}x_i = y_6, \quad x_1 = y_3,$$

from which it must follow that

$$(21) \quad \sum_1^6 a_i y_i = x_2, \quad \sum b_i y_i = \sum_0^5 b_{3i} x_i, \quad \sum_1^6 c_i y_i = x_4, \quad \sum_1^6 d_i y_i = \sum b_{4i} x_i.$$

Substituting the values of y_i from (20) on the left side of these equations and comparing the coefficients of the x 's on both sides we find the values of a_i, b_i, c_i, d_i expressed rationally in terms of the b_{ik} . We then calculate the Grassmann coordinates α_{ik}, β_{ik} . The work is rather long and tedious, but affords a valuable check on the correctness of the method we have pursued; in fact, the α_{ik}, β_{ik} thus found are seen to satisfy the fundamental relations (5,a), (5,b), and (5,c).

The singular loci of the V_4^4 having been found, our work is completed. If we had started with the equation (2) for $n=5$ we should have failed, the analytic work being too complicated.

5. The self-dual V_4^4 associated with a Schläfli hexad. In a former paper* it was proved that if on a generic V_4^4 in S_6 any two of the 10 fundamental double lines are bispatial, they are all bispatial and the spread is self-dual. In the case of the V_4^4 here considered a similar theorem holds: *If any one of the 15 fundamental double lines is bispatial, they are all bispatial and the V_4^4 is self-dual.*

Let the double line be $y_1 = y_2 = y_3 = y_5 = 0$; transforming the origin to a point ρ on this line we set $y_i = y'_i$, $y_4 = y'_4 + \rho y'_6$, $i = 1, 2, 3, 5, 6$. The equation (3) may then be written, dropping the primes,

$$\phi_2 y_6^2 + \phi_3 y_6 + \phi_4 = 0,$$

where

$$\begin{aligned} \phi_2 = & [(\alpha_{61} + \rho\alpha_{41})y_1 + (\alpha_{62} + \rho\alpha_{42})y_2][(\beta_{63} + \rho\beta_{43})y_3 + (\beta_{65} + \rho\beta_{45})y_5] \\ & - [(\beta_{61} + \rho\beta_{41})y_4 + (\beta_{62} + \rho\beta_{42})y_5][(\alpha_{63} + \rho\alpha_{43})y_3 + (\alpha_{65} + \rho\alpha_{45})y_5]. \end{aligned}$$

In order that the point ρ shall be bispatial the discriminant of this form must be of rank 2. We have

$$\begin{aligned} \Delta = & [(\alpha_{65} + \rho\alpha_{45})(\beta_{63} + \rho\beta_{43}) - (\alpha_{63} + \rho\alpha_{43})(\beta_{65} + \rho\beta_{45})][(\beta_{61} + \rho\beta_{41})(\alpha_{62} + \rho\alpha_{42}) \\ & - (\beta_{62} + \rho\beta_{42})(\alpha_{61} + \rho\alpha_{41})] = 0. \end{aligned}$$

Since every point ρ is to be bispatial we must have

$$\begin{aligned} \alpha_{45}\beta_{43} - \alpha_{43}\beta_{45} = \alpha_{42}\beta_{41} - \alpha_{41}\beta_{42} = 0, \quad \alpha_{65}\beta_{63} - \alpha_{63}\beta_{65} = \alpha_{62}\beta_{61} - \alpha_{61}\beta_{62} = 0, \\ \alpha_{65}\beta_{43} - \alpha_{43}\beta_{65} + \alpha_{45}\beta_{63} - \alpha_{63}\beta_{45} = 0, \quad \alpha_{42}\beta_{61} - \alpha_{61}\beta_{42} + \alpha_{62}\beta_{41} - \alpha_{41}\beta_{62} = 0, \end{aligned}$$

which may also be written

$$(22,a) \quad \frac{\beta_{41}}{\alpha_{41}} = \frac{\beta_{42}}{\alpha_{42}} = \frac{\beta_{61}}{\alpha_{61}} = \frac{\beta_{62}}{\alpha_{62}}, \quad (22,b) \quad \frac{\beta_{45}}{\alpha_{45}} = \frac{\beta_{43}}{\alpha_{43}} = \frac{\beta_{65}}{\alpha_{65}} = \frac{\beta_{63}}{\alpha_{63}};$$

hence three conditions must be satisfied, the second set being identical with the first, as follows from (6). If also the double line $y_3 = y_5 = 0$, $y_4 = y_6 = 0$ is bispatial, we get the following two sets of conditions:

$$(22,c) \quad \frac{\beta_{13}}{\alpha_{13}} = \frac{\beta_{15}}{\alpha_{15}} = \frac{\beta_{23}}{\alpha_{23}} = \frac{\beta_{25}}{\alpha_{25}}, \quad (22,d) \quad \frac{\beta_{41}}{\alpha_{41}} = \frac{\beta_{61}}{\alpha_{61}} = \frac{\beta_{42}}{\alpha_{42}} = \frac{\beta_{62}}{\alpha_{62}};$$

that is, no new conditions are added, if account is taken of equations (6). It will not be necessary to carry out the work for the 13 remaining double lines as no new conditions are found. If now we set

* R, pp. 358-360.

$$\frac{\beta_{41}}{\alpha_{41}} = \frac{\beta_{42}}{\alpha_{42}} = \frac{\beta_{61}}{\alpha_{61}} = \frac{\beta_{62}}{\alpha_{62}} = r, \quad \frac{\beta_{13}}{\alpha_{13}} = \frac{\beta_{15}}{\alpha_{15}} = \frac{\beta_{23}}{\alpha_{23}} = \frac{\beta_{25}}{\alpha_{25}} = s,$$

$$\frac{\beta_{45}}{\alpha_{45}} = \frac{\beta_{48}}{\alpha_{48}} = \frac{\beta_{65}}{\alpha_{65}} = \frac{\beta_{68}}{\alpha_{68}} = t,$$

the equation of the V_4^4 in y -coordinates is

$$(23) \quad \begin{vmatrix} Q_2 + Q_3 & Q_1 - Q_3 \\ rQ_2 + sQ_3 & tQ_1 - sQ_3 \end{vmatrix} = 0,$$

or, when expanded,

$$(23') \quad \begin{aligned} (r-s)Q_2Q_3 + (t-s)Q_1Q_3 + (t-r)Q_1Q_2 &= 0, \\ Q_1 &= y_3(\alpha_{43}y_4 + \alpha_{63}y_6) + y_5(\alpha_{45}y_4 + \alpha_{65}y_6), \\ Q_2 &= y_1(\alpha_{41}y_4 + \alpha_{61}y_6) + y_2(\alpha_{42}y_4 + \alpha_{62}y_6), \\ Q_3 &= y_1(\alpha_{31}y_3 + \alpha_{51}y_5) + y_2(\alpha_{32}y_3 + \alpha_{52}y_5). \end{aligned}$$

The equation of the V_4^4 in tangential coordinates is

$$(24) \quad \begin{aligned} [(t-r)U_3 + (s-t)U_2 + (s-r)U_1]^2 - 4(s-t)(s-r)U_1U_2 &= 0, \\ U_1 &= \frac{u_3(\alpha_{65}u_4 + \alpha_{54}u_6) + u_5(\alpha_{43}u_6 + \alpha_{26}u_4)}{\alpha_{46}\alpha_{35}}, \\ U_2 &= \frac{u_1(\alpha_{62}u_4 + \alpha_{24}u_6) + u_2(\alpha_{41}u_6 + \alpha_{16}u_4)}{\alpha_{46}\alpha_{12}}, \\ U_3 &= \frac{u_1(\alpha_{52}u_3 + \alpha_{23}u_5) + u_2(\alpha_{31}u_5 + \alpha_{15}u_3)}{\alpha_{36}\alpha_{12}}. \end{aligned}$$

Hence the order and class of V_4^4 are equal, as we wished to prove.

6. The singularities of the self-dual V_4^4 . Since the conditions (22,a) and the resulting conditions (22,b), (22,c), and (22,d) imply that the three equations (15) and (17) are indeterminate, it follows that there will be an infinite number of double planes instead of only two. These planes are the generators of the 3-dimensional quadric $Q_1=Q_2=Q_3=0$, which may also be written

$$\frac{y_1}{y_2} = \frac{\alpha_{42}y_4 + \alpha_{62}y_6}{\alpha_{14}y_4 + \alpha_{16}y_6} = \frac{\alpha_{32}y_3 + \alpha_{52}y_5}{\alpha_{13}y_3 + \alpha_{15}y_5}.$$

Setting each of these ratios equal to a variable α we have the ∞^1 generating planes. The six fundamental flats intersect this quadric in six 2-dimensional quadrics which are all bispatial. We may therefore state the following theorem:

*The self-dual V_4^4 associated with a Schläfli hexad has a 3-dimensional quadric as locus of double points. The six fundamental flats intersect this quadric in six 2-dimensional quadrics which are all bispatial. There are ∞^7 such self-dual V_4^4 's.**

* In R, p. 359, the fact was overlooked that the 2-dimensional quadric $x_1/x_2 = x_3/x_4 = x_5/x_6$ is a double locus on the self-dual V_4^4 . We have then five 2-dimensional bispatial quadrics, one in each of the five fundamental flats.

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ON QUASI-COMMUTATIVE MATRICES*

BY

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1. Introduction. In quantum mechanics there appear infinite matrices p and q with the property that

$$pq - qp = c,$$

where c is a scalar matrix. It is well known[†] that a relation of this type can not be satisfied by finite matrices. However, in the calculation of commutation formulas for polynomials in p and q no use is made of the fact that c is a scalar but merely that it is commutative with both p and q .[‡] And there do exist pairs of finite matrices x, y of the same order such that $xy - yx$ is not zero and is commutative with both x and y . Such matrices will be called *quasi-commutative matrices* and either may be said to be quasi-commutative with the other.

In a certain sense the algebra of polynomials in a pair of quasi-commutative matrices is homeomorphic to the algebra arising in quantum mechanics. It is hoped to discuss such algebras in some detail in a later paper. In the present paper we shall make a brief study of quasi-commutative matrices whose elements belong to the complex number field.

The concept of quasi-commutativity is an extension or generalization of commutativity, and as would be expected, some of the results obtained are generalizations of known theorems concerning commutative matrices.

The problem of determining quasi-commutative matrices is that of finding matrices x, y, z ($\neq 0$) which satisfy the equations

$$xy - yx = z, \quad xz = zx, \quad yz = zy.$$

If z is an assigned matrix, there may or may not exist matrices x and y such that (x, y, z) is a solution of these equations. In §3, we shall characterize those matrices z for which these equations do admit a solution.

If x is a given matrix, it is clear that there always exist matrices commutative with x but it is not evident whether there exist matrices quasi-commutative with x . It will be shown (§4) that in most cases there do exist such matrices, and a necessary and sufficient condition for their existence is ob-

* Presented to the Society, December 27, 1932; received by the editors June 21, 1933.

† See, e.g., Birtwistle, *The New Quantum Mechanics*, p. 67.

‡ See a previous paper, *On commutation formulas in the algebra of quantum mechanics*, these Transactions, vol. 31 (1929), pp. 793-806.

tained. The general form of a matrix quasi-commutative with x is also given for the case in which x has a single elementary divisor corresponding to each root.

In §5, we prove a theorem about the roots of any scalar polynomial $\psi(x, y)$ in quasi-commutative matrices x and y . It is shown that if the roots of x and y are λ_i and μ_i respectively, then the roots of $\psi(x, y)$ are all of the form $\psi(\lambda_i, \mu_i)$. This is merely an extension of a known theorem concerning commutative matrices.*

2. Commutative matrices. In this section we shall make some preliminary remarks and then mention a few known properties of commutative matrices which will be needed in later sections.

Let x be a given matrix of order n , with the elementary divisors $(\lambda - \lambda_i)^{p_i}$ ($i = 1, 2, \dots, r$). Then by a proper choice of basis, x may be expressed in the canonical form†

$$(1) \quad \begin{vmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_r \end{vmatrix},$$

where X_i is the matrix of order p_i ‡,

$$(2) \quad \begin{vmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{vmatrix} \quad (i = 1, 2, \dots, r).$$

If now we write $X_i = \lambda_i e_i + \eta_i$, the matrices e_i and η_i may be called respectively the *partial idempotent element* and the *partial nilpotent element*§ of x corresponding to the elementary divisor $(\lambda - \lambda_i)^{p_i}$.

* Frobenius, *Über vertauschbare Matrizen*, Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin, 1896, pp. 601-614.

† Cf. Bôcher, *Higher Algebra*, p. 289.

‡ We shall think of X_i as a matrix of order p_i or one of order n at pleasure. It will be clear from the context as which it is being considered.

§ Cf. J. H. M. Wedderburn, *The automorphic transformation of a bilinear form*, Annals of Mathematics, (2), vol. 23 (1921), pp. 122-134. In general a matrix a ($\neq 0$) such that $a^2 = a$ is said to be *idempotent*, a matrix b with the property that $b^r = 0$, $b^{r-1} \neq 0$, is *nilpotent of index r* . The index of η_i is clearly p_i .

Let λ_i ($i = 1, 2, \dots, l$) be the *distinct* roots of x and denote by ϕ_i the sum of all the partial idempotent elements of x corresponding to elementary divisors involving the same root λ_i . The matrix ϕ_i is called the *principal idempotent element* of x belonging to the root λ_i . It is of considerable importance that the principal elements of a matrix x may be expressed as scalar polynomials in x .*

If, in (1), we combine those blocks X_i corresponding to elementary divisors with the same root λ_i into a single block $X^{(i)}$, we may express x in the form

$$(3) \quad \begin{vmatrix} X^{(1)} & 0 & \dots & 0 \\ 0 & X^{(2)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & X^{(l)} \end{vmatrix}.$$

The principal idempotent element of x belonging to λ_i is in this case a matrix of order n with 1 in each element of the principal diagonal corresponding to the position of the diagonal of the submatrix $X^{(i)}$ in x , and zeros elsewhere.

We now discuss briefly the form of a matrix y commutative with a matrix x in the canonical form (1). If we set $e_i y e_j = Y_{ij}$, we may write y in the form

$$(4) \quad y = \sum_{i,j} Y_{ij} = \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1r} \\ Y_{21} & Y_{22} & \dots & Y_{2r} \\ \dots & \dots & \dots & \dots \\ Y_{r1} & Y_{r2} & \dots & Y_{rr} \end{vmatrix},$$

where Y_{ij} is a rectangular matrix of p_i rows and p_j columns.† The equation $xy = yx$ is then equivalent to the set of equations,

$$(5) \quad X_i Y_{ij} = Y_{ij} X_j \quad (i, j = 1, 2, \dots, r).$$

Let now s and t be fixed values of i and j , and consider the single equation for Y_{st} ,

$$(6) \quad X_s Y_{st} = Y_{st} X_t.$$

The following facts are known concerning the equation (6).‡ If $\lambda_s \neq \lambda_t$, the only solution is $Y_{st} = 0$. If $\lambda_s = \lambda_t$ and $p_s \geq p_t$, the general solution is of the form

* Wedderburn, loc. cit., p. 126.

† By actual definition, Y_{ij} is a matrix of order n , but there will be no confusion as it has non-zero elements only in the rectangular block indicated in the array.

‡ H. Kreis, *Contribution à la Théorie des Systèmes Linéaires*, Zurich Thesis, 1906. See also Hilton, *Homogeneous Linear Substitutions*, Oxford, 1914, chapter 5.

$$(7) \quad \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_{p_i-1} & a_{p_i} \\ 0 & a_1 & a_2 & a_3 & \cdots & a_{p_i-2} & a_{p_i-1} \\ 0 & 0 & a_1 & a_2 & \cdots & a_{p_i-3} & a_{p_i-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{vmatrix},$$

where a_1, a_2, \dots, a_{p_i} are arbitrary parameters. Similarly if $\lambda_s = \lambda_i$ and $p_s \leq p_i$, Y_{si} takes the form

$$(8) \quad \begin{vmatrix} 0 & \cdots & 0 & a_1 & a_2 & a_3 & \cdots & a_{p_s-1} & a_{p_s} \\ 0 & \cdots & 0 & 0 & a_1 & a_2 & \cdots & a_{p_s-2} & a_{p_s-1} \\ 0 & \cdots & 0 & 0 & 0 & a_1 & \cdots & a_{p_s-3} & a_{p_s-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & a_1 & a_2 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & a_1 \end{vmatrix}.$$

We may now write at once the general form of a matrix y commutative with x , as we only need to solve each of the r^2 equations (5) for the Y_{ij} by these rules, and substitute in (4).

The following application will be of importance in the sequel. Let (x, y, z) be a solution of the equations

$$(9) \quad xy - yx = z, \quad xz = zx, \quad yz = zy.$$

If S is any non-singular matrix of the same order, then clearly $(SxS^{-1}, SyS^{-1}, SzS^{-1})$ is also a solution of these equations. Hence we may, without loss of generality, assume that x is in canonical form (3).^{*} Set $Y^{(ij)} = \phi_i y \phi_j$, $Z^{(ij)} = \phi_i z \phi_j$, where the ϕ_i are the principal idempotent elements of x . The first of equations (9) may be replaced by the set of equations

$$X^{(i)} Y^{(ij)} - Y^{(ij)} X^{(j)} = Z^{(ij)} \quad (i, j = 1, 2, \dots, l).$$

But x and z are commutative and hence, by the above results, $Z^{(ij)} = 0$ if $i \neq j$ and thus also $Y^{(ij)} = 0$ if $i \neq j$. Hence y and z must be of the types

^{*} Cf. Bôcher, op. cit., p. 283.

$$y = \begin{vmatrix} Y^{(11)} & 0 & \dots & 0 \\ 0 & Y^{(22)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & Y^{(ll)} \end{vmatrix}, \quad z = \begin{vmatrix} Z^{(11)} & 0 & \dots & 0 \\ 0 & Z^{(22)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & Z^{(ll)} \end{vmatrix},$$

where the elements not in the diagonal blocks are necessarily zero. Thus the equations (9) are equivalent to the equations

$$X^{(i)}Y^{(ii)} - Y^{(ii)}X^{(i)} = Z^{(ii)}, \quad X^{(i)}Z^{(ii)} = Z^{(ii)}X^{(i)}, \quad Y^{(ii)}Z^{(ii)} = Z^{(ii)}Y^{(ii)} \\ (i = 1, 2, \dots, l).$$

If $l > 1$, the solution (x, y, z) of the equations (9) is therefore *reducible* and may be said to be the *direct sum* of the solutions $(X^{(i)}, Y^{(ii)}, Z^{(ii)})$ ($i = 1, 2, \dots, l$). We have thus shown that if (x, y, z) is a solution of the equations (9) and x (or y) has more than one distinct root, the solution is reducible.

Since $\phi_i \phi_j = \phi_j \phi_i = 0$ ($i \neq j$) and $y = \sum \phi_i y \phi_i$, we see that $\phi_i y = y \phi_i = Y^{(ii)}$. This proves the

THEOREM 1. *If x and y are quasi-commutative matrices, the principal idempotent elements of either matrix are commutative with the other.*

This theorem is of course obvious for commutative matrices as the principal elements of a matrix are polynomials in that matrix.

3. **Characterization of the matrix z .** Let (x, y, z) be a solution, in matrices of order n , of the equations (9). We shall assume, as we may, that z is in canonical form. If z has more than one distinct root, the solution (x, y, z) is reducible as x and y are commutative with z . Hence let z have the single root α . From the first of equations (9), we see that

$$\text{trace } z = 0 = n\alpha^*,$$

that is, $\alpha = 0$, and z is nilpotent.

Let the elementary divisors of z be

$$\lambda^{n_1}, \lambda^{n_2}, \dots, \lambda^{n_k} \quad \left(n_1 \geq n_2 \geq \dots \geq n_k; \sum_{i=1}^k n_i = n \right).$$

We shall find it convenient to use the symbol $[n_1, n_2, \dots, n_k]$, called the *characteristic* of z , to denote the degrees of these elementary divisors when arranged in the order indicated.† We shall now prove the following theorem:

* The trace of a square matrix is defined as the sum of the elements in the principal diagonal.

† Cf. Bôcher, op. cit., p. 287.

THEOREM 2. If z is a given matrix of order n , necessary and sufficient conditions that there exist matrices x and y of order n satisfying the equations

$$(9) \quad xy - yx = z, \quad xz = zx, \quad yz = zy$$

are that z be nilpotent and that it have a characteristic of the type $[n_1, n_2, \dots, n_k]$, where $n_k = 1$ and $n_i - n_{i+1} = 0$ or 1 ($i = 1, 2, \dots, k-1$).

We first establish the necessity of these conditions. It has already been shown that z must be nilpotent. Assume that z does not have a characteristic of the form stated in the theorem but that there exist matrices x and y satisfying the equations (9). We may write the characteristic of z in the more explicit form

$$[n_{11}, n_{12}, \dots, n_{1k_1}, n_{21}, n_{22}, \dots, n_{2k_2}, \dots, n_{l1}, n_{l2}, \dots, n_{lk_l}, \dots, n_{m1}, n_{m2}, \dots, n_{mk_m}]$$

where $n_{i1} = n_{i2} = \dots = n_{ik_i}$ ($i = 1, 2, \dots, m$); $n_{i1} - n_{i+1,1} = 1$ ($i = 1, 2, \dots, l-1$); but either $l = m$, $n_{m1} \geq 2$, or $n_{l1} - n_{l+1,1} > 1$. In either case $n_{11} \geq 2$.

We shall now assume that z is in canonical form. Let $f_{n_{ir}}$ be the partial idempotent element of z corresponding to the elementary divisor $\lambda^{n_{ir}}$, and set $f_{n_{ir}} x f_{n_{js}} = X^{n_{ir} n_{js}}$, $f_{n_{ir}} y f_{n_{js}} = Y^{n_{ir} n_{js}}$, $f_{n_{ir}} z f_{n_{js}} = Z^{n_{ir} n_{js}}$. Then $Z^{n_{ir} n_{js}} = 0$ unless $i = j$, $r = s$, and $Z^{n_{ir} n_{ir}}$ is the partial nilpotent element of z corresponding to the elementary divisor $\lambda^{n_{ir}}$.

Now $X^{n_{ir} n_{js}}$ is a rectangular matrix of n_{ir} rows and n_{js} columns, and since x is commutative with z , it will be of the general type (7) if $i \leq j$ and of the type (8) if $i \geq j$. We shall find it convenient to denote the element of $X^{n_{ir} n_{js}}$ corresponding to a_1 in (7) or (8) by $a_{n_{js}}^{n_{ir}}$, that corresponding to a_2 by $b_{n_{js}}^{n_{ir}}$. Similarly these elements in $Y^{n_{ir} n_{js}}$ will be denoted by $\alpha_{n_{js}}^{n_{ir}}$ and $\beta_{n_{js}}^{n_{ir}}$ respectively. We shall also let $X_{ab}^{n_{ir} n_{js}}$ denote the element in the a th row and b th column of $X^{n_{ir} n_{js}}$, and similarly for y and z .

Let $i \leq l$, $p \leq k_i$ be fixed positive integers. Then from the first of equations (9) we get the equation

$$(10) \quad \sum_{j=1}^m \sum_{\mu=1}^{k_j} [X^{n_{ip} n_{j\mu}} Y^{n_{j\mu} n_{ip}} - Y^{n_{ip} n_{j\mu}} X^{n_{j\mu} n_{ip}}] = Z^{n_{ip} n_{ip}},$$

for the part of z in the (n_{ip}, n_{ip}) block. But $n_{ip} \geq 2$ and hence $Z_{12}^{n_{ip} n_{ip}} = 1$. Thus from (10) we have

$$(11) \quad \sum_{j=1}^m \sum_{\mu=1}^{k_j} \sum_{\nu=1}^{n_{j\mu}} [X_{1\nu}^{n_{ip} n_{j\mu}} Y_{\nu 2}^{n_{j\mu} n_{ip}} - Y_{1\nu}^{n_{ip} n_{j\mu}} X_{\nu 2}^{n_{j\mu} n_{ip}}] = 1.$$

It may be verified from the form of x and y that $X_{\nu 2}^{n_{ip} n_{j\mu}} = Y_{\nu 2}^{n_{j\mu} n_{ip}} = 0$ if $\nu > 2$,

the theorem and exhibit a pair of matrices x and y satisfying the equations (9). There is no loss of generality in assuming that the elementary divisors of z are not all linear, as in this case any pair of commutative matrices will satisfy the equations. Let the characteristic of z be $[n_1, n_2, \dots, n_t, n_{t+1}, \dots, n_k]$, where $k \geq t+1$, $n_t = 2$, $n_{t+1} = \dots = n_k = 1$, $n_i - n_{i+1} = 0$ or 1 ($i = 1, 2, \dots, t-1$). If f_{n_i} is the partial idempotent element of z corresponding to the elementary divisor λ^{n_i} , we may again break any matrix A of order n into rectangular sub-matrices $A^{n_i n_j} = f_{n_i} A f_{n_j}$, and may call this the (n_i, n_j) block of A .

Let $e_{pq}^{n_i n_j}$ denote the matrix of order n for which the element in the p th row and q th column of the (n_i, n_j) block is 1 and all other elements are zero. Then

$$e_{pq}^{n_i n_j} e_{rs}^{n_l n_m} = \begin{cases} e_{ps}^{n_i n_m}, & \text{if } j = l, q = r, \\ 0, & \text{if } j \neq l \text{ or } q \neq r. \end{cases}$$

With this notation we have

$$z = \sum_{i=1}^t Z^{n_i n_i} = \sum_{i=1}^t \sum_{j=1}^{n_i-1} e_{i, i+1}^{n_i n_i}.$$

If now we set

$$A_i = \sum_{j=1}^{n_i-1} e_{i, i+n_{i+1}-n_i+1}^{n_i n_{i+1}} \quad (i = 1, 2, \dots, t),$$

$$B_i = \sum_{j=1}^{n_{i+1}} e_{i, i+n_i-n_{i+1}+1}^{n_{i+1} n_i} \quad (i = 1, 2, \dots, t),$$

it may be verified that each A_i and B_i is commutative with z and further that

$$(14) \quad \begin{aligned} A_i B_j &= \begin{cases} Z^{n_i n_i}, & i = j, \\ 0, & i \neq j, \end{cases} \\ B_i A_j &= \begin{cases} Z^{n_{i+1} n_{i+1}}, & i = j \neq t, \\ 0, & i \neq j \text{ or } i = j = t. \end{cases} \end{aligned}$$

Hence if we set

$$x = \sum_{j=1}^t j A_j, \quad y = \sum_{j=1}^t B_j,$$

then x and y are commutative with z and by the relations (14) we find

$$xy - yx = \sum_{j=1}^t j Z^{n_j n_j} - \sum_{j=1}^{t-1} j Z^{n_{j+1} n_{j+1}} = \sum_{j=1}^t Z^{n_j n_j} = z.$$

Thus we have exhibited a solution of the equations (9), and the proof of Theorem 2 is completed.

The following corollary follows readily.

COROLLARY. *There exist no quasi-commutative matrices of order two.*

For if $n=2$, the only choice of a characteristic for z which satisfies the conditions of Theorem 2 is $[1, 1]$, that is, $z=0$, and x and y are commutative.

4. Matrices quasi-commutative with a given matrix. Let x be an assigned matrix of order n . We shall in this section find necessary and sufficient conditions that there exist a matrix y quasi-commutative with x , and shall determine the general form of all such matrices for the special case in which x has a single elementary divisor corresponding to each root.

We first consider the case in which x has just one elementary divisor. There is no loss of generality in assuming that the root of x is zero,* hence if x is put in canonical form we have

$$(15) \quad x = \sum_{i=1}^{n-1} e_{i,i+1} \cdot \dagger$$

If $n=1$ or 2 , there exist no matrices quasi-commutative with x . Hence assume $n \geq 3$. Now if y and z are matrices such that (x, y, z) is a solution of equations (9), z is commutative with x and must also be nilpotent. Hence it must be of the form

$$(16) \quad z = a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1},$$

where the a_i are scalars.

If we write $y = \sum y_{ij} e_{ij}$, the equation $xy - yx = z$ becomes

$$\begin{aligned} \sum_i e_{i,i+1} \sum_{j,k} y_{jk} e_{jk} - \sum_{j,k} y_{jk} e_{jk} \sum_i e_{i,i+1} \\ = \sum_r y_{r+1,1} e_{r1} + \sum_{r,s} (y_{r+1,s+1} - y_{rs}) e_{rs,s+1} \\ = a_1 \sum_i e_{i,i+1} + a_2 \sum_i e_{i,i+2} + \cdots + a_{n-1} e_{1n}. \end{aligned}$$

From this it follows that $y_{21} = y_{31} = \cdots = y_{n1} = 0$. Thus if $i > j$, $y_{i+1,i+1} = y_{ij} = 0$. If $i \leq j$, $y_{i+1,i+1} = y_{ij} + a_{j-i+1}$, from which we see that

$$y_{ij} = y_{1,j-i+1} + (i-1)a_{j-i+1} \quad (i \leq j).$$

* For if x and y are quasi-commutative, so are $x - \lambda$ and y , where λ is any scalar matrix.

† The matrix unit e_{pq} is a matrix with 1 at the intersection of the p th row and q th column, and zeros elsewhere. The rule for multiplying these units is $e_{pq}e_{rs} = 0$ ($q \neq r$), $e_{pq}e_{qs} = e_{ps}$. It will be convenient to let $e_{pq} = 0$ if either p or q is greater than n or less than 1.

If then we set $y_{1j} = y_j$, y is of the form*

$$(17) \quad \begin{vmatrix} y_1 & y_2 & y_3 & y_4 & \cdots & y_n \\ 0 & y_1 + a_1 & y_2 + a_2 & y_3 + a_3 & \cdots & y_{n-1} + a_{n-1} \\ 0 & 0 & y_1 + 2a_1 & y_2 + 2a_2 & \cdots & y_{n-2} + 2a_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & y_1 + (n-1)a_1 \end{vmatrix}.$$

If x is the matrix (15), then y and z defined by (17) and (16), where $y_1, y_2, \dots, y_n, a_1, a_2, \dots, a_{n-1}$ are arbitrary parameters, are the most general matrices satisfying the first two of equations (9). We wish now to make necessary restrictions on y and z in order that they may be commutative. Let y given by (17) be expressed in the form $y = Y_1 + Y_2$, where $Y_1 = y_1 + y_2x + y_3x^2 + \cdots + y_nx^{n-1}$. Then y will be commutative with z if and only if Y_2 and z are commutative. Since the elements of Y_2 depend only upon the parameters a_i , the only restriction will be on these parameters, that is, on the matrix z .

In (16) suppose $a_k \neq 0$ ($1 \leq k \leq n-1$) but $a_i = 0$ ($i < k$). If positive integers p, q are defined by

$$n = pk + q, \quad 0 \leq q < k,$$

then it is known† that z has q elementary divisors of degree $p+1$ and $k-q$ of degree p . Hence by Theorem 2, there can exist matrices x and y satisfying equations (9) only if $p=1$, that is, $k \geq (n+1)/2$. Let k be the smallest integer satisfying this inequality. Then the elements of z are all zero except in the block of $n-k$ rows and $n-k$ columns in the upper right hand corner. Similarly Y_2 has non-zero elements only in the block of $n-k+1$ rows and $n-k$ columns in the upper right hand corner. Hence $zY_2 = Y_2z = 0$, and we have thus established the following theorem:

THEOREM 3. *If x is the matrix (15) of order $n \geq 3$ and k is the smallest integer not less than $(n+1)/2$, then the general form of a matrix y quasi-commutative with x is given by (17) where $a_i = 0$ ($i=1, 2, \dots, k-1$), and $a_k, a_{k+1}, \dots, a_{n-1}, y_1, y_2, \dots, y_n$ are arbitrary parameters.*

* Cf. R. Weitzenböck, *Über die Matrixgleichung $Ax + xB = C$* , Akademie van Wetenschappen te Amsterdam, Proceedings, vol. 35 (1932), pp. 54-59. The form (17) obtained for y is a special case of a formula given by Weitzenböck.

† H. Kreis, op. cit., p. 47. See also D. E. Rutherford, *On the canonical form of a rational integral function of a matrix*, Proceedings of the Edinburgh Mathematical Society, (2), vol. 3 (1932), pp. 135-143.

This theorem is sufficient to give the form of y even if x has more than one root, provided it has only one elementary divisor corresponding to each root. For by the results of §2, the general form of such a y is the direct sum of matrices of the type prescribed by this theorem. If x is not of this simple type it seems to be difficult to give the general form of a matrix quasi-commutative with x and we shall now limit ourselves to a consideration of the conditions under which such matrices exist.

Let x be a matrix of order n and let us separate the elementary divisors of x into two sets A_1 and A_2 . By a proper choice of basis x may then be expressed as the direct sum of two matrices, thus

$$x = \begin{vmatrix} X_1 & 0 \\ 0 & X_2 \end{vmatrix}$$

where the matrix X_1 has the elementary divisors A_1 and X_2 the elementary divisors A_2 . If now Y_1 is a matrix quasi-commutative with X_1 , then

$$y = \begin{vmatrix} Y_1 & 0 \\ 0 & 0 \end{vmatrix}$$

will be quasi-commutative with x . Hence there will exist a matrix quasi-commutative with a given matrix x if there exists a matrix quasi-commutative with a matrix the set of whose elementary divisors is a subset of the elementary divisors of x . We may now prove the following lemma:

LEMMA. *Let x be a matrix of order n with a single root λ_1 . A necessary and sufficient condition that there exist a matrix y quasi-commutative with x is that n be greater than two and the elementary divisors of x be not all linear.*

The necessity of these conditions follows from the Corollary to Theorem 2 and from the fact that if the elementary divisors of x are all linear, x is a scalar matrix and is thus commutative with all matrices of order n .

We now prove that these conditions are sufficient. If $n > 2$ and the elementary divisors of x are not all linear then (i) x has an elementary divisor of degree ≥ 3 , or (ii) x has at least two elementary divisors of degree 2, or (iii) x has one elementary divisor of degree 2 and at least one of degree 1. In the first case, Theorem 3 establishes the existence of a matrix y quasi-commutative with x . The existence of such a matrix y in the cases (ii) and (iii) is shown by the two examples

$$x = \lambda_1 + e_{12} + e_{34}, \quad y = e_{24},$$

and

$$x = \lambda_1 + e_{12}, \quad y = e_{23},$$

respectively.

The following theorem follows readily.

THEOREM 4. *A necessary and sufficient condition that there exist a matrix y quasi-commutative with a given matrix x is that for some root λ_i of x the sum of the degrees of the elementary divisors of x associated with λ_i be greater than two, and at least one of these elementary divisors be not linear.*

5. Roots of a polynomial in quasi-commutative matrices. In this section we shall prove the following theorem which is well known for the case of commutative matrices.*

THEOREM 5. *If x and y are quasi-commutative matrices with principal idempotent elements R_i and S_j ($i, j = 1, 2, \dots$) and corresponding roots λ_i and μ_j respectively, then the roots of any scalar polynomial $\psi(x, y)$ in x and y are $\psi(\lambda_i, \mu_j)$, where i and j take only those values for which $R_i S_j \neq 0$.†*

We first prove two lemmas.

LEMMA 1. *If $\psi(x, y)$ is any scalar polynomial in the quasi-commutative matrices x and y , and $z = xy - yx$, then*

$$\psi(x, y) = \psi_1(x, y) + z\psi_2(x, y, z),$$

where ψ_1 is of the form $\sum a_{ij} x^i y^j$, the a_{ij} being scalars.

It is clear that by repeated substitutions of the type $yx = xy - z$, $\psi(x, y)$ can be reduced to the form $\sum a_{ijk} z^i x^j y^k$. Hence we only need to set $\psi_1 = \sum_{j,k} a_{0jk} x^j y^k$, $z\psi_2 = \psi - \psi_1$.

LEMMA 2. *If x and y are quasi-commutative matrices and $z = xy - yx$, then*

$$x^{m_1} y^{n_1} x^{m_2} y^{n_2} \dots x^{m_k} y^{n_k} = \sum_{l=0}^P a_l z^l x^{P-l} y^{Q-l},$$

where the m_i and n_i are any positive integers and $P = \sum_{i=1}^k m_i$, $Q = \sum_{i=1}^k n_i$.

The sum of the exponents of x which appear explicitly in a given term may be called the *degree* of the term in x , and similarly for y . For example, the term $z^4 x^2 y^4 x^2 y$ is of degree 4 in x and 5 in y . If we think of z as being replaced by $xy - yx$, then the total degree in x and y would be 17. For convenience, let us call this the *weight* of the term.

As above, we may express $x^{m_1} y^{n_1} x^{m_2} y^{n_2} \dots x^{m_k} y^{n_k}$ in the form

$$(18) \quad \sum b_{ijk} z^i x^j y^k,$$

* Frobenius, loc. cit. The method of this section is a modification of that used by Wedderburn, loc. cit., p. 127.

† It is not necessary that x and y be quasi-commutative in order that the roots of $\psi(x, y)$ shall be of the form $\psi(\lambda_i, \mu_j)$. Cf. G. S. Bruton, *Certain aspects of the theory of equations for a pair of matrices*, and M. H. Ingraham, *A study of certain related pairs of square matrices*. Abstracts of these papers appear in the Bulletin of the American Mathematical Society, vol. 38 (1932), p. 633.

by repeated substitutions of $xy - z$ for yx . Each time this substitution is made in a term we get two terms; in one the degree in x and in y is the same as before the substitution, in the other the degree of each has been reduced by one. The weight is invariant under a substitution of this form. Hence each term of (18) is of the type $b_{ijk}z^i x^j y^k$, where $j = P - l$, $k = Q - l$, $2i + j + k = P + Q$, that is, $i = l$.

We may now proceed with the proof of the theorem. The principal elements R_i and S_i are polynomials in x and y respectively, and by Theorem 1 are commutative with each other and with x , y and $z = xy - yx$. If we set $T_{ij} = R_i S_j$, then $T_{ij} T_{pq} = 0$ if $i \neq p$ or $j \neq q$, $T_{ij}^2 = T_{ij}$, $\sum T_{ij} = 1$. Further those T_{ij} which are not zero are linearly independent, for if $\sum a_{ij} T_{ij} = 0$, then $R_p \sum a_{ij} T_{ij} R_q = a_{pq} T_{pq} = 0$ and thus $T_{pq} = 0$ unless $a_{pq} = 0$.

Let us now write

$$x = \sum_{i,j} [\lambda_i - (x - \lambda_i)] T_{ij}, \quad y = \sum_{i,j} [\mu_j - (y - \mu_j)] T_{ij}.$$

The matrices $(x - \lambda_i) T_{ij}$ and $(y - \mu_j) T_{ij}$ are then nilpotent. If $\psi(x, y)$ is a polynomial in x and y , we have, by the first lemma,

$$\psi(x, y) = \psi_1(x, y) + z \psi_2(x, y, z),$$

where $\psi_1(x, y) = \sum a_{ij} x^i y^j$. Since no interchange of order is necessary, we may write, as in the commutative case,

$$\psi_1(x, y) = \sum_{r,s} \psi_{rs}^{ij} (x - \lambda_i)^r (y - \mu_j)^s,$$

where the ψ_{rs}^{ij} are scalar constants. Thus

$$\psi(x, y) = \sum_{i,j} \left[\psi_1(\lambda_i, \mu_j) T_{ij} + \sum_{r,s} \psi_{rs}^{ij} (x - \lambda_i)^r (y - \mu_j)^s T_{ij} \right] + z \psi_2,$$

with the understanding that r and s are not to be zero simultaneously.

Let us set

$$A = \sum_{i,j} \psi_1(\lambda_i, \mu_j) T_{ij}, \quad B = \sum_{i,j} \sum_{r,s} [\psi_{rs}^{ij} (x - \lambda_i)^r (y - \mu_j)^s T_{ij}] + z \psi_2(x, y, z).$$

Then $\psi(x, y) = A + B$, and A and B are commutative. It will be shown below that B is nilpotent. Hence the roots of $\psi(x, y)$ are the roots of A and these are of the form $\psi_1(\lambda_i, \mu_j)$ where $T_{ij} \neq 0$.^{*} But $\psi_1(\lambda_i, \mu_j) = \psi(\lambda_i, \mu_j)$ and the theorem is established. We shall now complete this proof by showing that B is nilpotent.

^{*} Wedderburn, loc. cit.

Let

$$B_1 = \sum_{i,j} \sum_{r,s} \psi_{rs}^{ij} (x - \lambda_i)^r (y - \mu_j)^s T_{ij} = \sum_{i,j} A_{ij} T_{ij}.$$

Since T_{ij} is commutative with each A_{ij} , we see that

$$B_1^k = \sum_{i,j} A_{ij}^k T_{ij}.$$

We can thus show that B_1 is nilpotent by showing that each $A_{ij} T_{ij}$ is nilpotent. Let $x_1 = x - \lambda_i$, $y_1 = y - \mu_j$; then x_1 and y_1 are quasi-commutative and $x_1 y_1 - y_1 x_1 = z$. Let ρ_1 be the index of the nilpotent matrix $x_1 R_i$, ρ_2 that of $y_1 S_j$, and t that of z . Let $N = t + \max(\rho_1, \rho_2)$ and consider

$$(A_{ij} T_{ij})^{2N} = \left[\sum_{r,s} \psi_{rs}^{ij} x_1^r y_1^s \right]^{2N} T_{ij}.$$

The right hand side will, when expanded, consist of a sum of terms of the general type $a x_1^{m_1} y_1^{n_1} x_1^{m_2} y_1^{n_2} \cdots x_1^{m_k} y_1^{n_k} T_{ij}$. But by Lemma 2, this may be put in the form

$$\sum_l a_l z^l x_1^{P-l} y_1^{Q-l} T_{ij} = \sum_l a_l z^l (x_1 R_i)^{P-l} (y_1 S_j)^{Q-l} T_{ij}.$$

Here $P = \sum m_i$, $Q = \sum n_i$, and as r and s are not both zero, either $P \geq N$ or $Q \geq N$. Now the term in this last sum of which a_l is the coefficient is zero provided $l \geq t$ or $P-l \geq \rho_1$ or $Q-l \geq \rho_2$. Suppose, for example, that $P \geq N$. Then the term containing a_l is zero provided $l \geq t$ or $l \leq P - \rho_1$. But $P - \rho_1 \geq N - \rho_1 \geq t$, and thus all l are included. Hence $(A_{ij} T_{ij})^{2N} = 0$ and B_1 is nilpotent of index, say, r . It follows that

$$B^r = [(B_1 + z\psi_2)^r]^t = [B_1^r + z(\quad)]^t = 0.$$

Hence B is nilpotent and the proof of the theorem is completed.

It may be noted that the fact that $z = xy - yx$ is nilpotent is a special case of this theorem. For if we choose $\psi(x, y) = xy - yx$, the roots of ψ must be of the form $\lambda_i \mu_j - \mu_j \lambda_i = 0$.

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ON THE DISTRIBUTIONS OF THE ZEROS OF SUMS OF EXPONENTIALS OF POLYNOMIALS*

BY

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1. INTRODUCTION

Certain work by other investigators suggests the problem of determining the distribution of the zeros of a function of the form

$$(1) \quad f(z) = \sum_{j=0}^J \exp(\lambda_{jN} z^N + \cdots + \lambda_{j1} z + \lambda_{j0}),$$

where J and N are positive integers, and the λ 's are real or complex constants. The present paper gives the chief results of a study of this problem. In order to add precision to the problem, and in order to exclude certain extreme cases which are of minor interest, it is assumed (1) that we do not have $\lambda_{0N} = \lambda_{1N} = \cdots = \lambda_{JN}$; (2) that we do not have $\lambda_{0n} = \lambda_{1n} = \cdots = \lambda_{Jn} \neq 0$ for any $n < N$; (3) that for no two distinct values, j' and j'' , of j do we have all of the N relations $\lambda_{j'n} = \lambda_{j''n}$, $n = 1, 2, \cdots, N$. For the sake of brevity, a function of the form (1) which satisfies these conditions will be called an *E-function*. The integer N will be called the *exponent* of the *E-function*.

The problem discussed here is essentially a generalization, in one direction, of a problem that has already been the subject of numerous studies. C. E. Wilder, Tamarkin, Pólya, Schwengeler, and others† have studied the distributions of the zeros of functions of the form

$$f(z) = \sum_{j=0}^J A_j(z) \exp(\lambda_j z),$$

where J is a positive integer, the λ 's are constants, and the $A_j(z)$'s are analytic functions which behave essentially as powers of z when $|z|$ is large. Our generalization consists of replacing the linear exponents $\lambda_j z$ by the general polynomials $\lambda_{jN} z^N + \cdots + \lambda_{j0}$. At the present time we shall not consider the still more general case in which we have non-constant coefficients $A_j(z)$ of the type described above; for the theory is complicated at best, and the chief

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† The previous work is conveniently summarized in the following expository paper: Langer, *On the zeros of exponential sums and integrals*, Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 213-239. As this paper contains a rather full bibliography, it is unnecessary to give one here.

of the new phenomena, due to the generalization of the exponents, appear even when the coefficients are constants.* Broadly speaking, we find that the results for the general value of N are similar to, but more complicated than, the known results for the special case in which $N=1$. The methods used here do not differ fundamentally from those that have been used in the earlier studies; but, naturally, the analysis is considerably more intricate, both formally and otherwise.

2. NORMAL FORM OF $f(z)$

We have written $f(z)$ in the form (1) in order to display the structure of the function in the clearest possible manner. In order to be able to work with the function effectively, however, we shall rewrite it in a certain normal form.

The numbers $\lambda_{0N}, \dots, \lambda_{jN}$ may be all distinct or not. In either case, by properly collecting terms if these λ 's are not all distinct, we can write $f(z)$ in the form

$$(2) \quad f(z) = \sum_{m=0}^M f_m(z) \exp [g_m(z) + \mu_m z^N],$$

where M is a positive integer, the numbers μ_0, \dots, μ_M are distinct, $g_m(z)$ is a polynomial which, if it is not zero, is of degree less than N , and $f_m(z)$ is either the constant 1 or an E -function having an exponent $N_m < N$.

If any one of the functions $f_m(z)$ is an E -function, we arrange it as we have just arranged $f(z)$. Thus we write

$$(3) \quad f_m(z) = \sum_{n=0}^{M_m} f_{mn}(z) \exp [g_{mn}(z) + \mu_{mn} z^{N_m}],$$

with stipulations similar to the above.

Likewise, if any one of the functions $f_{mn}(z)$ is an E -function, we write it in the form

$$f_{mn}(z) = \sum_{p=0}^{M_{mn}} f_{mnp}(z) \exp [g_{mnp}(z) + \mu_{mnp} z^{N_{mn}}],$$

with similar stipulations.

We continue this process of arranging $f(z)$ until it automatically terminates after a finite number of steps. It will be observed that the resulting normal form of $f(z)$ is not unique; for a given sum of exponentials of polynomials, which is not a constant and not the exponential of a polynomial, can be separated in various ways into two factors, one of which is an E -function and the other of which is the exponential of a polynomial. For example, if we have $f_m(z) \exp g_m(z) = \exp z^2 + \exp (z^2 + z)$, we can set

* Constant coefficients are effectively provided for by the constants λ_{jk} .

$$f_m(z) = e^{-\lambda z} + e^{(1-\lambda)z}, \quad g_m(z) = z^2 + \lambda z,$$

where λ is any constant. However, for our purposes the normal form of $f(z)$ is effectively unique.

3. CRITICAL POLYGONS AND CRITICAL RAYS

The distribution of the zeros of $f(z)$ is closely related to certain geometrical figures which will now be defined.

Let the points μ_0, \dots, μ_M be plotted in the complex plane. These points are distinct, and there are at least two of them. Let the smallest convex polygon that contains these points in its interior or on its boundary be drawn. If the points μ_0, \dots, μ_M are collinear, the polygon is to be regarded, in an obvious way, as having just two sides, which are coincident and face in opposite directions. We call this polygon the *primary critical polygon* for $f(z)$, and we denote it by the symbol P .

For the present, we assign subscripts so that μ_0, \dots, μ_M are the vertices of P in counter-clockwise order, an arbitrarily chosen vertex being called μ_0 . If $M' < M$, the points $\mu_{M'+1}, \dots, \mu_M$ are in the interior of P or on the sides of P between the vertices. Let the side of P that follows the typical vertex μ_α in counter-clockwise order be denoted by L_α . From any point on L_α draw a normal exterior to P . Let ϕ_α ($0 \leq \phi_\alpha < 2\pi$) be the angle between the positive real axis and this outward-drawn normal. Let the vertex denoted by μ_0 be selected so that the ϕ 's increase with their subscripts. From the origin we now draw the $(M'+1)N$ rays $R_\alpha^{(\beta)}$ represented by the equations

$$(4) \quad R_\alpha^{(\beta)} : \text{amp } z = -(\phi_\alpha + 2\beta\pi)/N; \alpha = 0, \dots, M'; \beta = 0, \dots, N-1.$$

We call these rays the *primary critical rays* for $f(z)$. It is to be noted that if the value of β is fixed, the $M'+1$ rays corresponding to the several values of α , in increasing order, succeed one another in clockwise order and are contained in a sector of angular opening $2\pi/N$, and that the N sectors of this kind, corresponding to the several values of β in increasing order, succeed one another in clockwise order without overlapping.

The primary critical rays divide the plane into $(M'+1)N$ sectors, each of which has its vertex at the origin, is bounded by two of the rays, and has none of the rays in its interior. We denote the one of these sectors that is bounded on the clockwise side by the ray $R_\alpha^{(\beta)}$ by the symbol $S_\alpha^{(\beta)}$. Each of these sectors is understood to be an open point set.

As just above, let α be an arbitrarily chosen one of the integers $0, \dots, M'$. If the function $f_\alpha(z)$ is not a constant, we construct its primary

critical polygon P_α , its primary critical rays*

$$R_{\alpha\beta}^{(\gamma)}: \beta = 0, \dots, M'_\alpha; \gamma = 0, \dots, N_\alpha - 1,$$

and the associated sectors $S_{\alpha\beta}^{(\gamma)}$.

Let α be one of the integers $0, \dots, M'$, and let β be one of the integers $0, \dots, M'_\alpha$ (assuming that $f_\alpha(z)$ is not a constant). If the function $f_{\alpha\beta}(z)$ is not a constant, we construct its primary critical polygon $P_{\alpha\beta}$, its primary critical rays $R_{\alpha\beta\gamma}^{(\delta)}$, and the associated sectors $S_{\alpha\beta\gamma}^{(\delta)}$.

We continue this process of constructing polygons, rays and sectors until it automatically terminates, as it must, after a finite number of steps. It is to be noted that if $m > M'$, we do not construct the figures for $f_m(z)$; and that if $0 \leq \alpha \leq M'$, $n > M'_\alpha$, we do not construct the figures for $f_{\alpha n}(z)$. A similar remark applies also to the further cases.

We have defined the primary critical rays for $f(z)$; now we proceed to define critical rays of "higher order" for $f(z)$. A *secondary critical ray* for $f(z)$ is a primary critical ray for a function $f_\alpha(z)$, $\alpha = 0, \dots, M'$, which lies in one of the sectors $S_\alpha^{(0)}, \dots, S_\alpha^{(N-1)}$. A *tertiary critical ray* for $f(z)$ is a secondary critical ray for a function $f_\alpha(z)$, $\alpha = 0, \dots, M'$, which lies in one of the sectors $S_\alpha^{(0)}, \dots, S_\alpha^{(N-1)}$. Critical rays of other orders are defined similarly. An essential feature of these definitions is the fact that we have the same subscript in the symbols $f_\alpha(z)$, $S_\alpha^{(0)}, \dots, S_\alpha^{(N-1)}$. It is understood that the definitions of the critical rays of higher orders of the functions $f_\alpha(z)$ are precisely analogous to the definitions of the critical rays of higher order of $f(z)$; hence the definitions of the latter rays are complete. In this way we get a finite set of critical rays for $f(z)$, and these critical rays are classified as primary, secondary, tertiary, etc. At the very least there are two primary critical rays. Whether or not there are any critical rays of higher order depends on the structure of the particular function under consideration.

4. ZERO-FREE REGIONS

The first result concerning the distribution of the zeros of $f(z)$ is stated in the following:

THEOREM 1. *There exists a set of half-strips†, equal in number to the critical rays of $f(z)$, each extending in the direction of a different one of these critical rays, such that each zero of $f(z)$ is a point of one or more of these half-strips.*

* M'_α has the same significance in regard to $f_\alpha(z)$ that the previously defined symbol M' has in regard to $f(z)$.

† By a half-strip we shall always mean the open set of points between two parallel straight lines and on one side of a line perpendicular to these.

The theorem is an immediate consequence of the following

LEMMA. *There exists a set of half-strips, equal in number to the critical rays of $f(z)$, each extending in the direction of a different one of these critical rays, and there exist two positive constants, A and B , such that if $|z| \geq A$, and if z is not in any one of the half-strips, we have the relation*

$$(5) \quad |f(z)| \geq \exp [-B|z|^N].$$

The lemma will be proved by a process of induction. The reasoning in the following §4.1 proves the lemma directly for the case in which $N=1$. It will be shown that if $N>1$, the truth of the lemma is a consequence of the assumed truths of the corresponding lemmas concerning E -functions having exponents less than N .

The reasoning just referred to consists of two main steps. In the first place, we shall show that if we enclose each primary critical ray of $f(z)$ in a sector, with vertex at the origin and of arbitrarily small angular opening, and if we construct certain half-strips each extending in the direction of a different non-primary critical ray, we have a relation of the form (5), provided $|z|$ is large and provided z is not in any one of these sectors or half-strips. In the second place, we shall show that a relation such as (5) obtains at each sufficiently distant point of a small sector enclosing a primary critical ray, provided the point does not lie in a certain half-strip extending in the direction of the ray.

As most of the analysis used in the proof possesses no unusual features, many of the details will be left to the reader.

4.1. PROOF OF THE LEMMA

Let α be an arbitrarily chosen one of the integers $0, \dots, M'$, and let β be an arbitrarily chosen one of the integers $0, \dots, N-1$. Let ϵ be a positive number such that 2ϵ is less than the angular opening of the sector $S_{\alpha}^{(\beta)}$. Consider the sector $\Sigma_{\alpha}^{(\beta)}$ that is defined by the appropriate one of the following relations:

$$\begin{aligned} &\alpha = 1, \dots, M'; \beta = 0, \dots, N-1: \\ &\quad \Sigma_{\alpha}^{(\beta)}: -(\phi_{\alpha} + 2\beta\pi - N\epsilon)/N \leq \arg z \leq -(\phi_{\alpha-1} + 2\beta\pi + N\epsilon)/N; \\ (6) \quad &\alpha = 0; \beta = 1, \dots, N-1: \\ &\quad \Sigma_{\alpha}^{(\beta)}: -(\phi_0 + 2\beta\pi - N\epsilon)/N \leq \arg z \leq -(\phi_{M'} + 2\beta\pi - 2\pi + N\epsilon)/N; \\ &\alpha = \beta = 0: \\ &\quad \Sigma_{\alpha}^{(\beta)}: -2\pi + \epsilon - \phi_0/N \leq \arg z \leq -2\pi - \epsilon - (\phi_{M'} - 2\pi)/N. \end{aligned}$$

Each point of $\Sigma_{\alpha}^{(\beta)}$, except the origin, is a point of $S_{\alpha}^{(\beta)}$.

If the sector $S_\alpha^{(\beta)}$ contains no critical ray of $f(z)$, by the hypothesis for the induction we have a relation of the following form, provided z is in $\Sigma_\alpha^{(\beta)}$ and $|z|$ is sufficiently large:

$$(7) \quad |f_\alpha(z) \exp g_\alpha(z)| \geq \exp [-B|z|^{N-1}], \quad B \text{ a positive constant.}$$

If $S_\alpha^{(\beta)}$ contains one or more critical rays of $f(z)$, we take ϵ so small that $\Sigma_\alpha^{(\beta)}$ contains all of these rays in its interior. Then to each of these rays there corresponds a half-strip, extending in the direction of the ray, such that if z is in $\Sigma_\alpha^{(\beta)}$, but is not in any one of these half-strips, and if $|z|$ is sufficiently large, we have a relation of the form (7). We denote the set of points in $\Sigma_\alpha^{(\beta)}$, and not in any of these half-strips, by the symbol $T_\alpha^{(\beta)}$. If $S_\alpha^{(\beta)}$ contains no critical rays of $f(z)$, we use $T_\alpha^{(\beta)}$ as an alternative symbol for the sector $\Sigma_\alpha^{(\beta)}$.

We write $f(z)$ in the form

$$f(z) = \{\exp(\mu_\alpha z^N)\} \left\{ f_\alpha(z) \exp g_\alpha(z) + \sum_{m=0}^M{}' f_m(z) \exp [g_m(z) + (\mu_m - \mu_\alpha)z^N] \right\},$$

where the prime on the summation sign indicates that the term for $m=\alpha$ is to be omitted. It is easy to show, from the geometry of the polygon P , that in the sufficiently distant part of $\Sigma_\alpha^{(\beta)}$ we have

$$\left| \sum_{m=0}^M{}' f_m(z) \exp [g_m(z) + (\mu_m - \mu_\alpha)z^N] \right| \leq \exp [-B|z|^N],$$

where B is a positive constant. It follows from the last relation, and from (7), that in the sufficiently distant part of $T_\alpha^{(\beta)}$ we have a relation of the form (5). This completes the first of the two main parts of the proof of the lemma.

In the second part of the proof it is convenient to employ a new assignment of the subscripts. Choose an arbitrary side of the polygon P , and denote it by the symbol L . We assign subscripts so that $\mu_0, \dots, \mu_{M''}$ are the μ 's that lie on L , μ_0 and $\mu_{M''}$ being the clockwise and counter-clockwise extremities of L , respectively.

We write the function $f(z)$ in the form

$$(8) \quad f(z) = [h(z) + k(z)] \exp [\tfrac{1}{2}(\mu_0 + \mu_{M''})z^N],$$

where*

* Of course, it may happen that $M''=M$. In this event the function $k(z)$ does not appear, and the next few steps in the proof are simply to be omitted.

$$(9) \quad \begin{aligned} h(z) &= \sum_{m=0}^{M''} f_m(z) \exp \left[g_m(z) + \left(\mu_m - \frac{\mu_0 + \mu_{M''}}{2} \right) z^N \right], \\ k(z) &= \sum_{m=M''+1}^M f_m(z) \exp \left[g_m(z) + \left(\mu_m - \frac{\mu_0 + \mu_{M''}}{2} \right) z^N \right]. \end{aligned}$$

From any point on L draw a normal exterior to P , and let ϕ ($0 \leq \phi < 2\pi$) denote the angle between the positive real axis and this outward-drawn normal. It is clear that if $m > M''$, and if $\text{amp} \left[\mu_m - \frac{1}{2}(\mu_0 + \mu_{M''}) \right]$ is suitably defined,

$$\phi + \frac{\pi}{2} + \sigma \leq \text{amp} \left[\mu_m - \frac{1}{2}(\mu_0 + \mu_{M''}) \right] \leq \phi + \frac{3\pi}{2} - \sigma,$$

where σ is a certain number that satisfies the relations $0 < \sigma \leq \pi/2$.

Let ϵ be a positive number less than σ . Consider any one of the sectors U_β that are defined by the relations

$$\begin{aligned} U_\beta : (-\phi - \sigma - 2\beta\pi + \epsilon)/N \leq \text{amp } z \leq (-\phi + \sigma - 2\beta\pi - \epsilon)/N, \\ \beta = 0, \dots, N-1. \end{aligned}$$

It is easy to show that if z is in U_β , and if $|z|$ is sufficiently large, we have a relation of the form

$$|k(z)| \leq \exp [-B|z|^N], \quad B \text{ a positive constant.}$$

We now write the function $h(z)$ in the form

$$\begin{aligned} h(z) &= f_0(z) \exp [g_0(z) + \frac{1}{2}(\mu_0 - \mu_{M''})z^N] \\ &\quad \cdot \left\{ 1 + \sum_{m=1}^{M''} \frac{f_m(z)}{f_0(z)} \exp [g_m(z) - g_0(z) + (\mu_m - \mu_0)z^N] \right\}; \end{aligned}$$

and, in order to simplify the formal side of the exposition slightly, we assume, for the moment, that the primary critical ray

$$\text{amp } z = -(\phi + 2\beta\pi)/N$$

is not a critical ray of the function $f_0(z)$.

By the hypothesis for the induction, if the angular opening of the sector U_β is taken sufficiently small, if z is in U_β , and if $|z|$ is sufficiently large, we have a relation of the form

$$|f_0(z)| \geq \exp [-B_0|z|^{N_0}], \quad B_0 \text{ a positive constant.}$$

There exist positive numbers, C_m and D_m , such that, for all values of z ,

$$|f_m(z) \exp [g_m(z) - g_0(z)]| \leq \exp [C_m|z|^{N-1} + D_m].$$

It follows, therefore, that if z is in U_β , and $|z|$ is sufficiently large, we have the following relation,* for $m=1, \dots, M''$:

$$\left| \frac{f_m(z)}{f_0(z)} \exp [g_m(z) - g_0(z) + (\mu_m - \mu_0)z^N] \right| \leq \exp \{ B_0 |z|^{N_0} + C_m |z|^{N-1} + D_m + \Re[(\mu_m - \mu_0)z^N] \}.$$

Let us write

$$z = re^{i\theta}, \quad \mu_m - \mu_0 = R_m e^{i(\phi + \pi/2)}, \quad m = 1, \dots, M'',$$

where r is real and non-negative, and R_m is positive. Let q be a real number such that $M''e^q < 1$. We now write the M'' relations

$$(10) \quad R_m r^N \cos(N\theta + \phi + \pi/2) + B_0 r^{N_0} + C_m r^{N-1} + D_m \leq q,$$

and proceed to investigate the several regions defined by them.

The boundary of the region defined by the typical relation (10) is represented by the equation

$$(11) \quad N\theta + \phi = \arcsin \left\{ \frac{1}{R_m} [B_0 r^{N_0-N} + C_m r^{-1} + (D_m - q)r^{-N}] \right\}.$$

For our purposes it is convenient to write this last relation, for large values of r , in the form

$$(12) \quad r[\theta - (-\phi - j\pi)/N] = c_0 + \frac{c_1}{r} + \frac{c_2}{r^2} + \dots,$$

where j is any integer, and the c 's are constants. Equation (12) represents a curve which is asymptotic to a line parallel to the ray $\theta = -(\phi + j\pi)/N$, or to the ray itself.

Giving j successively the values $0, \dots, 2N-1$, we get, from (12), the representations of the $2N$ branches of the curve represented by (11). The branch corresponding to the even value 2β of j is asymptotic to a parallel to the bisector of U_β , or to the bisector itself. We are not directly concerned with the branches that correspond to odd values of j .

In the sufficiently distant part of the sector U_β there are M'' branches of curves of the type just described, corresponding to the several values $1, \dots, M''$ of m .

As $R_1, \dots, R_{M''}$ are all positive, the region (10) lies on the counter-clockwise sides of the branches of the boundary that correspond to even values of j . The region lies on the clockwise sides of the branches that correspond to odd values of j .

* If u and v are real, and $w = u + iv$, we write $u = \Re w$, $v = \Im w$.

It follows from the above that we can draw a line parallel to the bisector of U_β , such that in the distant part of the sub-sector bounded by this line and the counter-clockwise side of U_β the function

$$1 + \sum_{m=1}^{M'''} \frac{f_m(z)}{f_0(z)} \exp [g_m(z) - g_0(z) + (\mu_m - \mu_0)z^N]$$

is bounded away from zero.*

Now consider the function

$$(13) \quad f_0(z) \exp [g_0(z) + \frac{1}{2}(\mu_0 - \mu_{M'''})z^N].$$

If z is in U_β , and if $|z|$ is sufficiently large, the modulus of this function is not less than

$$\exp \{ -B'_0 r^{N-1} + \Re[\frac{1}{2}(\mu_0 - \mu_{M'''})z^N] \},$$

where B'_0 is a suitably chosen positive constant. By reasoning similar to that used just above, we show that in U_β there is a sub-sector,* bounded by a line parallel to the bisector of U_β and by the counter-clockwise side of U_β , in the distant part of which the function (13) is bounded away from zero.

It follows that in U_β there is a sub-sector,* bounded by a line parallel to the bisector of U_β and by the counter-clockwise side of U_β , in the distant part of which the function $h(z)$ is bounded away from zero.

Assuming provisionally that the ray $\arg z = -(\phi + 2\beta\pi)/N$ is not a critical ray of $f_{M'''}(z)$, we show, in an entirely similar way, that in U_β there is a sub-sector,* bounded by a line parallel to the bisector of U_β and by the clockwise side of U_β , in the distant part of which $h(z)$ is bounded away from zero.

It is easy to show that we have the results just stated even when the ray $\arg z = -(\phi + 2\beta\pi)/N$ is a critical ray of one or both of the functions $f_0(z)$, $f_{M'''}(z)$. By the hypothesis for the induction, the sector then contains a half-strip, extending in the direction of the bisector of U_β , such that in the distant parts of the regions within the sector and outside the half-strip, we have relations of the form

$$|f_0(z)| \geq \exp [-B|z|^{N_0}], \quad |f_{M'''}(z)| \geq \exp [-B|z|^{N_{M'''}}],$$

where B is a positive constant. Now we have only to confine our attention to values of z which lie in one or the other of the regions just described. With this understanding about the values of z under discussion, the preceding reasoning is valid without any essential change, and we are led again to the result stated in the preceding paragraph.

It follows from (8), and what has been proved concerning the functions

* The sub-sector is taken as including the points on its boundary.

$h(z)$ and $k(z)$, that in U_β there are two sub-sectors,* each bounded by a line parallel to the bisector of U_β and by a different one of the sides of the sector, such that in the sufficiently distant parts of these sub-sectors we have a relation of the form (5).

This completes the proof of the lemma.

5. DISTRIBUTION OF THE ZEROS WITHIN A CRITICAL HALF-STRIP

We shall call the half-strips determined in the preceding paragraphs, within which the zeros of $f(z)$ must lie, *critical half-strips*. It is natural to classify the several critical half-strips as primary, secondary, tertiary, etc., according to the classification of the several critical rays to which they correspond.

It has not yet been shown that $f(z)$ has any zeros at all. We shall now prove that zeros do actually exist, and we shall obtain asymptotic formulas giving the distributions of the zeros in the various critical half-strips. Specifically, the remainder of the paper will be devoted to proving the following two theorems.

THEOREM 2. *The number of zeros of $f(z)$ in the interior of a rectangle bounded by segments of length r of the sides of a primary critical half-strip, by the end of the half-strip, and by a segment congruent to the end, is equal, for r large, to*

$$(lr^N/(2\pi))[1 + O(1/r)],$$

where l is the length of the side of the primary critical polygon that corresponds† to the half-strip.

THEOREM 3. *Let $Z_f(r)$ and $Z_{f_{\alpha\ldots\lambda\mu}}(r)$ denote, respectively, the numbers of zeros of $f(z)$ and $f_{\alpha\ldots\lambda\mu}(z)$ in the interior of a rectangle bounded (1) by segments of length r of the sides of a critical half-strip which is primary for $f_{\alpha\ldots\lambda\mu}(z)$ and non-primary for $f(z)$, $f_\alpha(z)$, \ldots , $f_{\alpha\ldots\lambda}(z)$ ‡; (2) by the end of the half-strip; (3) by a segment congruent to the end. Then, if r is large, we have*

$$Z_f(r) = Z_{f_{\alpha\ldots\lambda\mu}}(r)[1 + O(1/r)].$$

* See footnote on p. 349.

† Each side of the polygon determines, through the direction of the outward-drawn normal, a set of primary critical rays, and each primary critical half-strip extends in the direction of one of these rays. This is the correspondence referred to.

‡ It is to be observed that a non-primary critical ray for $f(z)$ is a critical ray for a definite function $f_\alpha(z)$; if it is a non-primary critical ray for $f_\alpha(z)$, it is a critical ray for a definite function $f_{\alpha\beta}(z)$; and so on. The ray is a primary critical ray for a definite function $f_{\alpha\ldots\lambda\mu}(z)$. Also, it is to be observed that the critical half-strip for $f(z)$, corresponding to the ray, can be considered as the critical half-strip for each of the functions $f_\alpha(z)$, $f_{\alpha\beta}(z)$, \ldots , $f_{\alpha\ldots\lambda\mu}(z)$, corresponding to the ray.

We shall prove these theorems by a process of induction. Our reasoning proves Theorem 2 directly for the case in which $N=1$. We shall show that if $N>1$, the theorems are consequences of the corresponding theorems concerning E -functions having exponents less than N .*

5.1. PROOF OF THEOREM 2

We here employ the assignment of subscripts and the notation that were used in the latter part of §4.1.

Consider a particular primary critical half-strip, say the one that extends in the direction of the bisector of the sector U_β .

Consider a rectangle, R , the vertices of which, in counter-clockwise order, are denoted by V_1, V_2, V_3, V_4 , respectively. The rectangle is taken so that (1) V_1V_2 is a segment of the clockwise side of the half-strip; (2) V_3V_4 is a segment of the counter-clockwise side of the half-strip; (3) $f(z)$ does not vanish on either of the segments V_2V_3, V_4V_1 . We take the half-strips so that $f(z)$ does not vanish on either side of any one of them; hence $f(z)$ does not vanish on the boundary of R . Let the complex number corresponding to the typical vertex V_i of R be z_i .

The number, $Z_f(R)$, of zeros of $f(z)$ contained in R is given by the formula

$$2\pi Z_f(R) = \text{variation of amp } f(z) \text{ along the curve } V_1V_2V_3V_4V_1.$$

For the sake of brevity, we use a self-explanatory notation in which this formula becomes

$$\begin{aligned} 2\pi Z_f(R) &= [\text{v.a. } f(z), V_1V_2V_3V_4V_1] \\ (14) \quad &= [\text{v.a. } f(z), V_1V_2] + [\text{v.a. } f(z), V_2V_3] \\ &\quad + [\text{v.a. } f(z), V_3V_4] + [\text{v.a. } f(z), V_4V_1]. \end{aligned}$$

We shall estimate the value of each of the terms in the right-hand member of (14), assuming that R lies in the distant part of the half-strip.

First consider the term $[\text{v.a. } f(z), V_1V_2]$. We write $f(z)$ in the form

$$\begin{aligned} f(z) &= \{f_{M''}(z)\} \{\exp [g_{M''}(z) + \mu_{M''} z^N]\} \left\{1 + \sum_{m=0}^{M''-1} \frac{f_m(z)}{f_{M''}(z)} \exp [g_m(z) \right. \\ &\quad \left. - g_{M''}(z) + (\mu_m - \mu_{M''})z^N]\right\} \left\{1 + \frac{k(z)}{h(z)}\right\} \equiv F_1 F_2 F_3 F_4. \end{aligned}$$

The functions $h(z), k(z)$ are defined by equations (9). We now have the following relation:

* The widths of the half-strips, and the locations of the ends of the half-strips are somewhat arbitrary. The formulas stated in the theorems imply, of course, that the half-strips have been definitely chosen.

$$[\text{v.a. } f(z), V_1 V_2] = \sum_{i=1}^4 [\text{v.a. } F_i, V_1 V_2].$$

It has been shown in §4.1 that $|F_3 - 1| < 1$, and that $|F_4 - 1| < 1$, when z is on the distant part of the clockwise side of the half-strip. Therefore, if R is sufficiently distant,

$$-\pi < [\text{v.a. } F_3, V_1 V_2] < \pi,$$

$$-\pi < [\text{v.a. } F_4, V_1 V_2] < \pi.$$

We have immediately

$$[\text{v.a. } F_3, V_1 V_2] = \Im[g_{M''}(z_2) + \mu_{M''} z_2^N] - \Im[g_{M''}(z_1) + \mu_{M''} z_1^N].$$

It remains to consider the term $[\text{v.a. } f_{M''}(z), V_1 V_2]$. If $f_{M''}(z)$ is constant, this term is zero. Henceforth assume that the function is non-constant. The function is of the form*

$$(15) \quad f_{M''}(z) = \sum_{j=0}^J \exp(a_{jN_{M''}} z^{N_{M''}} + \dots + a_{j0}).$$

Let z' be the point collinear with z_1 and z_2 that is nearest the origin; let z'' be a particular one of the two points that are collinear with z_1 and z_2 and are such that $|z' - z''| = 1$. We write

$$z = z' + (z'' - z')t,$$

thereby changing the independent variable from z to t . We write

$$f_{M''}(z) \equiv \phi(t) = \sum_{j=0}^J \exp(b_{jN_{M''}} t^{N_{M''}} + \dots + b_{j0}),$$

where the b 's are functions of z' , z'' , and the a 's of (15). Let $b_{jk} = b_{jk}' + ib_{jk}''$, where b_{jk}' and b_{jk}'' are real.

We now consider the functions

$$A(t) = \sum_{j=0}^J \exp(b_{jN_{M''}}' t^{N_{M''}} + \dots + b_{j0}') \cos(b_{jN_{M''}}'' t^{N_{M''}} + \dots + b_{j0}''),$$

$$B(t) = \sum_{j=0}^J \exp(b_{jN_{M''}}' t^{N_{M''}} + \dots + b_{j0}') \sin(b_{jN_{M''}}'' t^{N_{M''}} + \dots + b_{j0}'').$$

When the points z , z_1 , z_2 are collinear, $A(t)$ and $iB(t)$ are, respectively, the real and imaginary parts of $\phi(t) \equiv f_{M''}(z)$. Consequently, for such z 's we have the relation

* It is not implied that the J here is the same as the J in equation (1). The same remark applies in several other places in the following work.

$$(16) \quad \tan \text{amp } f_{M''}(z) = \tan \text{amp } \phi(t) = \frac{B(t)}{A(t)}.$$

Now $f_{M''}(z) \neq 0$ on the distant part of the clockwise side of the half-strip. Hence, $A(t)$ and $B(t)$ cannot vanish simultaneously on the part of the real axis in the t -plane that corresponds to the distant part of this side of the half-strip. If either $A(t)$ or $B(t)$ is identically zero, $\text{amp } f_{M''}(z)$ is constant on the segment V_1V_2 . Henceforth let us assume that neither $A(t)$ nor $B(t)$ is identically zero. It follows from (16), and the properties of the tangent function, that $\text{amp } f_{M''}(z)$ cannot vary by as much as π on any distant segment of the side of the half-strip without $A(t)$ vanishing on the corresponding segment of the real axis in the t -plane. Therefore, if we can show that $A(t)$ has not more than, say, ν zeros on the segment in the t -plane that corresponds to the segment V_1V_2 , it will follow that

$$-(\nu + 1)\pi < [\text{v.a. } f_{M''}(z), V_1V_2] < (\nu + 1)\pi.$$

Thus the problem of estimating the value of $[\text{v.a. } f_{M''}(z), V_1V_2]$ is related to the problem of estimating the number of zeros of $A(t)$ on a certain distant segment on the real axis in the t -plane.

If $A(t)$ is the exponential of a polynomial, it has no zeros. Henceforth assume that $A(t)$ is not the exponential of a polynomial. Then $A(t)$ is a function of the same type as the function $f(z)$ which is the subject of this entire study, except for the essential difference that, instead of the exponent N , we have here the smaller exponent $N_{M''}$. Therefore, the zeros of $A(t)$, if there are any, are confined to certain half-strips in the t -plane. We are interested in those zeros, if there are any, which lie on a certain segment on the distant part of a particular half of the real axis. If no one of the critical half-strips for $A(t)$ contains this half of the real axis, there are no zeros on the segment in question (provided the rectangle R is taken sufficiently distant in the z -plane). If, on the other hand, the above-mentioned half of the real axis in the t -plane is contained in some critical half-strip for $A(t)$, we draw a rectangle, R' , in the latter half-strip (the rectangle having two of its sides on the sides of the half-strip, containing in its interior the segment corresponding to V_1V_2 , and being taken so that $A(t)$ does not vanish on the boundary of R'), and we estimate the number of zeros of $A(t)$ in R' by means of the theorems concerning $A(t)$ corresponding to our Theorems 2 and 3. The number of zeros on the segment corresponding to V_1V_2 does not exceed the number of zeros in R' . By the hypothesis for the induction, we have the following expression for the number of zeros of $A(t)$ contained in the rectangle R' :

$$\frac{a|t_2 + \tau''|^n}{2\pi} [1 + O(1/|t_2 + \tau''|)] - \frac{a|t_1 - \tau'|^n}{2\pi} [1 + O(1/|t_1 - \tau'|)],$$

where a is a positive constant, n is a positive integer not greater than $N_{M''}$, t_1 and t_2 are the values of t that correspond to z_1 and z_2 , respectively, and $t_2 + \tau''$ and $t_1 - \tau'$ are the values of t corresponding to the points at which the real t -axis intersects the boundary of R' ; $|\tau'|$ and $|\tau''|$ may be taken arbitrarily small, but not zero.

We have now completed the estimation of the value of the term $[v.a. f(z), V_1 V_2]$. It is evident that the term $[v.a. f(z), V_3 V_4]$ can be discussed in an altogether similar manner. No details of this discussion need be given here.

Next we shall estimate the value of the term

$$[v.a. f(z), V_2 V_3] = 3[\frac{1}{2}(\mu_0 + \mu_{M''})z_3^N] - 3[\frac{1}{2}(\mu_0 + \mu_{M''})z_2^N] \\ + [v.a. (h(z) + k(z)), V_2 V_3].$$

Let z_0 be the point at which the bisector of the sector U_β intersects the straight line through V_2 and V_3 . We write

$$z_3 - z_2 = \zeta, \quad z = z_0 + \zeta t,$$

thus changing the independent variable from z to t . We also write

$$h(z) \equiv \psi(t), \quad k(z) \equiv \omega(t).$$

The function $\psi(t) + \omega(t)$ is of the form

$$\psi(t) + \omega(t) = \sum_{j=0}^J \exp(\gamma_{jN} t^N + \dots + \gamma_{j0}),$$

where the γ 's are constants which depend on z_0 , ζ , and the λ 's in (1). Let $\gamma_{jn} = \gamma'_{jn} + i\gamma''_{jn}$, where γ'_{jn} and γ''_{jn} are real.

We now consider the functions

$$G(t) = \sum_{j=0}^J \exp(\gamma'_{jN} t^N + \dots + \gamma'_{j0}) \cos(\gamma''_{jN} t^N + \dots + \gamma''_{j0}),$$

$$H(t) = \sum_{j=0}^J \exp(\gamma'_{jN} t^N + \dots + \gamma'_{j0}) \sin(\gamma''_{jN} t^N + \dots + \gamma''_{j0}).$$

If z , z_2 , and z_3 are collinear, $G(t)$ and $iH(t)$ are, respectively, the real and imaginary parts of the function $\psi(t) + \omega(t)$.

For $0 \leq m \leq M''$, we have

$$\mu_m - \frac{1}{2}(\mu_0 + \mu_{M''}) = \rho_m \exp[i(\phi + \pi/2)],$$

where ρ_m is real. Also, we have

$$z_0 = |z_0| \exp [-i(\phi + 2\beta\pi)/N].$$

Hence

$$(17) \quad \left(\mu_m - \frac{\mu_0 + \mu_{M'}}{2} \right) z^N = i\rho_m |z_0|^N + \dots + \rho_m \zeta^N t^N \exp [i(\phi + \pi/2)].$$

Let T be any positive constant. The relation (17) shows that when $|t| \leq T$, and when $|z_0|$ is sufficiently large, we have a relation of the form

$$|\psi(t)| \leq \exp [B_1 |z_0|^{N-1}],$$

where B_1 is a positive constant. We know that in the distant part of U_ρ we have

$$|k(z)| \leq \exp [-B_2 |z|^N],$$

where B_2 is a positive constant; hence, when $|t| \leq T$, and when $|z_0|$ is sufficiently large, we have

$$|\omega(t)| \leq \exp [-B_3 |z_0|^N],$$

where B_3 is a positive constant. It follows that when $|t| \leq T$, and when $|z_0|$ is sufficiently large, we have relations of the form

$$(18) \quad |G(t)| \leq \exp [B |z_0|^{N-1}], \quad |H(t)| \leq \exp [B |z_0|^{N-1}],$$

where B is a positive constant.

When z is collinear with z_2 and z_3 we have the relation

$$\tan \text{amp } [h(z) + k(z)] = \frac{H(t)}{G(t)}.$$

Because of the way in which the rectangle R has been taken, $G(t)$ and $H(t)$ do not vanish simultaneously at any point on the segment corresponding to V_2V_3 . If either of the functions $G(t)$, $H(t)$ is identically zero, $\text{amp } [h(z) + k(z)]$ is constant along V_2V_3 . Henceforth assume that neither $G(t)$ nor $H(t)$ is identically zero. Then, if $G(t)$ vanishes not more than, say, ν times on the segment in the t -plane that corresponds to V_2V_3 , the following relation holds:

$$-(\nu + 1)\pi < [\text{v.a. } (h(z) + k(z)), V_2V_3] < (\nu + 1)\pi.$$

We have the same relation if $H(t)$ does not vanish more than ν times on the segment corresponding to V_2V_3 .

We know that $h(z)$ is bounded away from zero when z is on the distant part of the clockwise side of the half-strip. The same is, therefore, true of the function $h(z) + k(z)$. Hence, if $|z_0|$ is sufficiently large, and if t_2 is the value of

t that corresponds to the value z_2 of z , one at least of the numbers $|G(t_2)|$, $|H(t_2)|$ is greater than a certain fixed positive number δ . To fix the ideas, suppose that $|G(t_2)| > \delta$; similar reasoning applies if $|G(t_2)| \leq \delta$ and $|H(t_2)| > \delta$.

We wish to establish an upper bound for the number of zeros of $G(t)$ on the segment corresponding to V_2V_3 . For this purpose we employ a result due to Jensen, which we state for our purposes in the following restricted form*:

Let $G(t)$ be an integral function, such that

$$|G(t)| \leq M(\tau) \text{ for } |t - t_2| \leq \tau.$$

Let $G(t)$ vanish at the points $t^{(1)}, t^{(2)}, \dots, t^{(v)}$, such that

$$0 \leq |t^{(n)} - t_2| < \tau \quad (n = 1, 2, \dots, v).$$

Then

$$(19) \quad |(t^{(1)} - t_2)(t^{(2)} - t_2) \dots (t^{(v)} - t_2)| \geq |G(t_2)| \tau^v / M(\tau).$$

Choose a positive number τ large enough so that the segment in the t -plane corresponding to V_2V_3 is contained in the interior of the circle $|t - t_2| = \tau$. The value of τ may be, and is, taken to be independent of $|z_0|$. Denote by t_3 the value of t corresponding to z_3 . We are interested in those zeros of $G(t)$, if there are any, that are within or on the circle $|t - t_2| = |t_3 - t_2|$. We take the $t^{(n)}$'s of the above theorem to be just these zeros. We then have, by (18) and (19),

$$|t_3 - t_2|^v \geq \frac{|G(t_2)| \tau^v}{M(\tau)} > \frac{\delta \tau^v}{\exp [B |z_0|^{N-1}]},$$

or

$$(20) \quad v < \frac{B |z_0|^{N-1} - \log \delta}{\log \frac{\tau}{|t_3 - t_2|}},$$

the logarithms being real. This completes our estimation of the value of term [v.a. $f(z)$, V_2V_3]. It is to be noted that τ is a constant greater than $|t_3 - t_2|$, so the only variable in the right-hand member of (20) is $|z_0|$.

The term [v.a. $f(z)$, V_4V_1] can be discussed in a way that is entirely similar to the way in which we have discussed [v.a. $f(z)$, V_2V_3]. No details of this discussion need be given here.

To complete the proof of Theorem 2 we write out the complete expression for the number of zeros contained in the rectangle R , using the estimates we

* Bieberbach, *Lehrbuch der Funktionentheorie*, vol. II, p. 109.

have found for the several terms in (14). We shall consider the side V_4V_1 of R as fixed, and the side V_2V_3 as variable; in particular we shall confine our attention to cases in which the sides V_1V_2 and V_3V_4 are long. Collecting our results, we see that the number of zeros in R is given by the equation

$$2\pi Z_f(R) = \Im[\tfrac{1}{2}(\mu_{M''} - \mu_0)(z_3^N + z_2^N)] + \Im[g_{M''}(z_2) - g_0(z_3)] \\ + [\text{v.a. } f_{M''}(z), V_1V_2] + [\text{v.a. } f_0(z), V_3V_4] + [\text{v.a. } (h(z) + k(z)), V_2V_3] + W,$$

where W is a quantity that is less, in absolute value, than a fixed number.

Now, in terms of the notation used before,

$$\mu_{M''} - \mu_0 = |\mu_{M''} - \mu_0| \exp [i(\phi + \pi/2)], \\ z_2^N = (z_0 + \zeta t_2)^N = |z_0|^N \exp(-i\phi) + \dots, \\ z_3^N = (z_0 + \zeta t_3)^N = |z_0|^N \exp(-i\phi) + \dots,$$

and hence

$$\Im[\tfrac{1}{2}(\mu_{M''} - \mu_0)(z_3^N + z_2^N)] = |\mu_{M''} - \mu_0| \cdot |z_0|^N \\ + \text{terms in lower powers of } |z_0|.$$

We have previously seen that if $|z_0|$ is sufficiently large we have relations of the forms

$$|[\text{v.a. } f_{M''}(z), V_1V_2]| \leq \alpha_1 |z_0|^{N-1}, \\ |[\text{v.a. } f_0(z), V_3V_4]| \leq \alpha_2 |z_0|^{N-1}, \\ |[\text{v.a. } (h(z) + k(z)), V_2V_3]| \leq \alpha_3 |z_0|^{N-1},$$

where the α 's are positive constants. Also, if $|z_0|$ is large,

$$|\Im[g_{M''}(z_2) - g_0(z_3)]| \leq \alpha_4 |z_0|^{N-1},$$

where α_4 is a positive constant.

It follows from the relations we have obtained that when $|z_0|$ is sufficiently large we have

$$(21) \quad Z_f(R) = \frac{|\mu_{M''} - \mu_0| \cdot |z_0|^N}{2\pi} [1 + O(1/|z_0|)].$$

Equation (21) contains our fundamental result; from it Theorem 2 follows at once.

5.2. PROOF OF THEOREM 3

We revert to the notation used in §3.

Suppose that the function $f_\alpha(z)$ has a critical ray lying in one of the sectors $S_\alpha^{(0)}, \dots, S_\alpha^{(N-1)}$. We write

$$F_\alpha(z) = 1 + \sum_{m=0}^M \frac{f_m(z)}{f_\alpha(z)} \exp [g_m(z) - g_\alpha(z) + (\mu_m - \mu_\alpha)z^N],$$

the prime on the summation sign indicating that the term for $m=\alpha$ is to be omitted. The zeros of $f(z)$ are the same as those of the function $f_\alpha(z)F_\alpha(z)$.

Our reasoning depends essentially on the following two theorems from the theory of integral functions*:

1. Let the integral function $F(z)$ be of finite order ρ . Let z_1, z_2, z_3, \dots denote the non-zero zeros of $F(z)$, and let h be any real number greater than ρ . Then the series

$$\sum_{i=1}^{\infty} |z_i|^{-h}$$

converges.

2. Let the integral function $F(z)$ be of finite order ρ . Let h be any positive number greater than ρ , and let ϵ be any positive number. About each of the non-zero zeros, z_i , of $F(z)$ as center describe a circle Γ_i of radius $|z_i|^{-h}$. Then if the point z is outside each of the circles Γ_i , and if $|z|$ is sufficiently large, we have the relation

$$|F(z)| > \exp [-|z|^{h+\epsilon}].$$

It is to be noted that the first theorem insures that z 's such as are referred to in the second theorem exist.

Obviously, the order of $f_\alpha(z)$ is not greater than N_α . Let h be a positive number greater than N_α . Let ϵ be a positive number less than unity. By the second theorem cited above, if about each non-zero zero z_i of $f_\alpha(z)$ as center we describe a circle Γ_i of radius $|z_i|^{-h}$, and if we take z outside of each of these circles, and such that $|z|$ is sufficiently large, we have

$$|f_\alpha(z)| > \exp [-|z|^{N_\alpha+\epsilon}].$$

It follows readily from considerations similar to those used in the proof of Theorem 1 that if z is in the sufficiently distant part of a certain sector S that encloses the critical ray under consideration, we have a relation of the form

$$|f_\alpha(z)| \cdot |F_\alpha(z) - 1| \leq \exp [-B|z|^N],$$

where B is a positive constant.

Let k be any positive number less than unity. It is clear, from the foregoing, that we can find a positive number K such that if z is in the sector S , is outside of each of the circles Γ_i , and is such that $|z| \geq K$, we have

$$(22) \quad |F_\alpha(z) - 1| \leq k < 1.$$

* Bieberbach, *Lehrbuch der Funktionentheorie*, vol. II, p. 243 and p. 268.

We denote the set of all such points z by the symbol Ω .

Now consider any simple closed regular curve Γ that is composed entirely of points of the set Ω , and which does not pass through any zero of $f(z)$ or of $f_\alpha(z)$. The number, $Z_f(\Gamma)$, of zeros of $f(z)$ within Γ is given by the formula

$$(23) \quad Z_f(\Gamma) = \frac{1}{2\pi} [\text{v.a. } f_\alpha(z), \Gamma] + \frac{1}{2\pi} [\text{v.a. } F_\alpha(z), \Gamma].$$

The first term in the right-hand member of (23) is the number of zeros of $f_\alpha(z)$ within Γ . By (22), we have

$$-\frac{1}{2} < \frac{1}{2\pi} [\text{v.a. } F_\alpha(z), \Gamma] < \frac{1}{2}.$$

Hence the number of zeros of $f(z)$ within Γ is equal to the number of zeros of $f_\alpha(z)$ in the same region. This is our fundamental result.

Consider a rectangle bounded (1) by segments of length r of the sides of the half-strip corresponding to the critical ray under consideration; (2) by the end of the half-strip; (3) by a segment congruent to the end. Let V_1, V_2, V_3, V_4 denote the vertices of the rectangle in counter-clockwise order, the segment V_4V_1 being the end of the half-strip. We consider three other rectangles, $V_1V_2^{(0)}V_3^{(0)}V_4$, $V_1V_2^{(1)}V_3^{(1)}V_4$, and $V_1V_2^{(2)}V_3^{(2)}V_4$. The segments $V_2^{(0)}V_3^{(0)}$, $V_2^{(1)}V_3^{(1)}$, V_2V_3 , $V_2^{(2)}V_3^{(2)}$ are assumed to be crossed in that order as we proceed outward along the half-strip. The segments $V_2^{(0)}V_3^{(0)}$, $V_2^{(1)}V_3^{(1)}$, $V_2^{(2)}V_3^{(2)}$, $V_2^{(0)}V_3^{(2)}$, $V_3^{(0)}V_3^{(2)}$ are assumed to consist entirely of points of the set Ω .* Furthermore, the boundaries of the rectangles $V_2^{(0)}V_3^{(0)}V_3^{(1)}V_4^{(0)}$, $V_2^{(0)}V_2^{(2)}V_3^{(2)}V_3^{(0)}$ are assumed not to pass through any zeros of $f(z)$. Let the symbols R, R_0, R_1, R_2 , respectively, denote the rectangles in the order in which they have been named. Let R'_1, R'_2 denote the rectangles $V_2^{(0)}V_2^{(1)}V_3^{(1)}V_3^{(0)}$, $V_2^{(0)}V_2^{(2)}V_3^{(2)}V_3^{(0)}$, respectively.

Let $Z_f(R)$ denote the number of zeros of $f(z)$ within R . Similar symbols will be used in similar senses without further explanation.

Now obviously,

$$\frac{Z_f(R'_1)}{Z_{f_\alpha}(R_0) + Z_{f_\alpha}(R'_1)} \leq \frac{Z_f(R)}{Z_{f_\alpha}(R)} \leq \frac{Z_f(R'_2)}{Z_{f_\alpha}(R_0) + Z_{f_\alpha}(R'_1)} + \frac{Z_f(R_0)}{Z_{f_\alpha}(R_0) + Z_{f_\alpha}(R'_1)}.$$

It has been shown that

$$Z_f(R'_j) = Z_{f_\alpha}(R'_j), \quad j = 1, 2.$$

* It is a simple consequence of the first theorem cited above, and of our other results, that this condition can be satisfied.

By the hypothesis for the induction, we have, for $j=1, 2$,

$$Z_{f_a}(R'_j) = \frac{ar_j^n}{2\pi}[1 + O(1/r_j)] - \frac{ar_0^n}{2\pi}[1 + O(1/r_0)]$$

where a is a positive constant, n is a positive integer not greater than N_a , and r_0, r_1, r_2 are the lengths of the segments $V_1V_2^{(0)}, V_1V_2^{(1)}, V_1V_2^{(2)}$, respectively.

We hold r_0 fixed. By the first theorem cited above, if we make the segment V_1V_2 sufficiently long, we can make $r_2 - r_1$ arbitrarily small without violating any of our previous stipulations. Now an easy and obvious calculation gives us the result stated in Theorem 3.

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IDEAL THEORY AND ALGEBRAIC DIFFERENTIAL EQUATIONS*

BY

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INTRODUCTION

J. F. Ritt† introduced the idea of irreducible system of algebraic differential equations and showed that every system of such equations is equivalent to a finite set of irreducible systems.

One of the objects of this paper is to develop a special type of abstract ideal theory which has Ritt's theorem as a consequence. The elements of our ideals are polynomials in unknowns y_1, \dots, y_n and a certain number of their derivatives. Following Ritt, we call these polynomials *forms*. The coefficients in these forms are assumed to be elements of a *differential field* \mathfrak{F} of characteristic zero.‡ A *differential field* is a commutative field (as in abstract algebra) whose elements a, b, \dots have unique derivatives a_1, b_1, \dots which are elements of the field. These derivatives must satisfy the rules $(a+b)_1 = a_1 + b_1$ and $(ab)_1 = a_1b + ab_1$.§ The totality of these forms with coefficients in \mathfrak{F} is a differential ring \mathcal{R} .|| We consider *differential ideals*, which are ideals containing together with any element its derivative.¶ An example given by Ritt shows that there exists a differential ideal of \mathcal{R} having no finite subset, such that every element of the ideal is a linear combination of elements of the subset and their derivatives with forms of \mathcal{R} as coefficients.**

Certain results of Ritt suggested that we consider, as our purpose permits, only differential ideals which have the property that if they contain an ele-

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† J. F. Ritt, *Differential Equations from the Algebraic Standpoint*, Colloquium Publications of this Society, vol. 14. Cf. p. 14.

‡ For definitions of terms of abstract algebra see B. L. van der Waerden, *Moderne Algebra*.

§ Abstract differential fields have been treated by R. Baer, *Algebraische Theorie der differenzierbaren Funktionenkörper*, I, Heidelberger Akademie der Wissenschaften, Sitzungsberichte, Mathematische-Naturwissenschaftliche Klasse, 1927-1928, and by the author, *Differential fields and ideals of differential forms*, *Annals of Mathematics*, vol. 34 (1933), pp. 509-517. They have been used by O. Ore, *Formale Theorie der linearen Differentialgleichungen*, *Journal für Mathematik*, vol. 167 (1932), pp. 221-234, and vol. 168 (1932), pp. 233-252.

|| Raudenbush, loc. cit., p. 514. In the definition of differential field, substitute ring for field to obtain the definition of differential ring.

¶ Raudenbush, loc. cit., p. 516.

** Ritt, loc. cit., p. 12.

ment a of \mathcal{R} , they contain any element b of \mathcal{R} such that a positive power of b is a . We call these differential ideals *perfect differential ideals*. We show that every perfect differential ideal of \mathcal{R} is the intersection of a finite number of prime perfect differential ideals.

The use of perfect differential ideals was suggested by the following two results of Ritt:

(a) *Every infinite system of forms has a finite subsystem whose manifold of solutions is identical with that of the infinite system.**

(b) *Let F_1, \dots, F_r ; G be forms such that G has every solution of the system F_1, \dots, F_r . Then some power of G is a linear combination of the F_i and a certain number of their derivatives with forms for coefficients.†*

We obtain abstract theorems that specialize to a combination of these results of Ritt. For instance, we show that every perfect differential ideal of \mathcal{R} has a finite subset such that every form of the ideal has a power which is a linear combination of the forms of the subset and their derivatives with forms of \mathcal{R} for coefficients. The proof of this basis theorem is like the proof of Ritt's result (a) in fundamental respects, but there are essential differences. We also obtain an abstract generalization of Ritt's result (b). The conciseness of the proof of this theorem is an indication of the simplicity of our theory.

Having established the basis theorem, the development of our ideal theory follows approximately the well known methods of E. Noether.‡

PERFECT DIFFERENTIAL IDEALS

1. We consider a fixed differential ring \mathcal{R} of characteristic zero.

The intersection of any arbitrary set of differential ideals is a differential ideal. For let a be any element of the intersection. Then a is an element of every ideal of the set; hence the derivative a_1 is in the intersection. The intersection, which is known to be an ideal, is then a differential ideal. The intersection of any arbitrary set of perfect differential ideals is a perfect differential ideal. Let a and b be elements of \mathcal{R} such that a is in the intersection and some power of b is a . Then a is in every ideal of the set, hence also b . Therefore the intersection is a perfect differential ideal.

Let σ be an arbitrary set of elements of \mathcal{R} . We notice that \mathcal{R} is a perfect differential ideal. The intersection of the differential ideals containing σ will be called the *differential ideal* $[\sigma]$ *determined by* σ . $[\sigma]$ is uniquely defined. The intersection of all perfect differential ideals containing σ we call the *perfect differential ideal* $\{\sigma\}$ *determined by* σ . $\{\sigma\}$ is uniquely defined.

* Ritt, loc. cit., p. 10.

† Ritt, loc. cit., p. 108.

‡ E. Noether, *Idealtheorie in Ringbereichen*, Mathematische Annalen, vol. 86 (1921), pp. 24-66.

Let α be any set of elements of \mathcal{R} . We shall denote by α' the set consisting of all elements of \mathcal{R} which have a positive integral power in α . Using the set σ of the preceding paragraph, we define σ_n recursively as follows:

$$\begin{aligned}\sigma_1 &= [\sigma], \\ \sigma_n &= [\sigma'_{n-1}] \quad (n = 2, 3, 4, \dots).\end{aligned}$$

Let β denote the totality of elements of the sets σ_n . Then β is a perfect differential ideal and is contained in $\{\sigma\}$, hence is $\{\sigma\}$. This means that any element t of $\{\sigma\}$ is in some σ_n with a sufficiently large subscript.

LEMMA. *If a differential ideal δ contains a positive integral power a^p of an element a it contains the positive integral power $a_1^{2^{p-1}}$ of the derivative a_1 of a .*

δ contains $(a^p)_1 = pa^{p-1}a_1$ hence δ contains $a^{p-1}a_1$.* Assume that δ contains $a^{p-r}a_1^r$, where $r < p$. Then δ contains

$$a_1(a^{p-r}a_1^r)_1 - sa_2(a^{p-r}a_1^r) = (p-r)a^{p-r-1}a_1^{r+2},$$

where $a_2 = (a_1)_1$; hence δ contains $a^{p-r-1}a_1^{r+2}$. Applying this result $p-1$ times to $a^{p-1}a_1$ we find that δ contains $a_1^{2^{p-1}}$.

Let t be any element of $\{\sigma\}$ not in σ_1 . There is a least positive integer $n > 1$ such that σ_n contains t . As an element of σ_n , t is equal to a linear homogeneous expression in a finite number of elements of σ'_{n-1} and a finite number of derivatives of elements of σ'_{n-1} with elements of the ring or integers for coefficients.† But, by the lemma, each of these elements has a power in σ_{n-1} . Let r be their number and s the maximum of the powers to which each must be raised to give an element of σ_{n-1} . Then t^{r+s+1} is in σ_{n-1} , for each term of the same power of the linear expression contains an s th power of some one of the elements and hence each term is in σ_{n-1} .

This power of t by the same reasoning has a power in σ_{n-2} . Hence t has a power in σ_{n-2} . Continuing this process a finite number of steps gives the

THEOREM 1. *If t is any element of a perfect differential ideal $\{\sigma\}$ of a differential ring \mathcal{R} of characteristic zero determined by a set σ , then some positive integral power of t is in the differential ideal $[\sigma]$ determined by σ .‡*

2. § LEMMA. *If a perfect differential ideal π contains the product ab of any two elements a and b then it contains the product $a_p b_q$ of any derivatives of a and b .||*

* If \mathcal{R} were of characteristic p we could not draw this conclusion.

† If n is an integer and a an element of the ring, $1a = a$, $-1a = -a$, $na = (n-1)a + a$, $na = an$.

‡ The theorem is not true for rings of non-zero characteristic.

§ The results of this and the next article are independent of Theorem 1 and true for non-zero characteristic.

|| p may be zero; $a_0 = a$ and we shall speak of the zero derivative of a . $a_p = (a_p - 1)_1$.

Assume that $a_m b_n$ is in π . Then π contains

$$a_{m+1} b_n (a_m b_n)_1 - a_{m+1} b_{n+1} (a_m b_n) = a_{m+1}^2 b_n^2.$$

Hence, by the definition of a perfect differential ideal, π contains $a_{m+1} b_n$. Similarly, π contains $a_m b_{n+1}$. Since, by hypothesis, π contains $a_0 b_0$, the lemma is obtained by induction.

THEOREM 2. *The intersection $\{\sigma, a\} \wedge \{\sigma, b\}$ of the perfect differential ideals determined by the sets obtained by adjoining elements a and b , respectively, to the set σ of elements is the perfect differential ideal $\{\sigma, ab\}$ determined by the set obtained by adjoining the product ab to σ .**

Every element of $\{\sigma, ab\}$ is in the intersection. We have only to show that any element t of the intersection is in $\{\sigma, ab\}$.

By Theorem 1 some power of t , say t^r , is in $[\sigma, a]$, and some power, say t^s , is in $[\sigma, b]$. Hence t^{r+s} is in $\{\sigma, ab\}$ since each term of the product of the linear expression for t^r , in terms of the elements of σ and a and their derivatives, and for t^s , in terms of the elements of σ and b , contains either elements of σ or a product $a_p b_q$ of derivatives of a and b . By the definition of perfect differential ideal, $\{\sigma, ab\}$ contains t .

DECOMPOSITION OF PERFECT DIFFERENTIAL IDEALS

3. We shall say that a perfect differential ideal π which is determined by a set σ has σ as a *basis*. If every perfect differential ideal of a differential ring \mathcal{R} has a finite basis, we say that \mathcal{R} is a *differential ring with a basis theorem*.

THEOREM 3. *Let*

$$\pi_1 \leq \pi_2 \leq \pi_3 \leq \dots$$

be an infinite sequence of perfect differential ideals of a differential ring with a basis theorem such that each ideal contains its predecessor in the sequence. There exists an integer n such that

$$\pi_n = \pi_{n+1} = \dots .^\dagger$$

Let π be the totality of elements in the ideals of the sequence. Let a be any element of π . Then a is contained in some ideal of the sequence with a sufficiently high subscript. Therefore π contains a_1 or any element b having a power equal to a and hence is a perfect differential ideal. π has a finite basis

* A more general theorem could be proved but this is sufficient to our purpose.

† Cf. van der Waerden, loc. cit., vol. II, p. 25.

which must be contained in some ideal of the sequence with a sufficiently large subscript n . But $\pi_n = \pi$, hence $\pi_n = \pi_{n+1} = \dots$.

A perfect differential ideal π will be called *reducible* if there exist perfect differential ideals α and β such that π is a proper subset of α and of β and is their intersection $\alpha \wedge \beta$. If a perfect differential ideal is not reducible, it is said to be *irreducible*.*

THEOREM 4. *A perfect differential ideal which is irreducible is prime.†*

We show that a perfect differential ideal which is not prime is reducible. Let π be a perfect differential ideal which is not prime. There exist two elements a and b such that π contains ab but neither a nor b . Form the perfect differential ideals $\{\pi, a\}$ and $\{\pi, b\}$. Each contains π as a proper subset. Their intersection $\{\pi, a\} \wedge \{\pi, b\}$ by §2 is $\{\pi, ab\}$ but since ab is in π , the intersection is π . Hence π is reducible.

THEOREM 5. *In a differential ring with a basis theorem, any perfect differential ideal is the intersection of a finite set of irreducible or prime perfect differential ideals.*

We suppose that the theorem is not true. Then there exists a perfect differential ideal π which is not the intersection of a finite number of irreducible perfect differential ideals. π must be reducible. Hence π is the intersection of two perfect differential ideals α and β each containing π as a proper subset. At least one of the perfect differential ideals α and β is not the intersection of a finite number of irreducible perfect differential ideals. Let π_1 denote this perfect differential ideal. By the same reasoning π_1 is a proper subset of a perfect differential ideal π_2 which is not the intersection of a finite number of irreducible perfect differential ideals. Continuing in this manner we obtain an infinite sequence of perfect differential ideals, each containing its predecessor as a proper subset. This contradiction of Theorem 3 proves the theorem.

In such a finite set of irreducible or prime perfect differential ideals, we may delete in turn all ideals which contain other ideals of the set. The remaining set will be called an *essential set*.

THEOREM 6. *If a perfect differential ideal π is the intersection of each of two essential sets $\alpha_1, \dots, \alpha_r$ and β_1, \dots, β_s , then $r=s$ and the α 's coincide with the β 's after a suitable rearrangement.*

* This use of the word "irreducible" is analogous to its use in algebra. Cf. van der Waerden, loc. cit., vol. II, p. 36.

† An ideal is prime if it contains together with the product of any two elements at least one of the elements.

α_1 is contained in some β_i . If not, each β_i would contain an element b_i not in α_1 . By a repeated application of Theorem 2, π contains the product $b_1 \cdots b_n$. Hence α_1 contains this product, which contradicts the fact that α_1 is prime.

We may suppose that α_1 is contained in β_1 after a suitable rearrangement of the β 's. β_1 is contained in some α which must be α_1 . For if β_1 were contained in α_k , $k \neq 1$, then α_1 would be contained in α_k contradicting the assumption that the α 's form an essential set. Hence α_1 is β_1 .

α_2 is contained in some β which cannot be β_1 . Suppose that it is β_2 . Then β_2 is in α_2 and is α_2 . Continuing in this manner the theorem is proved.

THE BASIS THEOREM

4. In what follows we will need the following

LEMMA. *If the perfect differential ideal $\{\sigma\}$ has a finite basis, it has a finite basis consisting of elements of σ .*

Let s_1, \dots, s_n be the elements of a finite basis of $\{\sigma\}$. Each s_i as an element of $\{\sigma\}$ is an element of $\{\sigma_i\}$ where σ_i is a suitably chosen finite subset of σ . Every s_i is in $\{\sigma_1, \dots, \sigma_n\}$, hence $\{\sigma\}$ is $\{\sigma_1, \dots, \sigma_n\}$.

5. We prove the following theorem:

THEOREM 7. *The differential ring \mathcal{R} of forms in a finite set of indeterminates y_1, \dots, y_n with coefficients in a differential field \mathcal{F} of characteristic zero is a differential ring with a basis theorem.*

We suppose that the theorem is not true and force a contradiction.

LEMMA. *Let Σ be a perfect differential ideal without a finite basis. Let F_1, \dots, F_s be forms such that by multiplying each form of Σ by some product of non-negative powers of F_1, \dots, F_s a system Λ is obtained such that $\{\Lambda\}$ has a finite basis. Then $\{\Sigma, F_1 \cdots F_s\}$ has no finite basis.*

Suppose, as we may by the lemma of the preceding article, that $\{H_1, \dots, H_t; F_1 \cdots F_s\}$ is $\{\Sigma, F_1 \cdots F_s\}$, where H_1, \dots, H_t are forms of Σ . Let a finite basis of $\{\Lambda\}$ be chosen from the forms of Λ and let K_1, \dots, K_r be forms of Σ such that the forms which they yield, after the above described multiplications, form this basis of $\{\Lambda\}$. Let Π be the totality of H 's and K 's. Then $\{\Pi, F_1 \cdots F_s\}$ is $\{\Sigma, F_1 \cdots F_s\}$ and $\{\Pi\}$ contains $\{\Lambda\}$.

Since Σ has no finite basis, there exists a form L of Σ not in $\{\Pi\}$. Some $F_1^{g_1} \cdots F_s^{g_s} L$ is in $\{\Lambda\}$ and hence in $\{\Pi\}$. Consequently, if g is the maximum of the g_i 's, $F_1^g \cdots F_s^g L^g$ and hence $F_1 \cdots F_s L$ are in $\{\Pi\}$. L is in $\{\Pi, F_1 \cdots F_s\}$ by our assumption and obviously in $\{\Pi, L\}$; hence by §2 it is in $\{\Pi, F_1 \cdots F_s L\}$ which is $\{\Pi\}$. This contradiction proves the lemma.

LEMMA. Let Σ and $\{\Sigma, F_1 \cdots F_s\}$ be perfect differential ideals having no finite basis sets. Then at least one of the perfect differential ideals $\{\Sigma, F_1\}, \cdots, \{\Sigma, F_s\}$ has no finite basis.

We may limit ourselves to the case of $s=2$. Let $\{\Sigma, F_1\}$ and $\{\Sigma, F_2\}$ be $\{\Phi_1, F_1\}$ and $\{\Phi_2, F_2\}$ respectively, where Φ_1 and Φ_2 are finite sets taken, according to the lemma of the preceding article, as subsets of Σ . $\{\Sigma, F_1\}$ and $\{\Sigma, F_2\}$ are also $\{\Phi_1, \Phi_2, F_1\}$ and $\{\Phi_1, \Phi_2, F_2\}$ respectively. $\{\Sigma, F_1 F_2\}$ is the intersection $\{\Sigma, F_1\} \wedge \{\Sigma, F_2\}$ by §2 and hence also $\{\Phi_1, \Phi_2, F_1 F_2\}$ which contradiction proves the lemma.

We consider the totality of perfect differential ideals of \mathcal{R} without finite basis sets.* We form a *basic set*† for each. By a lemma of Ritt's†, we know that there is a perfect differential ideal Σ without a finite basis whose basic sets are not of higher rank‡ than the basic sets of any other perfect differential ideal without a finite basis. Let

$$(1) \quad A_1, \cdots, A_r$$

be a basic set of Σ . Then A_1 is not an element of \mathfrak{F} , as otherwise Σ would have unity as a finite basis.

For every form of Σ not in (1), let a remainder§ with respect to (1) be found. Let Λ be the system composed of the forms of (1) and the products of the forms of Σ not in (1) by the products $S_1^{a_1} \cdots S_r^{a_r} I_1^{b_1} \cdots I_r^{b_r}$ of the separants|| S_i and the initials|| I_i of the forms of (1) used in their reduction.|| Let Ω be the system composed of (1) and the remainders of the forms of Σ not in (1).

$\{\Omega\}$ has a finite basis. If not, Ω would have non-zero forms not in (1). Such forms would be reduced† with respect to (1) and $\{\Omega\}$ would have lower basic sets than Σ , contradicting our assumption. Consequently $\{\Lambda\}$ has a finite basis, for $\{\Lambda\}$ is $\{\Omega\}$.

The lemmas show that some $\{\Sigma, S_i\}$ or some $\{\Sigma, I_i\}$ has no finite basis. But for every i , S_i and I_i are distinct from zero, and reduced with respect to (1). Hence the basic sets of $\{\Sigma, S_i\}$ and $\{\Sigma, I_i\}$ are lower than (1). This contradiction proves the theorem.

* Cf. Ritt, loc. cit. In what follows we use the concepts and theorems of §§2 to 5 in this book. The reader will have no difficulty seeing that these articles with slight changes in language apply to our abstract forms.

† Ritt, loc. cit., p. 6.

‡ Ritt, loc. cit., p. 4.

§ Ritt, loc. cit., p. 9 and p. 7 for the existence. Notice that in case of non-zero characteristic a separant may vanish.

|| Ritt, loc. cit., p. 7.

THEOREM 8. *Any infinite system of forms contains a finite subset such that every form of the system has a power which is a linear combination of the forms of the subset and their derivatives with forms for coefficients.*

This follows at once from Theorems 1 and 7, and the lemma of the preceding article.

ANALOGUE OF THE HILBERT-NETTO THEOREM

We prove the following

THEOREM 9. *Let Σ and G be a system of forms and a form respectively of the differential \mathcal{R} of forms in a finite set of indeterminates y_1, \dots, y_n and with coefficients in a differential field \mathcal{F} of characteristic zero. If G is not in $\{\Sigma\}$ then there exists a set of elements a_1, \dots, a_n of an extension of \mathcal{F} such that every form of Σ vanishes when the a 's are substituted for the indeterminates and such that G does not vanish for the same substitution.*

Let Π_1, \dots, Π_n be prime perfect differential ideals whose intersection is $\{\Sigma\}$. Some Π_i , say Π' , does not contain G . By a theorem of the author's dissertation,* there exists a set of elements a_1, \dots, a_n of an extension of \mathcal{F} such that Π' is the set of forms of \mathcal{R} that vanish when the a 's are substituted for the indeterminates. The a 's are then solutions of Σ but not of G .

In what follows, we suppose that \mathcal{F} is a differential field of functions of a complex variable x meromorphic on an open region \mathcal{A} . Let Π' have A_1, \dots, A_p as its basic set. A_1 is of class greater than zero.† Let S_i and I_i be the separant and initial, respectively, of A_i . We show that the basic set has analytic solutions, when regarded as polynomials in the y_i , that they contain‡, for which G and no separant or initial vanishes. We suppose that every analytic solution of the basic set is a solution of $T = S_1 \cdots S_p I_1 \cdots I_p G$. Then by the Hilbert-Netto theorem for polynomials, T is in Π' . This contradicts the fact that Π' is prime, for Π' can contain no separant or initial of the forms of its basic set, and was chosen so as not to contain G . For a suitable value of x the values of the analytic functions in such a solution provide initial conditions for a regular§ analytic solution of the basic set which is not a solution of G . By a theorem of Ritt's||, such a solution is a solution of Π' and hence of Σ . This together with Theorem 1 gives Ritt's result (b).

* Raudenbush, loc. cit., p. 517, Theorem V.

† Ritt, loc. cit., p. 3.

‡ Certain of the y_i may be indeterminate.

§ Ritt, loc. cit., p. 20.

|| Ritt, loc. cit., p. 25. The theorem is true if the system is not closed, provided it is an ideal.

DIFFERENTIABLE FUNCTIONS DEFINED IN CLOSED SETS. I†

BY
HASSLER WHITNEY‡

1. Introduction. In a recent paper§ the author has shown that if a function $f(x)$ defined in a closed set A in n -space E satisfies certain conditions involving Taylor's formula (in finite form), i.e. if it is "of class C^m in A ," then its definition can be extended over E so that it will have continuous partial derivatives through the m th order. In this paper we restrict ourselves to the one-dimensional case. (For the above theorem in this case, see §4.) Let x_0, \dots, x_m be distinct points of A . If $P(x) = c_0 + \dots + c_m x^m$ is the polynomial of degree at most m such that $P(x_i) = f(x_i)$ ($i = 0, \dots, m$), the m th difference quotient of $f(x)$ at these points is $\Delta_0 \dots_m f = \Delta^m f(x) = m! c_m$. The main object of this paper is to prove (see §§2 and 3 for definitions)

THEOREM I. *A necessary and sufficient condition that $f(x)$ be of class C^m in A is that $\Delta^m f(x)$ converge in A .*

This theorem furnishes a direct definition of the differentiability of a function; the former definition (see §3) involved the existence of other functions $f_1(x), \dots, f_m(x)$.

The necessity of the condition is easily proved. The definition of $f(x)$ being extended over the x -axis E , consider any $m+1$ points x_0, \dots, x_m ($x_0 < x_1 < \dots < x_m$). Define $P(x)$ as above. As $f(x_i) - P(x_i) = 0$ ($i = 0, \dots, m$) there is a point x' ($x_0 < x' < x_m$) such that $(d^m/dx^m)[f(x') - P(x')] = 0$. But $d^m P(x)/dx^m = m! c_m = \Delta_0 \dots_m f$; hence $\Delta_0 \dots_m f = d^m f(x')/dx^m$. Therefore if x_0, \dots, x_m are in A and are sufficiently near a point x^* of A , $\Delta_0 \dots_m f = d^m f(x')/dx^m = d^m f(x^*)/dx^m$ approximately, and $\Delta^m f(x)$ converges in A (in fact, in E). This may be proved also from (2.6) for $s = m$.

We note that, for $f(x) = f_0(x)$ to be of class C^m in a general closed set A , it is not sufficient that there exist functions $f_s(x)$ ($s = 1, \dots, m$) in A such that $df_s(x)/dx = f_{s+1}(x)$ there. As an example, set $f_0(0) = 0$ and $f_0(x) = 1/2^{2^i}$ ($1/2^i \leq x \leq 3/2^{i+1}$, $i = 1, 2, \dots$), and set $f_1(x) \equiv 0$ and $f_2(x) \equiv 0$ in the same point set A .

The majority of the paper is devoted to the proof of Theorem I. In the

† Presented to the Society, October 28, 1933; received by the editors July 27, 1933.

‡ National Research Fellow.

§ Analytic extensions of differentiable functions defined in closed sets, these Transactions, vol. 36 (1934), pp. 63-89; this paper will be referred to as A.E.

last section we study Taylor's formula in finite form, when it holds in closed sets, and when its validity implies differentiability of the given function.

2. **Difference quotients.**† If x_0, \dots, x_m are distinct numbers, set‡

$$(2.1) \quad u_{ij} = x_j - x_i, \quad r_{ij} = |u_{ij}|, \quad \alpha_{01\dots m}^i = \frac{1}{u_{0i} \dots u_{i-1,i} u_{i+1,i} \dots u_{mi}}.$$

Given a function $f(x)$, we define the m th difference quotient by the formula

$$(2.2) \quad \Delta^m f(x) = \Delta(x_0, x_1, \dots, x_m; f) = \Delta_{01\dots m} f = m! \sum_{i=0}^m \alpha_{01\dots m}^i f(x_i).$$

In particular, $\Delta_0 f = f(x_0)$, $\Delta_1 f = [f(x_1) - f(x_0)]/(x_1 - x_0)$. $\Delta_{0\dots m}$ is symmetric in the points x_0, \dots, x_m .

If $i \geq 2$,

$$\frac{1}{u_{01}} (\alpha_{12\dots s}^i - \alpha_{02\dots s}^i) = \frac{1}{u_{01}} \left(\frac{1}{u_{1i} \dots} - \frac{1}{u_{0i} \dots} \right) = \frac{1}{u_{01}} \frac{u_{0i} - u_{1i}}{u_{0i} u_{1i} \dots} = \alpha_{012\dots s}^i;$$

hence

$$(2.3) \quad \begin{aligned} \frac{s}{u_{01}} (\Delta_{12\dots s} - \Delta_{02\dots s}) &= \frac{s!}{u_{01}} \left[-\alpha_{02\dots s}^0 f(x_0) + \alpha_{12\dots s}^1 f(x_1) \right. \\ &\quad \left. + \sum_{i \geq 2} (\alpha_{12\dots s}^i - \alpha_{02\dots s}^i) f(x_i) \right] \\ &= s! \sum_{i=0}^s \alpha_{012\dots s}^i f(x_i) = \Delta_{012\dots s}. \end{aligned}$$

Suppose $*$ is a set of subscripts containing neither 0, 1, nor 2; then for some m ,

$$\Delta_{012*} = \frac{m}{u_{01}} (\Delta_{12*} - \Delta_{02*}) = \frac{m}{u_{02}} (\Delta_{12*} - \Delta_{01*}).$$

Solving for Δ_{01*} , we find

$$(2.4) \quad \Delta_{01*} = \frac{u_{02}}{u_{01}} \Delta_{02*} + \frac{u_{21}}{u_{01}} \Delta_{21*},$$

which may be written as follows: $u_{01}\Delta_{01*} + u_{12}\Delta_{12*} + u_{20}\Delta_{20*} = 0$.

Let x_0, \dots, x_s be distinct numbers. If we solve the equations $\sum_{i=0}^s$

† Compare Nörlund, *Differenzenrechnung*, Berlin, 1924, pp. 8-9. It is seen that $\Delta_{01\dots m} = m! \cdot [x_0 x_1 \dots x_m]$.

‡ In the equations below, the numbers 0, 1, \dots , when appearing as subscripts, are to be considered as variables. Thus, as a particular case of (2.1), $\alpha_{023}^0 = 1/(u_{20}u_{30})$; in the second equation of §6, $\sum_i 1/u_{ji} = 1/u_{0j} + \dots$. Without this notation, the equations would often get quite cumbersome.

$(x_i - x)^j z_i = \delta_{ij}$ ($j = 0, \dots, s$), x being any fixed number, we find $z_i = \alpha_0^i \dots \alpha_s^i$.
Hence

$$(2.5) \quad \sum_{i=0}^s \alpha_0^i \dots \alpha_s^i (x_i - x)^j = 0 \quad (j = 0, \dots, s-1),$$

$$\sum_{i=0}^s \alpha_0^i \dots \alpha_s^i (x_i - x)^s = 1.$$

Suppose $f(x) = f_0(x), \dots, f_m(x)$, $R(x', x) = R_0(x', x)$ satisfy (3.1) below for $s = 0$. Then (2.2) and (2.5) give

$$(2.6) \quad \Delta_{0 \dots s} f = s! \sum_{i=0}^s \alpha_0^i \dots \alpha_s^i \left[\sum_{j=0}^m \frac{f_j(x)}{j!} (x_i - x)^j + R(x_i, x) \right]$$

$$= f_s(x) + s! \sum_{j=s+1}^m \frac{f_j(x)}{j!} \sum_{i=0}^s \alpha_0^i \dots \alpha_s^i (x_i - x)^j + s! \sum_{i=0}^s \alpha_0^i \dots \alpha_s^i R(x_i, x).$$

If $f(x) = c_0 + \dots + c_m x^m$ is a polynomial of degree at most m , then (3.1) is satisfied with $f_m(x) = m! c_m$ and $R_s(x', x) = 0$. Setting $s = m$ in (2.6) gives

$$(2.7) \quad \Delta_{0 \dots m} f = m! c_m.$$

We say $\Delta^n f(x)$ converges in the set A if for each point x of A and every $\epsilon > 0$ there is a $\delta > 0$ such that if $x_0, \dots, x_m, x_0', \dots, x_m'$ are any two sets of distinct points of A , all within δ of x , then

$$|\Delta_{0 \dots m} f - \Delta_{0' \dots m'} f| < \epsilon.$$

$\Delta^n f(x)$ of course converges at all isolated points of A . We say $\Delta^n f(x) \rightarrow f_m(x)$ in A if $|\Delta_{0 \dots m} f - f_m(x)| < \epsilon$ whenever x_0, \dots, x_m are in A and within δ of x . Evidently if $\Delta^n f(x) \rightarrow f_m(x)$ in A , then $f_m(x)$ is continuous in the set of limit points of A .

DIFFERENTIABLE FUNCTIONS

3. Definition of differentiable functions. Let $f(x) = f_0(x)$ be defined in the closed set A . We say $f(x)$ is of class C^m in A (see A. E.) if there exist functions $f_1(x), \dots, f_m(x)$, $R(x', x) = R_0(x', x), \dots, R_m(x', x)$ in A such that

$$(3.1) \quad f_s(x') = \sum_{i=s}^m \frac{f_i(x)}{(i-s)!} (x' - x)^{i-s} + R_s(x', x) \quad (s = 0, \dots, m),$$

and for each s , each point x of A , and every $\epsilon > 0$ there is a $\delta > 0$ such that

$$(3.2) \quad \left| \frac{R_s(x'', x')}{(x'' - x')^{m-s}} \right| < \epsilon \quad (x', x'' \text{ in } A; |x' - x|, |x'' - x| < \delta).$$

If $f_i(x), \dots, f_m(x)$, $R_i(x', x)$ satisfy (3.1) and (3.2) for $s = i$, we say $f_i(x)$

can be expanded in a Taylor's formula to the $(m-i)$ th order locally uniformly in terms of $f_i(x), \dots, f_m(x)$. If $f(x)$ is defined throughout an open interval and has a continuous m th derivative there, then it is of class C^m , by Taylor's theorem.

4. **Extension of differentiable functions.** If $f_0(x)$ is of class C^m in terms of $f_0(x), \dots, f_m(x)$ in A , then the definitions of these functions can be extended throughout E so they will be continuous and so that $df_s(x)/dx = f_{s+1}(x)$ there ($s=0, \dots, m-1$) (see A. E., Lemma 2). As the proof can be given more simply in the one-dimensional case, we give it here. We can assume A is unbounded on both sides; otherwise, take a point a beyond A on either side, and set $f_s(x) \equiv 0$ ($s=0, \dots, m$) beyond a .

For each interval (a, b) of $E-A$, let $P(x)$ be the polynomial of degree at most $2m+1$ such that

$$(4.1) \quad \frac{d^s}{dx^s} P(a) = f_s(a), \quad \frac{d^s}{dx^s} P(b) = f_s(b) \quad (s = 0, \dots, m);$$

we set

$$(4.2) \quad f_s(x) = \frac{d^s}{dx^s} P(x) \quad \text{in } (a, b).$$

$df_s(x)/dx = f_{s+1}(x)$ ($s=0, \dots, m-1$) in $E-A$; we must show that this holds also at any point x_0 of A .

Suppose each $f_{s+1}(x)$ is continuous in E . Then given x_0 in A and $\epsilon > 0$, take $\delta > 0$ so small that

$$|f_{s+1}(x') - f_{s+1}(x_0)| < \frac{\epsilon}{2} \quad (|x' - x_0| < \delta).$$

By (3.1) and (3.2), we can also take δ so small that if a is in A , $|a - x_0| < \delta$, and

$$f_s(a) = f_s(x_0) + f_{s+1}(x_0)(a - x_0) + R'(a, x_0),$$

then $|R'(a, x_0)/(a - x_0)| < \epsilon/2$. Now take any point x within δ of x_0 . If x is in A , set $a = x$; otherwise, let a be the end point nearest x_0 of the interval of $E-A$ containing x . Now for some x' , $a \leq x' \leq x$,

$$f_s(x) = f_s(a) + f_{s+1}(x')(x - a).$$

Adding this to the last equation and dividing by $x - x_0$, we find

$$\frac{f_s(x) - f_s(x_0)}{x - x_0} = f_{s+1}(x_0) + [f_{s+1}(x') - f_{s+1}(x_0)] \frac{x - a}{x - x_0} + \frac{R'(a, x_0)}{x - x_0}.$$

As $|x' - x_0| < \delta$, $|x - a| \leq |x - x_0|$ and $|x - x_0| \geq |a - x_0|$,

$$\left| \frac{f_s(x) - f_s(x_0)}{x - x_0} - f_{s+1}(x_0) \right| < \epsilon \quad (|x - x_0| < \delta),$$

as required. (We have given here the details of A. E., Lemma 1.)

We must prove still that each $f_s(x)$ is continuous at each point x_0 of A ; it is of course true in $E - A$. As $f_s(x)$ is continuous in A , it is sufficient to prove that for every $\epsilon > 0$ there is a $\delta > 0$ such that if (a, b) is any interval of $E - A$ lying within δ of x_0 , then

$$|f_s(x) - f_s(a)| < \epsilon \quad (a \leq x \leq b).$$

Take $\epsilon' < \epsilon / [2(m+1)^2 K]$, where K is a number to be determined later. Let M be the maximum of $|f_i(x)|$ in A ($|x - x_0| \leq 1$, $i = 0, \dots, m$). Take $\delta < \epsilon / (2mM)$ and < 1 so small that (3.2) holds with ϵ replaced by ϵ' for any x, x' within δ of x_0 . Now take any interval (a, b) of $E - A$ lying within δ of x_0 . In (a, b) , $f(x)$ equals

$$P(x) = \sum_{i=0}^m \frac{f_i(a)}{i!} (x - a)^i + \sum_{i=m+1}^{2m+1} \frac{\gamma_i}{i!} (x - a)^i,$$

where the γ_i are determined by the relations

$$\frac{d^s}{dx^s} P(b) = \sum_{i=s}^m \frac{f_i(a)}{(i-s)!} (b-a)^{i-s} + \sum_{i=m+1}^{2m+1} \frac{\gamma_i}{(i-s)!} (b-a)^{i-s} = f_s(b);$$

hence

$$\sum_{i=m+1}^{2m+1} \frac{\gamma_i}{(i-s)!} (b-a)^{i-s} = f_s(b) - \sum_{i=s}^m \frac{f_i(a)}{(i-s)!} (b-a)^{i-s} = R_s(b, a).$$

Solving for the γ_i , we find

$$\gamma_i = \sum_{j=0}^m K_{ij} \frac{R_j(b, a)}{(b-a)^{i-j}},$$

where the K_{ij} depend on m alone. Set $K = \max |K_{ij}|$; then

$$|\gamma_i| \leq \sum_{j=0}^m \frac{K}{|b-a|^{i-m}} \left| \frac{R_j(b, a)}{(b-a)^{m-j}} \right| < \frac{(m+1)K}{|b-a|^{i-m}} \epsilon'.$$

Now if x is any point in (a, b) , then $|x-a| \leq |b-a|$, and

$$\begin{aligned}
 |f_s(x) - f_s(a)| &= \left| \frac{d^s}{dx^s} P(x) - f_s(a) \right| \\
 &= \left| \sum_{i=s+1}^m \frac{f_i(a)}{(i-s)!} (x-a)^{i-s} + \sum_{i=m+1}^{2m+1} \frac{\gamma_i}{(i-s)!} (x-a)^{i-s} \right| \\
 &< mM|x-a| + (m+1)K\epsilon' \sum_{i=m+1}^{2m+1} \frac{|x-a|^{i-s}}{|b-a|^{i-m}} \\
 &< mM\delta + (m+1)^2K|b-a|^{m-s}\epsilon' < \epsilon,
 \end{aligned}$$

as required.

THEOREM I, A PERFECT

5. A succession of lemmas culminates in Lemma 7, which is the sufficiency part of Theorem I for perfect sets.

LEMMA 1. *Let A be a closed set, and let $\Delta^s f(x)$ converge on A . Then we can define $f_s(x)$ on the set of limit points A^* of A so that the following is true. Given x in A^* and $\epsilon > 0$, we can choose a $\delta > 0$ so that if x_0, \dots, x_s is any set of distinct points of A lying within δ of x , then $|\Delta_0 \dots_s f_s(x)| < \epsilon$.*

The proof is simple.

LEMMA 2. *If $\Delta^s f_0(x)$ converges in the perfect set A , then for each point x of A and every $\epsilon > 0$ there is a $\delta > 0$ such that*

$$(5.1) \quad |\Delta_0 \dots_{t-1, t \dots s-1} - \Delta_0' \dots_{(t-1)', t \dots s-1}| < \epsilon$$

($0 \leq t \leq s$) whenever all the points concerned lie within δ of x .

This is trivial if $t=0$. We assume it holds for numbers $0, \dots, t-1$, and shall prove it for t . Given a point x , distinct from all former points, the equations

$$\begin{aligned}
 \Delta_0 \dots_{t-1, t \dots s-1, s} &= \frac{s}{u_{0s}} (\Delta_1 \dots_{t-1, t \dots s} - \Delta_0 \dots_{t-1, t \dots s-1}), \\
 \Delta_0' \dots_{(t-1)', t \dots s-1, s} &= \frac{s}{u_{0's}} (\Delta_1' \dots_{(t-1)', t \dots s} - \Delta_0' \dots_{(t-1)', t \dots s-1})
 \end{aligned}$$

give

$$\begin{aligned}
 \Delta_0 \dots_{t-1, t \dots s-1} - \Delta_0' \dots_{(t-1)', t \dots s-1} &= (\Delta_1 \dots_{t-1, t \dots s} - \Delta_1' \dots_{(t-1)', t \dots s}) \\
 &\quad - \frac{1}{s} (u_{0s} \Delta_0 \dots_{t-1, t \dots s} - u_{0's} \Delta_0' \dots_{(t-1)', t \dots s}).
 \end{aligned}
 \quad (5.2)$$

As $\Delta_0 \dots_s$ converges, we can take $M > 0$ and $\delta' < \epsilon/(4M)$ so that $|\Delta_0 \dots_s| < M$ whenever x_0, \dots, x_s are within δ' of x . By induction, we can take $\delta < \delta'$ so small that the first term on the right in (5.2) is in absolute value $< \epsilon/2$ whenever all points concerned are within δ of x . Now given the points x_0, \dots, x_{s-1} ,

$x_0', \dots, x_{(t-1)'}'$ within δ of x , let x_s be another such point; then (5.2) gives (5.1).

The lemma with $t=s$ shows that $\Delta^{s-1}f_0(x)$ converges in A .

LEMMA 3. If $\Delta^m f_0(x)$ converges in the perfect set A , then there are continuous functions $f_1(x), \dots, f_m(x)$ in A such that $\Delta^s f_0(x) \rightarrow f_s(x)$ in A ($s=1, \dots, m$).

We prove this successively for $s=m, m-1, \dots, 1$ with the help of Lemmas 1 and 2.

6. We proceed to the following lemma.

LEMMA 4. If $\Delta^p g(x) \rightarrow g_p(x)$ and $\Delta^1 g(x) \rightarrow g_1(x)$ in the perfect set A , then $\Delta^{p-1} g_1(x) \rightarrow g_p(x)$ in A .

Set $q=p-1$. If we apply the relation $\Delta^1 \phi(x_i) \rightarrow d\phi(x_i)/dx_i$ to $\alpha_0^i \dots \alpha_{i-1}^i$ as a differentiable function of x_i , we find

$$\frac{\alpha_0^{i'} \dots \alpha_{i-1}^{i'} - \alpha_0^i \dots \alpha_{i-1}^i}{u_{ii'}} = -\alpha_0^i \dots \alpha_{i-1}^i \sum_{j=1}^i \frac{1}{u_{ji}} + \epsilon(x_i'),$$

where $\epsilon(x_i') \rightarrow 0$ as $x_i' \rightarrow x_i$. Hence

$$\begin{aligned} \alpha_0^i \dots \alpha_{i-1}^i \sum_j \frac{1}{u_{ji'}} &= -\frac{\alpha_0^{i'} \dots \alpha_{i-1}^{i'}}{u_{ii'}} + \alpha_0^i \dots \alpha_{i-1}^i \sum_{j=1}^i \left(\frac{1}{u_{ji'}} - \frac{1}{u_{ji}} \right) + \epsilon(x_i') \\ &= -\frac{\alpha_0^{i'} \dots \alpha_{i-1}^{i'}}{u_{ii'}} + \zeta_i(x_0', \dots, x_q'), \end{aligned}$$

where $\zeta_i(x_0', \dots, x_q') \rightarrow 0$ as $x_j' \rightarrow x_j$ ($j=0, \dots, q$). Consider the $2q$ points $x_0, x_0', \dots, x_q, x_q'$. We have

$$\begin{aligned} \Delta_{0 \dots jj' \dots q} g &= p! \left[\sum_{i=0}^q \frac{\alpha_0^i \dots \alpha_{i-1}^i}{u_{ji'}} g(x_i) + \frac{\alpha_0^{j'} \dots \alpha_{j'-1}^{j'}}{u_{jj'}} g(x_{j'}) \right], \\ \frac{1}{p} \sum_{j=0}^q \Delta_{0 \dots jj' \dots q} g &= (p-1)! \left[\sum_{i=0}^q \alpha_0^i \dots \alpha_{i-1}^i g(x_i) \sum_{j=0}^q \frac{1}{u_{ji'}} + \sum_{i=0}^q \frac{\alpha_0^{i'} \dots \alpha_{i-1}^{i'}}{u_{ii'}} g(x_i') \right] \\ &= q! \sum_{i=0}^q \left[\alpha_0^i \dots \alpha_{i-1}^i \frac{g(x_i') - g(x_i)}{u_{ii'}} + g(x_i) \zeta_i(x_0', \dots, x_q') \right]. \end{aligned}$$

As $\Delta^1 g(x) \rightarrow g_1(x)$, this gives, letting $x_j' \rightarrow x_j$ ($j=0, \dots, q$),

$$(6.1) \quad \frac{1}{p} \lim \sum_{j=0}^q \Delta_{0 \dots jj' \dots q} g = q! \sum_{i=0}^q \alpha_0^i \dots \alpha_{i-1}^i g_1(x_i) = \Delta_{0 \dots q} g_1.$$

Now given a point x of A and an $\epsilon > 0$, take $\delta > 0$ so that if $x_0, \dots, x_i, x_p, \dots, x_q$ are within δ of x , then

$$|\Delta_{0\dots ii'\dots q}g - g_p(x)| < \epsilon.$$

Then if x_0, \dots, x_q are within δ of x , we find by adding points $x_{0'}, \dots, x_{q'}$ within δ of x and letting $x_{i'} \rightarrow x_i$ ($i = 0, \dots, q$) that

$$|\Delta_{0\dots q}g_1 - g_p(x)| < \epsilon,$$

as required.

LEMMA 5. If $\Delta^n f_0(x)$ converges in the perfect set A , then there are continuous functions $f_1(x), \dots, f_m(x)$ such that $\Delta_{p+q} f_0(x) \rightarrow f_{p+q}(x)$ in A .

This follows from Lemmas 3 and 4.

7. We now present the two final lemmas needed for the proof of Theorem I when A is perfect.

LEMMA 6. Let $g(x) = g_0(x), \dots, g_s(x)$ be defined in the perfect set A , and suppose $\Delta^s g(x) \rightarrow g_s(x)$. If $g(x)$ can be expanded in a Taylor's formula to the $(s-1)$ th order in terms of $g_0(x), \dots, g_{s-1}(x)$, then it can be expanded in a Taylor's formula to the s th order locally uniformly in terms of $g_0(x), \dots, g_s(x)$.

Given a point x of A and an $\epsilon > 0$, take $\delta > 0$ so that

$$(7.1) \quad |\Delta_{0\dots s} - g_s(x_0)| < \frac{s!}{2^s} \frac{\epsilon}{3}$$

whenever x_0, \dots, x_s are within δ of x (recall that $g_s(x)$ is continuous, by §2). Take any two points x_0 and x_s of A within δ of x ; we must show that $|R^{(s)}(x_s, x_0)|/r_{0s}^s < \epsilon$.

Take δ' so small that if $|x' - x_0| < \delta'$, then

$$\left| \frac{R^{(s-1)}(x', x_0)}{(x' - x_0)^{s-1}} \right| < \frac{r_{0s}}{2^{2s}} \frac{\epsilon}{3},$$

where $R^{(s-1)}(x', x_0) = g(x') - \sum_{j=0}^{s-1} g_j(x_0)(x' - x_0)^j/j!$. Take $M > |g_s(x_0)|$. Take a point x_{s-1} in A within δ' of x_0 and so close to x_0 that

$$\frac{r_{0,s-1}}{r_{0s}} < \frac{s!}{2^s M} \frac{\epsilon}{3} \quad \text{and} \quad < \frac{1}{2},$$

and (if $s > 2$) take in succession points x_{s-2}, \dots, x_1 in A so that

$$(7.2) \quad r_{0,i-1} < \frac{1}{2} r_{0i} \quad (i = 2, \dots, s-1);$$

let these points lie within δ of x . Then if $i < s$,

$$|\alpha_{0\dots s} R^{(s-1)}(x_i, x_0)| < \frac{1}{r_{0i}} \frac{r_{0i}^{s-1}}{r_{0i} \cdots r_{i-1,i} r_{i+1,i} \cdots r_{s-1,i}} \frac{r_{0s}}{2^{2s}} \frac{\epsilon}{3} < \frac{1}{2^s} \frac{\epsilon}{3}.$$

Now

$$\begin{aligned} \frac{1}{s!} \Delta_{0\dots s} &= \alpha^s g(x_s) + \sum_{i=0}^{s-1} \alpha^i \left[\sum_{j=0}^{s-1} \frac{g_j(x_0)}{j!} u_{0i}^j + R^{(s-1)}(x_i, x_0) \right] \\ &= \alpha^s g(x_s) - \alpha^s \sum_{j=0}^{s-1} \frac{g_j(x_0)}{j!} u_{0s}^j + \sum_{i=0}^{s-1} \alpha^i R^{(s-1)}(x_i, x_0), \end{aligned}$$

on account of (2.5). Therefore

$$\begin{aligned} (7.3) \quad R^{(s)}(x_s, x_0) &= g(x_s) - \sum_{j=0}^s \frac{g_j(x_0)}{j!} u_{0s}^j \\ &= \frac{\Delta_{0\dots s}}{s! \alpha^s} - \frac{g_s(x_0) u_{0s}^s}{s!} - \sum_{i=0}^{s-1} \frac{\alpha^i}{\alpha^s} R^{(s-1)}(x_i, x_0), \end{aligned}$$

and as $r_{is}/r_{0s} \leq (r_{0s} + r_{0i})/r_{0s} = 1 + r_{0i}/r_{0s}$,

$$\begin{aligned} \frac{|R^{(s)}(x_s, x_0)|}{r_{0s}^s} &\leq \frac{r_{0s} \cdots r_{s-1,s}}{s! r_{0s}^s} |\Delta_{0\dots s} - g_s(x_0)| + \frac{|g_s(x_0)|}{s!} \left| \frac{r_{0s} \cdots r_{s-1,s}}{r_{0s}} - 1 \right| \\ &\quad + \sum_{i=0}^{s-1} \frac{r_{0s} \cdots r_{s-1,s}}{r_{0s}} |\alpha^i R^{(s-1)}(x_i, x_0)| \\ &< \frac{2^s}{s!} \frac{s!}{2^s} \frac{\epsilon}{3} + \frac{M}{s!} \left[\left(1 + \frac{r_{00}}{r_{0s}} \right) \cdots \left(1 + \frac{r_{0,s-1}}{r_{0s}} \right) - 1 \right] + s \cdot 2^s \frac{1}{2^s} \frac{\epsilon}{3} < \epsilon, \end{aligned}$$

as required.

LEMMA 7. If $\Delta^n f(x)$ converges in the perfect set A , then $f_0(x) = f(x)$, $f_1(x), \dots, f_m(x)$ can be defined in A so that $f(x)$ is of class C^m in A in terms of the $f_s(x)$ ($s=0, \dots, m$).

We define $f_1(x), \dots, f_m(x)$ by means of Lemma 5. Taylor's formula for each $f_s(x)$ holds to the 0th order, as $f_s(x)$ is continuous (see §2). We prove in succession that it holds to the k th order for $k=1, \dots, m-s$. This completes the proof of the lemma, and therefore of Theorem I for the case that A is perfect.

P-SETS AND Q-SETS

8. We shall prove a lemma which will be needed in the next part. Let $A' = a_1, a_2, \dots$ be a set of isolated points, at least $m+1$ in number. With each point a , we shall associate m other points a_{i1}, \dots, a_{im} ; these m points

together with a_i we say form the Q -set $Q(a_i)$. Take a Q -set Q_i , and let $a_{i_1}, \dots, a_{i_\mu}$ be all those points such that $Q(a_{i_t}) = Q_i$; these points form the P -set P_i corresponding to Q_i . Each point of P_i is in Q_i . Each point a_i lies in just one P -set $P(a_i)$, as a_i is associated with just one Q -set $Q(a_i)$; however, a_i may lie in several Q -sets. Let $\delta(Q_i)$ be the greatest distance between pairs of points of Q_i .

LEMMA 8. *The P -sets and Q -sets may be so chosen that for any two points a_i and a_j ,*

$$(8.1) \quad \text{if } \frac{\delta(Q(a_i)) + \delta(Q(a_j))}{|a_j - a_i|} > 2m, \text{ then } P(a_i) = P(a_j).$$

We first associate sets of points with certain of the limit points of the points a_1, a_2, \dots as follows. Let c_i be a point such that there is a sequence of points of A' approaching it from one side, say the left, while there is a nearest point of A' to c_i on the other side of c_i . Let μ equal $m+1$, or the number of points a_j between c_i and the next limit point c_k to the right of c_i if that number is smaller, and let $a_{j_1}, \dots, a_{j_\mu}$ be the points nearest c_i on the right (counting from left to right). Let τ be the smallest of the numbers $|a_j - a_{i_s}|$ ($s, t = 1, \dots, \mu$) which are $> |a_{j_1} - c_i|$, if there are such. Let $a_1(c_i)$ be a point of A' to the left of c_i such that

$$(8.2) \quad |c_i - a_1(c_i)| < |a_{j_1} - c_i|, \text{ and } |a_{j_1} - a_1(c_i)| < \tau$$

if τ is defined. Let $a_2(c_i), \dots, a_m(c_i)$ be points of A' lying between c_i and $a_1(c_i)$.

We now define the Q -sets. Given a point a_i , we associate another point with it as follows. Suppose, Case I, there is a point a_j whose distance from a_i is less than or equal to the distance from any other a_k to a_i ; then we associate a_j with a_i , or that one of the pair a_j, a_k which lies to the left of a_i , if their distances from a_i are the same. Suppose, Case II, there is no such point. Then there is a limit point c_i nearer a_i than any point a_j . If there are two such points, we consider that one c_i on the left. The point we associate with a_i is then $a_1(c_i)$.

Suppose now we have associated a number of points with a_i , forming the set of points S . We associate the next point in a fashion much the same as above. If Case II has not occurred in associating the other points of S with a_i , we again have two cases to consider. Case I, there is a nearest point a_j to the set S ; we then associate this point with S (or the point a_k , as above). Case II, there is none; then take the point c_i as above, and associate $a_1(c_i)$ with S . At any time we employ Case II, we immediately associate also the

points $a_2(c_i), a_3(c_i), \dots$ with S , till we have the required $m+1$ points $Q(a_i)$.

Note that the point we associate with S does not depend on which point a_i of S we started with. Also if Case I has occurred each time in forming the subset S of $Q(a_i)$, then there is no point a_k not in S which lies between two points of S .

9. To prove that (8.1) holds take any two points a_i and a_j ; set $r_{ij} = |a_j - a_i|$.

(1) Suppose there are at most a finite number of points of A' between a_i and a_j . If $\delta(Q(a_i)) + \delta(Q(a_j)) > 2mr_{ij}$, then either $\delta(Q(a_i)) > mr_{ij}$ or $\delta(Q(a_j)) > mr_{ij}$, say the former. Then there is a first time when, on adding a point a_k to a set S in forming $Q(a_i)$, the distance from a_k to S is $> r_{ij}$.

(a) In forming S from a_i , Case I has occurred each time. For if Case II had occurred, say in adding the point $a_1(c_i)$ to the subset S_1 of S , then a_k would be some $a_s(c_i)$; but the distance from a_k to S is then at most the distance from $a_s(c_i)$ to $a_1(c_i)$ which is less than the distance from $a_1(c_i)$ to S_1 which is by hypothesis $\leq r_{ij}$.

(b) There is no point a_s whose distance from S is $\leq r_{ij}$. For suppose there were; then Case II must occur in adding $a_k = a_1(c_i)$ to S , and c_i is nearer S than any point a_r . (If Case I occurred, a_s or a nearer point, not a_k , would be added to S .) Say c_i lies to the left of S . Let a_p and a_q be the left and right-hand end points of S respectively. As there is a point a_s distant $\leq r_{ij}$ from S , $|a_p - c_i| < r_{ij}$. Suppose a_j is not in S . As there are no limit points between a_i and a_j , a_j lies to the right of S , and hence there is a first point a_r to the right of S . Then as a_i is in S , $|a_r - a_q| \leq r_{ij}$. But as a_q and a_r are among the first $m+1$ (or μ) points to the right of c_i , and $|a_p - c_i| < |a_r - a_q|$, (8.2) gives $|a_p - a_k| < |a_r - a_q| \leq r_{ij}$, a contradiction; therefore a_j is in S . As a_j is in S and $|a_p - c_i| < r_{ij}$, (8.2) gives $|a_p - a_k| < r_{ij}$, again a contradiction.

(c) S contains a_j . For otherwise (b) would be contradicted.

(d) In forming $Q(a_j)$, the points of S are chosen first. For suppose not. Then after perhaps adding some points of S to a_j , forming the set S' , we choose a point a_i not in S . By (b), the distance from a_i to S is $> r_{ij}$. As there is a point in S whose distance from S' is at most r_{ij} , a_i must have been chosen under Case II; then the distance from some point c_j to S' is $< r_{ij}$. But then as c_j is a limit point of points a_s , there is a point a_s whose distance from S is $< r_{ij}$, a contradiction.

Now in forming both $Q(a_i)$ and $Q(a_j)$, the points of S are chosen first. As the remaining points chosen depend only on S , $Q(a_i)$ and $Q(a_j)$ must coincide; hence a_i and a_j lie in the same P -set.

(2) Suppose there is a limit point of isolated points b between a_i and a_j . In forming $Q(a_i)$, the set S at any step is at a distance $\leq |b - a_i|$ from b ; hence in adding the next point a_k to S , its distance from S is $\leq |b - a_i|$ if Case I

occurs, and is $< 2|b - a_i|$ if Case II occurs, by (8.2). Therefore $\delta(Q(a_i)) < 2m|b - a_i|$. Similarly $\delta(Q(a_j)) < 2m|b - a_j|$. Adding,

$$\delta(Q(a_i)) + \delta(Q(a_j)) < 2m(|b - a_i| + |b - a_j|) = 2mr_{ij},$$

completing the proof.

Remark. Given a point a_i , if there exist m points a_{i_1}, \dots, a_{i_m} such that the m intervals between $a_i, a_{i_1}, \dots, a_{i_m}$ are all $\leq \rho$, or if there exists a point a not in $Q(a_i)$ within ρ of a_i , then $\delta(Q(a_i)) < 2m\rho$. This follows from the proof in (2).

THEOREM I, A CLOSED

Each isolated point of A is enclosed in an interval; this gives a perfect set B . The definition of $f(x)$ is extended over B . With the help of Lemma 8 it is shown that $\Delta^m f(x)$ now converges over B . By Lemma 7, $f(x)$ is of class C^m in B ; hence the same is true in A .

10. The sets A' and B . Let A_1 be the set of isolated points of the closed set A , let A_2 be the set of limit points of isolated points, and let A_3 be the remaining points of A . Let A' consist of A_1 , together with certain other points as follows. $A_1 + A_2$ being closed, let I be any open interval of $E - (A_1 + A_2)$ containing points of A_3 . If an end point a_i of I is in A_1 , then there is, in I , a nearest point $a_1(a_i)$ of A_3 to a_i . We associate this point with a_i , and also points $a_2(a_i), \dots, a_m(a_i)$ of A_3 in I , chosen so that

$$(10.1) \quad |a_s(a_i) - a_1(a_i)| < |a_1(a_i) - a_i| \quad (s = 2, \dots, m).$$

A' is a set of isolated points; we may name them a_1, a_2, \dots . A' is contained in $A_1 + A_3$.

For each point a_i of A_1 , let $d(a_i)$ be its distance from the rest of A , and let B_i be a closed interval of length $d(a_i)/2$, with a_i as center. Let the perfect set B be A plus all of these intervals. Arrange the points of A' into P -sets and Q -sets so as to obey Lemma 8. For each P -set P_i , let the corresponding P' -set P'_i contain the points of P_i , together with the points of any intervals B_j there may be which enclose points of P_i .

Given any set S of $m+1$ points in B , we shall define its complexity $\sigma(S)$ as follows. If all the points of S are in A , set $\sigma(S) = 0$. If S contains $p > 0$ points in $B - A$, and all these points lie in a single P' -set P'_i , let q be the number of remaining points of S which do not lie in the corresponding Q -set Q_i , and set $\sigma(S) = pq$. The complexity of S is in this case certainly $\leq m^2$. If S contains p points in $B - A$, and these points do not all lie in the same P' -set, set $\sigma(S) = m^2 + p - 1$. The complexity of any set S is $\leq m^2 + m$.

11. The following lemma together with Lemma 7 gives Theorem I.

LEMMA 9. Let $f(x)$ be defined in the closed set A so that $\Delta^m f(x)$ converges in A . Then A can be enclosed in a perfect set B , the definition of $f(x)$ can be extended over B , and $f_m(x)$ can be defined in B , so that $\Delta^m f(x) \rightarrow f_m(x)$ in B .

Define the sets A' , B etc. as above. We may assume there are at least $m+1$ points in A' . Define $f_m(x)$ at each point of $A_2 + A_3$ as in Lemma 1. Take a fixed interval B_i with center a_i ; we define $f(x)$ and $f_m(x)$ over B_i as follows. Let $Q_i = Q(a_i)$ be the corresponding Q -set. Let

$$(11.1) \quad R_i(x) = \gamma_0 + \cdots + \gamma_m x^m$$

be the polynomial of degree at most m such that $R_i(x) = f(x)$ at each point of Q_i ; then $\Delta(Q_i) = m! \gamma_m$, by (2.7). Set

$$(11.2) \quad f(x) = R_i(x), \quad f_m(x) = m! \gamma_m \text{ in } B_i.$$

The same polynomial $R_i(x)$ is used in defining $f(x)$ and $f_m(x)$ over each interval of the P' -set P'_i corresponding to Q_i ; hence if S is any set of $m+1$ points such that all of its points in $B-A$ lie in P'_i , and all remaining points lie in Q_i , then $f(x) = R_i(x)$ at each point of S , and therefore, by (2.7), $\Delta(S) = m! \gamma_m = \Delta(Q_i)$.

Each point x of A_3 is at a positive distance from $B-A$; by the definition of $f_m(x)$, $\Delta^m f(x) \rightarrow f_m(x)$ at such points. Each point x of $B - (A_2 + A_3)$ is in an interval B_i ; hence near x , $f(x)$ is a polynomial, and $\Delta^m f(x) \rightarrow f_m(x)$ there also. It remains to show that for each point x of A_2 and every $\epsilon > 0$ there is a $\delta > 0$ such that if S is any set of $m+1$ points of B within δ of x , then

$$(11.3) \quad |\Delta(S) - f_m(x)| < \epsilon.$$

By Lemma 1, we can take $\delta' > 0$ so that

$$(11.4) \quad |\Delta(S_0) - f_m(x)| < \frac{\epsilon}{(8m+8)^{m^2+m}}$$

for any set S_0 of $m+1$ points of A lying within δ' of x . Set $\delta = \delta'/(4m+2)$. We shall prove the following:

(A) If S is any set in B , of complexity $\sigma(S) = \sigma$, composed of sets of points S_1 in $B-A$ and S_2 in A , and if S_1 lies within δ of x and S_2 lies within δ' of x , then

$$(11.5) \quad |\Delta(S) - f_m(x)| < \epsilon_\sigma = \frac{\epsilon}{(8m+8)^{m^2+m-\sigma}}.$$

As $\sigma \leq m^2 + m$, (11.3) follows.

12. We note first that if b_i is in some interval of the P' -set P'_i , and b_i lies within δ of x , then Q_i lies within δ' of x . Say a_i is the center of B_i ; then a_i lies

within 2δ of x , a limit point of points of A' . Hence $\delta(Q(a_i)) < 4m\delta$, by the remark at the end of §9, and $Q_i = Q(a_i)$ lies within $(4m+2)\delta = \delta'$ of x .

We shall prove (A) first for $\sigma=0$, then for $\sigma>0$, using induction. Suppose $\sigma=0$. If S is in A , the fact follows from (11.4). If S contains points of $B-A$, then all these points lie in a single P' -set P'_i , and the rest of S lies in the corresponding Q -set Q_i ; hence $\Delta(S) = \Delta(Q_i)$. Q_i lies within δ' of x ; hence (11.4) holds with S_0 replaced by Q_i or by S , and therefore (11.5) holds.

Now suppose (11.5) is proved for all sets S' with $\sigma(S') < \sigma$; we shall prove it for any set S with $\sigma(S) = \sigma$. Suppose first $\sigma > m^2$; then the points of S in $B-A$ lie in at least two P' -sets. Let P'_i and P'_j be two of these sets, let b_i and b_j be points of S (in $B-A$) in P'_i and P'_j respectively, and let a_i and a_j be the centers of the corresponding intervals. Let a_k be a point of $Q(a_i)$ not lying in S . If $S' = S - b_i - b_j$, then, by (2.4),

$$(12.1) \quad \Delta(S) = \Delta(b_i, b_j, S') = \frac{a_k - b_i}{b_j - b_i} \Delta(b_i, a_k, S') + \frac{b_j - a_k}{b_j - b_i} \Delta(a_k, b_j, S').$$

The sets $S' + b_i + a_k$ and $S' + b_j + a_k$ each contain fewer points of $B-A$ than S ; hence their complexities are each $< \sigma$. Also $Q(a_i)$ and therefore a_k lie within δ' of x . Therefore, by induction,

$$(12.2) \quad |\Delta(b_i, a_k, S') - f_m(x)| < \epsilon_{\sigma-1}, \quad |\Delta(a_k, b_j, S') - f_m(x)| < \epsilon_{\sigma-1}.$$

As a_i and a_j lie in distinct P -sets, $\delta(Q(a_i)) + \delta(Q(a_j)) \leq 2mr_{ij}$, by (8.1). As $|b_i - a_i| \leq r_{ij}/4$ and $|b_j - a_j| \leq r_{ij}/4$, $|b_j - b_i| \geq r_{ij}/2$. As a_k and a_j lie in $Q(a_i)$ and $Q(a_j)$ respectively, $|a_j - a_k| \leq \delta(Q(a_i)) + \delta(Q(a_j)) + r_{ij} \leq (2m+1)r_{ij}$; hence $|b_j - a_k| < (2m+2)r_{ij}$. Also $|a_k - b_i| \leq \delta(Q(a_i)) + |a_i - b_i| < (2m+2)r_{ij}$; hence

$$(12.3) \quad \left| \frac{a_k - b_i}{b_j - b_i} \right| < 4m + 4, \quad \left| \frac{b_j - a_k}{b_j - b_i} \right| < 4m + 4.$$

This with (12.2) and (12.1) gives

$$\begin{aligned} |\Delta(S) - f_m(x)| &< \left| \frac{a_k - b_i}{b_j - b_i} \right| |\Delta(b_i, a_k, S') - f_m(x)| \\ &\quad + \left| \frac{b_j - a_k}{b_j - b_i} \right| |\Delta(a_k, b_j, S') - f_m(x)| \\ &< (8m+8)\epsilon_{\sigma-1} = \epsilon_\sigma, \end{aligned}$$

as required.

Suppose now $0 < \sigma \leq m^2$; then the points of S in $B-A$ lie in a single P' -set P'_i , and there are points of S not in $P'_i + Q_i$. Let b_i be a point of S in $B-A$, let a be a point of S not in $P'_i + Q_i$, and let a_k be a point of Q_i which is not in S . If $S' = S - b_i - a$, the sets $S' + b_i + a_k$ and $S' + a + a_k$ each have a smaller

complexity than S . a_k lies within δ' of x , and hence, by induction, (12.2) holds with b_i replaced by a . Let a_i be the center of the interval B_i containing b_i .

Suppose, (1), $a = a_j$ is in A' . Then $|a_j - b_i| > r_{ij}/2$. As a_k is in $Q_i = Q(a_i)$ while a_j is not, $|a_k - b_i| < |a_k - a_i| + r_{ij} < (2m+1)r_{ij}$, by the remark, and $|a_j - a_k| < (2m+1)r_{ij}$. Hence (12.3) holds with b_i replaced by $a_j = a$, and (11.5) follows just as before. Suppose, (2), a is in $A - (A' + A_2)$. From a , move toward a_i to the first point a' in $A_1 + A_2$. If a' is in A_1 , move back to the first point $a_1(a')$ in A_3 . Then $|a_1(a') - a'| \leq |a - a_i|$ and $|a_2(a') - a_1(a')| < |a_1(a') - a'| \leq |a - a_i|$ ($s=2, \dots, m$), by (10.1). Hence $\delta(Q_i) < 2m|a - a_i|$, by the remark, and $|a_k - b_i| < (2m+1)|a - a_i|$, and $|a - a_k| < (2m+1)|a - a_i|$. As $|a - b_i| > |a - a_i|/2$, (12.3) and (11.5) follow, as before. If a' is in A_2 , there are m points of A' nearer a_i than a , and again $\delta(Q_i) < 2m|a - a_i|$ and (11.5) follows. Suppose finally, (3), a is in A_2 . Again we must have $\delta(Q_i) < 2m|a - a_i|$ and (11.5) follows. This completes the proof of (A), therefore of Lemma 9, and therefore of Theorem I.

TAYLOR'S FORMULA

13. Conditions under which Taylor's formula is valid. Taylor's formula for $f(x)$ may hold to the m th order in certain closed sets even if $f(x)$ is not of class C^m (see §14). We find here a difference quotient condition equivalent to the validity of Taylor's formula, at least for perfect sets.

LEMMA 10. If $f(x) = f_0(x)$ can be expanded in a Taylor's formula to the m th order locally uniformly in terms of $f_0(x), \dots, f_m(x)$ in the closed set A , then these functions are continuous in A .

It is apparent from (3.1) and (3.2) with $s=0$ that $f_0(x)$ is continuous. Take any $s, 0 < s \leq m$. We shall assume $f_j(x)$ is continuous for $s < j \leq m$, if there are such values of j , and shall prove that $f_s(x)$ is continuous.

Let x_0, \dots, x_s be distinct points of A . If we subtract (2.6) with x replaced by x_0 from the same equation with x replaced by x_1 , we find

$$\begin{aligned} f_s(x_1) - f_s(x_0) &= s! \sum_{j=s+1}^m \left[\frac{f_j(x_0)}{j!} \sum_{i=0}^s \alpha^i u_{0i}^j - \frac{f_j(x_1)}{j!} \sum_{i=0}^s \alpha^i u_{1i}^j \right] \\ (13.1) \quad &+ s! \sum_{i=0}^s \alpha^i [R(x_i, x_0) - R(x_i, x_1)]. \end{aligned}$$

Given any limit point x_0 of A and any $\epsilon > 0$, take $\delta < \epsilon/[2^{s+3}(s+1)mM]$ (if $s < m$) and $< 1/2$ so small that (3.2) holds with x and ϵ replaced by x_0 and $\epsilon/[2^{s+3}(s+1)!]$ respectively, where $M = \max |f_j(x')|$ ($|x' - x_0| \leq 1, s < j \leq m$). If $s > 1$, take a point x_s of A within δ of x_0 , and take points x_{s-1}, \dots, x_2 of A so that $r_{0i} < r_{0, i+1}/3$ ($i=2, \dots, s-1$). Now take any point x_1 within δ

of x_0 , so that $r_{01} < r_{02}/3$ if $s > 1$. From (13.1) we see that $|f_s(x_1) - f_s(x_0)| < \epsilon$, as required (see the proof of Lemma 6).

Let x_0, \dots, x_s be an ordered set of points. We say they form an (x_0, ρ) -set ($\rho > 1$), if

$$(13.2) \quad r_{0,i-1} < \frac{r_{0i}}{\rho} \quad (i = 1, \dots, s).$$

THEOREM II. Let $f(x) = f_0(x), \dots, f_m(x)$ be defined in the closed set A . A necessary condition that a Taylor's expansion for $f(x)$ should hold to the m th order locally uniformly in terms of $f_0(x), \dots, f_m(x)$ is that for each (or some) $\rho > 1$, each s ($0 \leq s \leq m$), each point x of A , and each $\epsilon > 0$, there exist a $\delta > 0$, such that if x_0, \dots, x_s is any (x_0, ρ) -set of points lying within δ of x , then

$$|\Delta_0 \dots_s f - f_s(x)| < \epsilon.$$

By the last lemma, the $f_i(x)$ are continuous. Take M so that $|f_i(x')| < M$ for $|x' - x| < 1$. Take $\delta < \epsilon(\rho - 1)^s / [2(s+1)mM\rho^s]$ and < 1 so that $|f_s(x') - f_s(x)| < \epsilon/2$ ($|x' - x| < \delta$), and so that (3.2) holds with ϵ replaced by $\epsilon(\rho - 1)^s / [2(s+1)! \rho^s]$. Now take any (x_0, ρ) -set of points x_0, \dots, x_s lying within δ of x . Then

$$\frac{r_{0i}}{r_{ki}} < \frac{\rho}{\rho - 1}$$

for $k \neq i$. For if $k < i$, then $r_{0k} \leq r_{0,i-1} < r_{0i}/\rho$, hence $r_{ki} \geq r_{0i} - r_{0k} > r_{0i}(1 - 1/\rho)$, and $r_{0i}/r_{ki} < 1/(1 - 1/\rho) = \rho/(\rho - 1)$; if $k > i$, then $r_{0k} \geq r_{0,i+1} > \rho r_{0i}$, hence $r_{ki} \geq r_{0k} - r_{0i} > r_{0i}(\rho - 1)$, and $r_{0i}/r_{ki} < 1/(\rho - 1) < \rho/(\rho - 1)$. Replacing x by x_0 in (2.6) gives immediately $|\Delta_0 \dots_s f - f_s(x_0)| < \epsilon/2$; hence $|\Delta_0 \dots_s f - f_s(x)| < \epsilon$.

THEOREM III. If A is perfect, then the condition in Theorem II is also sufficient.

We shall prove successively for $s = 0, \dots, m$ that $f(x)$ can be expanded in a Taylor's formula to the s th order locally uniformly in terms of $f_0(x), \dots, f_s(x)$. Evidently $f_0(x)$ is continuous; hence this is true for $s = 0$. The proof for a general s follows the proof of Lemma 6; we need merely be careful to choose x_{s-1}, \dots, x_1 so that $r_{0,i-1} < r_{0i}/\rho$ ($i = 2, \dots, s$).

14. Taylor's formula and differentiability. We shall say the set A has the property Z_ρ at the point x ($\rho > 1$) if there is an $\eta > 0$ such that corresponding to any two points x_0 and x_1 of A within η of x , points x_2, \dots, x_s of A can be found such that

$$(14.1) \quad \frac{1}{\rho} < \left| \frac{x_i - x_j}{x_1 - x_0} \right| < \rho \quad (i, j = 0, \dots, s; i \neq j);$$

then $r_{ij}/r_{kl} < \rho^2$ for $i \neq j, k \neq l$. This condition is satisfied for instance by Cantor's set. s is any number $\leq m, m$ fixed.

THEOREM IV.* Let A be a closed set having the property Z_ρ for some $\rho = \rho(x)$ at each point x , and let $f(x) = f_0(x), \dots, f_m(x)$ be defined in A . A necessary and sufficient condition that $f(x)$ be of class C^m in terms of $f_0(x), \dots, f_m(x)$ is that Taylor's formula for $f(x)$ should hold to the m th order locally uniformly in terms of $f_0(x), \dots, f_m(x)$.

In short, in this case, Taylor's formula for $f_0(x)$ implies Taylor's formula for each $f_s(x)$.

The necessity of the condition being trivial, we turn to the sufficiency. By Lemma 10, $f_m(x)$ is continuous. It remains to prove that for any $s, 0 < s < m, f_s(x)$ may be expanded in a Taylor's formula to the $(m-s)$ th order locally uniformly in terms of $f_s(x), \dots, f_m(x)$. We shall prove this for s , assuming it for numbers $s+1, \dots, m$.

Let x_0, \dots, x_s be distinct points of A . Set

$$(14.2) \quad H_j = \sum_{i=0}^s \alpha^i u_{0i}^j,$$

$$(14.3) \quad H_j' = \sum_{i=0}^s \alpha^i u_{1i}^j = \sum_{i=0}^s \alpha^i (u_{0i} - u_{01})^j = \sum_{i=0}^s \alpha^i \sum_l (-1)^{j-l} \binom{j}{l} u_{0i}^l u_{01}^{j-l} \\ = \sum_l (-1)^{j-l} \binom{j}{l} H_l u_{01}^{j-l},$$

where \sum_l means summation over all values of l . We can write (if $s < m$)

$$\sum_{j=s+1}^m \frac{f_j(x_1)}{j!} H_j' = \sum_{j=s+1}^m \frac{1}{j!} \sum_{k=j}^m \frac{f_k(x_0)}{(k-j)!} u_{01}^{k-j} \sum_l (-1)^{j-l} \binom{j}{l} H_l u_{01}^{j-l} + R \\ = \sum_{k=s+1}^m \frac{f_k(x_0)}{k!} \sum_l u_{01}^{k-l} H_l \sum_{j=s+1}^k (-1)^{j-l} \binom{k}{j} \binom{j}{l} + R,$$

where

$$R = \sum_{j=s+1}^m \frac{1}{j!} H_j' R_j(x_1, x_0).$$

Now if $k \geq l > s$, then on replacing j by $k-j$ we find

* For the special case that A is a closed interval, see a paper by the author, *Derivatives, difference quotients and Taylor's formula*, Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 89-94; Theorem III.

$$\sum_{j=s+1}^k (-1)^{j-l} \binom{k}{j} \binom{j}{l} = \sum_j (-1)^{k-j-l} \binom{k}{j} \binom{k-j}{l-j} \\ = (-1)^{k-l} \binom{0}{k-l} = \delta_{kl},^*$$

and if $k > l = s$,

$$\sum_{j=s+1}^k (-1)^{j-l} \binom{k}{j} \binom{j}{l} = \delta_{ks} - (-1)^{s-s} \binom{k}{s} \binom{s}{s} = -\binom{k}{s}.$$

Therefore, as $H_l = 0$ ($l < s$) and $H_s = 1$,

$$(14.4) \quad \sum_{j=s+1}^m \frac{f_j(x_1)}{j!} H_j' = \sum_{k=s+1}^m \frac{f_k(x_0)}{k!} \left[-\binom{k}{s} u_{01}^{k-s} + H_k \right] + R.$$

Putting this in (13.1) gives

$$(14.5) \quad f_s(x_1) = \sum_{k=s}^m \frac{f_k(x_0)}{(k-s)!} u_{01}^{k-s} - \sum_{j=s+1}^m \frac{s!}{j!} \sum_{i=0}^j \alpha^i u_{1i}^j R_j(x_1, x_0) \\ + s! \sum_{i=0}^s \alpha^i [R(x_i, x_0) - R(x_i, x_1)].$$

Given a point x of A and an $\epsilon > 0$, take ρ and η corresponding to x , and take $\delta' < \eta$ so that (3.2) holds with δ and ϵ replaced by δ' and $\epsilon/[3m(s+1)! \rho^{2m}]$ and with s taking on the values $0, s+1, \dots, m$. Set $\delta = \delta'/(2\rho)$. Now if x_0 and x_1 are points of A within δ of x , we can add points x_2, \dots, x_s of A so that (14.1) holds and these points will lie within δ' of x . Then

$$\left| \frac{\alpha^s u_{1s}^s}{u_{01}^{m-s}} R_s(x_1, x_0) \right| = \frac{r_{1s}^s}{r_{0s} \cdots r_{i-1,s} r_{i+1,s} \cdots r_{s1} r_{01}^{m-s}} \frac{|R_s(x_1, x_0)|}{r_{01}^{m-s}} < \frac{\epsilon}{3m(s+1)!},$$

and similarly for the other remainder terms. Therefore $|R_s(x_1, x_0)|/r_{01}^{m-s} < \epsilon$, as required.

COROLLARY. If $m \leq 2$, Theorem IV holds for all closed sets.

The only value of s we may need in the above proof is $s = 1$; the condition Z_ρ is satisfied trivially if $s = 1$.

Example. Theorem IV does not hold for all closed sets, as we now show, using $m = 3$. Set $a_i = 1/2^i$, $b_i = 1/2^{2i}$, $c_i = 1/2^{3i}$; $b_i' = a_i + b_i$, $c_i' = a_i + c_i$, $d_i = a_i + b_i - c_i$ ($i = 1, 2, \dots$). Let A be the set of points $0, a_i, c_i', d_i, b_i'$. Set $f_0(0) = f_1(0) = f_2(0) = f_3(0) = 0$,

* See Netto, *Lehrbuch der Combinatorik*, Leipzig, 1927, §158, (27).

$$\begin{array}{llll}
f_0(a_i) = 0, & f_0(c'_i) = 0, & f_0(d_i) = 0, & f_0(b'_i) = b_i^2 c_i, \\
f_1(a_i) = 0, & f_1(c'_i) = 0, & f_1(d_i) = b_i^2 - b_i c_i, & f_1(b'_i) = b_i^2 + b_i c_i, \\
f_2(a_i) = 0, & f_2(c'_i) = 0, & f_2(d_i) = 2b_i, & f_2(b'_i) = 2b_i, \\
f_3(a_i) = 0, & f_3(c'_i) = 0, & f_3(d_i) = 0, & f_3(b'_i) = 0.
\end{array}$$

As $\Delta(0, a_i, c'_i, d_i) = 0$, while $\Delta(a_i, c'_i, d_i, b'_i) = 3! b_i^2 c_i / [b_i(b_i - c_i)c_i] \rightarrow 6$ as $i \rightarrow \infty$, $\Delta^3 f_0(x)$ does not converge at $x = 0$, and hence $f_0(x)$ is not of class C^3 , by Theorem I. However, Taylor's formula holds for $f_0(x)$ to the third order locally uniformly. For a calculation shows that $R(x, y) = 0$ whenever x and y are chosen from the points a_i, c'_i, d_i, b'_i , except that $R(b'_i, a_i) = R(b'_i, c'_i) = b_i^2 c_i$, $R(c'_i, d_i) = R(c'_i, b'_i) = b_i c_i (b_i - 2c_i)$; hence if x and y are chosen in any manner from the points a_i, c'_i, d_i, b'_i , $R(y, x)/(y-x)^3 \rightarrow 0$ as $i \rightarrow \infty$. Suppose now x_i and y_i are chosen from a_i, c'_i, d_i, b'_i , and from a_i, c'_i, d_i, b'_i respectively, $j \neq i$ (or $x_i = 0$ or $y_i = 0$). If k is the larger of the numbers i, j , then

$$|R(y_i, x_i)| < 2b_k^2 c_k + (b_k^2 + b_k c_k)(a_k + b_k) + b_k(a_k + b_k)^2,$$

and as $|y_i - x_i|^3 \geq a_k^3/8$, $R(y_i, x_i)/(y_i - x_i)^3 \rightarrow 0$ as $i, j \rightarrow \infty$ ($j \neq i$). Hence for some $\delta > 0$, if x and y are any two points of A within δ of 0, $|R(y, x)/(y-x)^3| < \epsilon$. This is true also at each isolated point of A ; hence Taylor's formula is valid.

Note that we may increase A to a perfect set by adding the intervals between a_i and c'_i and between d_i and b'_i , and giving the obvious definitions of $f_0(x), \dots, f_3(x)$ there. In this example, Taylor's formula holds to the required order for neither $f_1(x)$ nor $f_2(x)$.

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NORMAL DIVISION ALGEBRAS OVER A MODULAR FIELD*

BY
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1. Introduction. Let $\phi(\omega)=0$ have coefficients in a modular field F of characteristic p and be irreducible in F . Then $\phi(\omega)=0$ and the field $F(x)$ generated by any one of its roots x are called separable or inseparable according as $\phi(\omega)=0$ has not or has multiple roots. It is well known[†] that if $\phi(\omega)=0$ is inseparable, then

$$\phi(\omega) \equiv \sum_i \alpha_i \omega^{p^i} \quad (\alpha_i \text{ in } F),$$

and that there exist inseparable extensions $F(x)$ of F if and only if some quantity α of F is not the p th power of any quantity of F .

An infinite field F is called perfect if either F is non-modular or every quantity of F has the form β^p where p is the characteristic of F and β is in F . In any consideration of normal division algebras D over F the property that F is perfect is used only when we consider quantities of D and the minimum equations of these quantities. But if the degree n of D is not divisible by the characteristic p of F , then the assumption that F is perfect evidently has no value and is a needless extremely strong restriction on F .

In most of the papers on the structure of normal division algebras written recently in Germany[‡], the assumption has been that F is perfect. But I shall prove here that if F is perfect of characteristic p , then n is not divisible by p . Hence it is now necessary to consider algebras of degree p^e over F of characteristic p , where F is not perfect.

I shall give here a brief discussion of the validity of the major results on algebras over non-modular fields when F is assumed to be merely any infinite field. Moreover, I shall determine all normal division algebras of degree two over F of characteristic two, of degree three over F of characteristic three.[§]

2. The existence of a maximal separable sub-field of A . Let A be any normal division algebra of degree n over any field F , and let

$$(1) \quad u_1, \dots, u_m \quad (m = n^2)$$

* Presented to the Society, December 1, 1933; received by the editors November 22, 1933.

† Cf. B. L. van der Waerden's *Moderne Algebra* for the theory of modular fields.

‡ In particular the papers by R. Brauer.

§ I have also completed a determination of all normal division algebras of degree four over F of characteristic two and have offered this more complicated determination for publication in the *American Journal of Mathematics*.

be a basis of F . Then it is known* that if K is an algebraically closed extension of F , the algebra A_K over K is a total matrix algebra M . Let

$$(2) \quad e_{\alpha\beta} = v_j = \sum_{i=1}^m \mu_{ji} u_i, \quad u_i = \sum_{j=1}^m \lambda_{ij} v_j \quad (i, j = 1, \dots, m).$$

where $\alpha, \beta = 1, \dots, n$ and $j = (\alpha - 1)n + \beta$. The quantities λ_{ij}, μ_{ji} are then in K and $e_{\alpha\beta}$ corresponds to an n -rowed matrix with unity in the α th row and β th column and zero elsewhere.

The rank equation of A is the minimum equation of the quantity $x = \sum_{i=1}^m \xi_i u_i$ where the ξ_i are independent variables. Then it is known that we have the result†

THEOREM 1. *The rank equation of A is the characteristic equation of the matrix*

$$(3) \quad \|\zeta_{\alpha\beta}\| \quad (\alpha, \beta = 1, \dots, n)$$

where

$$(4) \quad \zeta_{\alpha\beta} = \sum_{i=1}^m \mu_{ji} \xi_i, \quad j = (\alpha - 1)n + \beta.$$

This equation has coefficients in $L = F(\xi_1, \dots, \xi_m)$ and is irreducible in L .

E. Noether and G. Köthe have given proofs‡ of

THEOREM 2. *Algebra A of degree n over an infinite field F has separable subfields $F(x)$ of degree n .*

Their proofs are not at all elementary while my very much earlier simpler proof§ for the case where F is non-modular holds and uses only Theorem 1. We may in fact prove

THEOREM 3. *The sub-fields $F(x)$ of Theorem 2 may be so chosen that x satisfies*

$$\omega^n + \lambda_1 \omega^{n-1} + \dots + \lambda_n = 0 \quad (\lambda_1 \neq 0, \lambda_i \text{ in } F).$$

For the rank equation $R(\omega; \xi_1, \dots, \xi_m)$ is satisfied by any matrix (3) when the corresponding values of ξ_1, \dots, ξ_m are given. Let β_1, \dots, β_n be n quantities of the infinite field F so chosen that $\beta_1, \dots, \beta_{n-1}$ are distinct

* Cf. van der Waerden's *Algebra*, II, p. 176.

† For proof of Theorem 1, see L. E. Dickson's *Algebren und ihre Zahlentheorie*, pp. 259-262. Dickson's proof uses only (2) and is an immediate consequence of his Theorem 5 without the argument of the unnecessary section 132.

‡ *Journal für Mathematik*, vol. 166 (1932), pp. 182-184, for Köthe's proof, and *Mathematische Zeitschrift*, vol. 37 (1933), pp. 514-541, p. 535 for Noether's proof.

§ *Bulletin of the American Mathematical Society*, vol. 36 (1930), pp. 649-650.

and $\beta_n \neq \beta_i$, $-(\beta_1 + \dots + \beta_{n-1})$ for $i = 1, \dots, n-1$. Then we solve (4) for the ξ_i and have proved the existence of ξ_{i0} in K for which $R(\omega; \xi_{10}, \dots, \xi_{m0}) = 0$ has distinct roots and the coefficient $\lambda_1(\xi_{10}, \dots, \xi_{m0})$ of ω^{n-1} is not zero. Let $D(\xi_1, \dots, \xi_n)$ be the discriminant of $R(\omega; \xi_1, \dots, \xi_m)$. Then

$$D(\xi_{10}, \dots, \xi_{m0}) \lambda(\xi_{10}, \dots, \xi_{m0}) \neq 0,$$

so that $D(\xi_1, \dots, \xi_m) \cdot \lambda(\xi_1, \dots, \xi_m) \neq 0$. But then there exist values ξ_{i1} of ξ_1, \dots, ξ_m in F such that $D(\xi_{11}, \dots, \xi_{m1}) \cdot \lambda(\xi_{11}, \dots, \xi_{m1}) \neq 0$ and hence such that the rank equation of A for $x = \sum \xi_{i1} u_i$ has distinct roots and coefficient of ω^{n-1} not zero.

The characteristic equation of the corresponding matrix (3) is an exact power of the minimum equation of x since x in the division algebra A has irreducible minimum equation. Since the characteristic equation has been shown to have distinct roots, it is the minimum equation of x and we have proved Theorems 2, 3.

3. Known theorems. In this section we shall state certain well known theorems on algebras over non-modular fields which hold for any infinite field. We first have

THEOREM 4. *Let D be a normal division algebra of degree n over F , and let Z be equivalent to any sub-field of D of degree n . Then $D \times Z = D_Z$ is a total matrix algebra.*

Wedderburn's proof* of this theorem holds for an arbitrary field. As an immediate consequence of Theorem 2 we have

THEOREM 5. *There exist separable splitting fields of D of degree n .*

We of course say that Z is a splitting field of D if D_Z is a total matrix algebra.

We also have Wedderburn's theorems:

THEOREM† 6. *Let A be a normal simple algebra of degree n^2 over F . Then $A = M \times D \sim D$, where M is a total matrix algebra and D is a normal division algebra whose degree is the index of A . Moreover D and M are uniquely determined apart from an interior automorphism of A .*

THEOREM‡ 7. *Let B be a normal simple algebra over F contained in any algebra A over F with the same modulus as B . Then $A = B \times C$ where C also has the same modulus as A .*

* For Theorems 10, 12, see Wedderburn's paper in these Transactions, vol. 22 (1921), pp. 129-135. The proof of Theorem 4 appears on p. 133 and the footnote to p. 134.

† Cf. L. E. Dickson's *Algebren*, p. 120.

‡ Proceedings of the Edinburgh Mathematical Society, vol. 25 (1906-07), pp. 1-3.

The proofs given by Wedderburn of the above Theorems 6, 7 also hold in view of Theorem 5. They may also be applied, as in the non-modular case, to give my

INDEX REDUCTION THEOREM.* *Let D be a normal division algebra of degree (index) n over any infinite field F , Z an algebraic field of degree r over F . Then the index of D_Z over Z is*

$$n' = n/s,$$

where the index reduction factor s divides r .

As a consequence we have the whole Brauer exponent theory as well as my

THEOREM† 8. *Let D be a normal division algebra of degree n over any infinite field F , p a prime divisor of n . Then there exists a field Z of degree r over F such that*

$$D = M \times B \sim B \quad (M \text{ total matrix}),$$

where B is a cyclic division algebra of degree p over its centrum Z .

THEOREM‡ 9. *Let Z_0 be in D so that the degree r of the field Z_0 divides n and let Z be equivalent to Z_0*

$$D_Z = M \times B,$$

as in the Index Reduction Theorem. Then the algebra B_0 over Z_0 of all quantities of D commutative with every quantity of Z_0 is equivalent to B over Z .

We may indeed say that almost all of the recent general theory on normal division algebras holds when F is any infinite field. The determination theorems on algebras of degree 2, 3, 4 do not hold however. We shall give here a determination in the cases $n=2$, 3, and, in a later American Journal paper, the case $n=4$. We shall require

THEOREM 10. *Let D be a normal division algebra of degree n over F , and let x in D have $\phi(\omega)=0$ of degree v as its minimum equation. Then*

$$\phi(\omega) \equiv (\omega - x_r)(\omega - x_{r-1}) \cdots (\omega - x_2)(\omega - x),$$

where the v factors may be permuted cyclically.

THEOREM§ 11. *Every root y in D of $\phi(\omega)=0$ is a transform $txt^{-1}=y$ of x by t in D .*

* On direct products, these Transactions, vol. 33 (1931), pp. 690-711.

† For probably the best proof of Theorem 8 see (1), (2) on p. 725 of the joint paper by H. Hasse and myself in these Transactions, vol. 34 (1932), pp. 722-726.

‡ On normal simple algebras, these Transactions, vol. 34 (1932), pp. 620-625.

§ Cf. Annals of Mathematics, vol. 30 (1929), pp. 322-338, Theorem 12.

THEOREM 12. Let $f(\omega) \equiv g(\omega) \cdot h(\omega)$ where f, g, h have coefficients in D and ω is a scalar variable. Then if $\omega - x$ is a right divisor of $f(\omega)$, $h(\omega) \equiv q(\omega)(\omega - x) + R$ where $R \neq 0$ is in D , then $\omega - RxR^{-1}$ is a right divisor of $g(\omega)$:

4. Algebras over perfect fields. We may now prove

THEOREM 13. Let D be a normal division algebra of degree n over a perfect modular field F of characteristic p . Then n is not divisible by p .

For by Theorem 8, if n is divisible by p then there exists an extension Z of finite degree over F , such that $D \times Z = M \times B$ where B is a cyclic division algebra of degree p over F . But it is known* that then Z is perfect. Moreover $B = (X, S, \gamma)$ where X is cyclic of degree p over Z and with generating automorphism S , γ in Z is not the norm $N(f)$ of any f in X . But Z is perfect, $\gamma = \delta^p = N(\delta)$, a contradiction.

5. Algebras of degree two. Let D be a normal division algebra of degree two over an infinite field F of characteristic two. By Theorem 2, algebra D contains a separable quadratic field $F(x)$, $x^2 = \lambda x + \mu$ where $\lambda \neq 0$, $\mu \neq 0$ are in F . We let $i = \lambda^{-1}x$ so that $i^2 = \lambda^{-2}(\lambda x + \mu) = i + \alpha$ where $\alpha = \mu\lambda^{-2} \neq 0$ is in F . The equation $\omega^2 = \omega + \alpha$ is cyclic and in fact has the roots $i, i+1$. By Theorem 12 there exists a quantity j in D such that $ji = (i+1)j$. But then $j^2i = ij^2$. Since $F(i)$ is a maximal sub-field of A , the quantity j^2 is in $F(i)$. But $F(j^2) < F(i)$ since $jj^2 = j^2j$, but $ji \neq ij$. Hence $j^2 = \gamma$ in F and we have proved

THEOREM 14. Every normal division algebra D of degree two over F of characteristic 2 is a cyclic algebra

$$\begin{aligned}(1, i, j, ij), \quad i^2 &= i + \alpha, \\ ii &= (i + 1)j, \quad j^2 = \gamma,\end{aligned}$$

with α and γ in F .

6. Algebras of degree three. We now let *three* be the degree of D and the characteristic of F . By Theorem 2 there exists a separable cubic sub-field $F(u)$ of F such that u has

$$\phi(\omega) \equiv \omega^3 + \alpha\omega^2 + \beta\omega + \gamma = 0,$$

with $\alpha \neq 0$ by Theorem 3. By Theorem 10 we have

$$\phi(\omega) \equiv (\omega - u_1)(\omega - u_2)(\omega - u_3)$$

where $u = u_1, u_2, u_3$ are evidently distinct and u_2, u_3 are transforms of u by

* Cf. E. Steinitz, *Algebraische Theorie der Körper*, p. 55.

quantities of F . If

$$x = u_2 u_1 - u_1 u_2$$

is zero then evidently $\phi(\omega)$ is a cyclic equation, D is a cyclic algebra. For $u_2 u_1 = u_1 u_2$ implies that u_2 is in $F(u_1)$.

Hence let $x \neq 0$. By Wedderburn's proof for the case where the characteristic of F is not three, we have

$$x u_1 = u_2 x, \quad x u_2 = u_3 x, \quad x u_3 = u_1 x,$$

so that $x^3 u_1 = u_1 x^3$ and x^3 is in F . Let then $x^3 = \delta$ in F .

The minimum equation of x with respect to F is

$$\psi(\omega) \equiv \omega^3 - \delta \equiv (\omega - x)^3 = 0,$$

so that $F(x)$ is inseparable and *Wedderburn's proof breaks down*. But let $v = u_1 x - x u_1 = (u_1 - u_2)x \neq 0$. Write $x = x_1$. Then $x_1 \neq u_1 x_1 u_1^{-1}$ since $(x_1 - u_1 x_1 u_1^{-1})u_1 = x u_1 - u_1 x = -v \neq 0$. Hence $\omega - u_1 x_1 u_1^{-1}$ is a right divisor of $\psi(\omega)$ but not of $\omega - x$, and, by Theorem 12, with $R = u_1 x_1 u_1^{-1} - x_1 = v u_1^{-1}$ we have $\omega - v x_1 v^{-1}$ a right divisor of $(\omega - x_1)^2$. We have obtained

$$(\omega - x_1)^2 \equiv (\omega^2 - 2x_1\omega + x_1^2) \equiv (\omega - x_2)(\omega - x_2), \quad x_2 = v x_1 v^{-1}.$$

Now

$$x_2 = v x_1 v^{-1} = (u_1 - u_2) x_1^2 x_1^{-1} (u_1 - u_2)^{-1} = (u_1 - u_2) x_1 (u_1 - u_2)^{-1}.$$

But

$$x_1(u_1 - u_2) = (u_2 - u_3)x_1, \quad (u_2 - u_3)^{-1}x_1 = x_1(u_1 - u_2)^{-1}$$

and

$$x_2 = (u_1 - u_2)(u_2 - u_3)^{-1}x_1.$$

If $x_2 = x_1$ then $u_1 - u_2 = u_2 - u_3$. But $3u_2 = 0$, $u_1 - 2u_2 + u_3 = u_1 + u_2 + u_3 = 0 = \alpha$, a contradiction. Hence $x_2 \neq x_1$. Also $x_3 + x_2 + x_1 = 0$, $x_3 + x_2 = 2x_1$, $x_3 - x_1 = x_1 - x_2 \neq 0$, $x_3 - x_2 = 2(x_1 - x_2) \neq 0$. Thus x_3, x_2, x_1 are all distinct and we have obtained a factorization in D of $\psi(\omega)$ into distinct factors in spite of the fact that $\psi(\omega) = 0$ is inseparable.

Moreover $(\omega - x_1)^3 \equiv (\omega - x_2)^3 \equiv (\omega - x_3)^3 \equiv (\omega - x_1)(\omega - x_2)(\omega - x_3)$, so that $(\omega - x_2)^2 - (\omega - x_1)(\omega - x_3)$ and $x_1 x_3 = x_2^2$.

If $x_2 x_1 - x_1 x_2 = 0$, then $x_2 \neq x_1$ is in $F(x_1)$, $(x_2 - x_1)^3 = x_2^3 - x_1^3 = 0$, a contradiction. Hence $y = x_2 x_1 - x_1 x_2 \neq 0$. By the Wedderburn proof*

$$y x_1 = x_2 y, \quad y x_2 = x_3 y, \quad y x_3 = x_1 y, \quad y^3 = \epsilon \text{ in } F.$$

* These Transactions (loc. cit.), 1921.

We let $z_1 = x_1y$, $z_2 = yz_1y^{-1} = yx_1yy^{-1} = yx_1$, a transform of z_1 by y . Also $yx_1 \neq x_1y$ so that $z_2 \neq z_1$. Thus $z_2z_1 - z_1z_2 = yx_1^2y - x_1y^2x_1 = (x_2^2 - x_3x_1)y^2 = 0$. Hence z_2 is commutative with z_1 , z_2 is in $F(z_1)$, $z_2 \neq z_1$ and $F(z_1)$ is cyclic. We have proved

THEOREM 15. *Every normal division algebra of degree three over any infinite field F is cyclic.*

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THE VALUE OF THE NUMBER $g(k)$ IN WARING'S PROBLEM*

BY

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1. Introduction. The number $g(k)$ is defined to be such that (a) every integer is a sum of $g(k)$ k th powers ≥ 0 ; (b) there is at least one integer which is not a sum of $g(k) - 1$ k th powers ≥ 0 . It is well known that $g(2) = 4$, $g(3) = 9$, but the exact value of $g(k)$ is not known when $k \geq 4$.

The number $G(k)$ is defined to be such that every integer $> C = C(k)$ is a sum of $G(k)$ k th powers ≥ 0 . Hardy and Littlewood‡ have proved that

$$G(k) \leq (k-2)2^{k-2} + k + 5 + \zeta_k,$$

where

$$\zeta_k = \left\lceil \frac{(k-2) \log 2 - \log k + \log(k-2)}{\log k - \log(k-1)} \right\rceil.$$

In this paper we obtain a similar bound for $g(k)$ when $k \geq 6$. We shall prove the

THEOREM. *Let L be a number $> k^*$ such that every integer $\leq L$ is a sum of s_3 k th powers ≥ 0 . Let*

$$D = (d+2)(k-1) - 2^{d+1} + 1/10, \quad d = [\log(k-1)/\log 2];$$

$$E = s_3 + \frac{3 \log k + \log 20 - \log(\log L - k \log k)}{\log k - \log(k-1)};$$

$$F = \log 2(\log k - \log(k-1))^{-1}; \quad H = (k-2)2^{k-2} + k;$$

$$Q = 2 + s_2 = 6 + \zeta_k; \quad R = (1 + (1-a)^{s-2})k2^{k-2} - DQ.$$

Then

$$(1) \quad g(k) \leq \left[\frac{1}{2}(H + FD + Q + E + ((H + FD + Q - E)^2 + 4F(ED + R))^{1/2}) \right] + 1.$$

The method of proof is as follows: We determine the constants as they occur at each step of the Hardy-Littlewood analysis as functions of k , s , and

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‡ G. H. Hardy and J. E. Littlewood, *Some problems of "partitio numerorum" (VI): Further researches in Waring's problem*, Mathematische Zeitschrift, vol. 23 (1925), pp. 1-37. See also E. Landau, *Vorlesungen über Zahlentheorie*, vol. 1, part 6 (referred to as L), and M. Gelbcke, *Zum Waringschen Problem*, Mathematische Annalen, vol. 105 (1931), pp. 637-652 (referred to as G).

ϵ , where ϵ is a small positive number. In this way we conclude that every integer $> C(k, s, \epsilon)$ is a sum of s k th powers ≥ 0 when $s \geq g_1(k, \epsilon)$ (Theorem 46). Then, using a theorem proved by L. E. Dickson,* we show that every integer $\leq C(k, s, \epsilon)$ is a sum of s k th powers ≥ 0 when $s \geq g_2(k, \epsilon)$ (Theorem 50). We choose ϵ as a function of k so that $g_1(k, \epsilon(k)) = g_2(k, \epsilon(k))$ and then $g(k) \leq g_1(k, \epsilon(k)) = g_2(k, \epsilon(k))$.

In Theorem 48 we give a general method for the determination of L and s_3 and prove that

$$s_3 < 2^k + \left(\frac{3}{2}\right)^k + 2\left(\frac{4}{3}\right)^k + 2\left(\frac{2}{3}\right)^k + 2\left(\frac{1}{2}\right)^k + \frac{k(2k+7)}{9} - 9$$

when $L = (k+1)^k - k^k > k^k$. It then follows from (1) that

$$\lim_{k \rightarrow \infty} \frac{g(k)}{k2^{k-1}} \leq \frac{1}{2}.$$

For particular values of k we may obtain better values of L and s_3 . For example, since $25 \cdot 2^8 = 6400$, $3^8 = 6561$, $26 \cdot 2^8 = 6656$, every integer from 6400 to 6656 is a sum of 185 8th powers ≥ 0 . Repeated application of Theorem 47 yields the result that every integer from 1 to $10^{713 \cdot 7}$ is a sum of 279 8th powers ≥ 0 . With $L = 10^{713 \cdot 7}$, $s_3 = 279$, we get $g(8) \leq 622$. Again, since $25 \cdot 2^8 + 9 \cdot 3^8 = 65449$, $4^8 = 65536$, $10 \cdot 3^8 = 65610$, $26 \cdot 2^8 + 9 \cdot 3^8 = 65705$, every integer from 65449 to 65705 is a sum of 120 8th powers ≥ 0 and this gives $L = 10^{3,920,000}$, $s_3 = 279$, $g(8) \leq 595$. It is obvious that the larger we can make L for a given s_3 the better will be the resulting bound for $g(k)$. In the table below we summarize the known results for $g(k)$ and $G(k)$ when $6 \leq k \leq 10$. The first line gives the bounds for $g(k)$ obtained by algebraic methods separately for each k ; the second gives the bounds obtained by the methods of this paper; the third gives the bounds for $G(k)$ obtained by the Hardy-Littlewood method; and the fourth gives the lower bounds for $g(k)$.

| k | 6 | 7 | 8 | 8 | 10 |
|-------------|-----|------|-------|------|--------|
| $g(k) \leq$ | 478 | 3806 | 31353 | — | 140004 |
| $g(k) \leq$ | 183 | 322 | 595 | 1177 | 2421 |
| $G(k) \leq$ | 87 | 193 | 425 | 949 | 2113 |
| $g(k) \geq$ | 73 | 143 | 279 | 548 | 1079 |

* L. E. Dickson, *Proof of a Waring theorem on fifth powers*, Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 549-553.

The numbers in the last line are probably the exact values of $g(k)$. In order to prove $g(10) = 1079$, for example, it would be necessary to prove some inequality like $G(10) \leq 700$. On the basis of an unproved hypothesis, Hardy and Littlewood (loc. cit.) have shown that $G(10)$ would be ≤ 21 . It seems likely, then, that a far less drastic assumption would be sufficient to prove $g(10) = 1079$ and this assumption may be capable of proof.

The possibility of evaluating the constants of the Hardy-Littlewood analysis was suggested by Professor L. E. Dickson. The case of fifth powers was considered in the author's doctor's dissertation written under Professor Dickson's direction at the University of Chicago.

2. Notation. We shall use the following notation throughout the paper.

Let

- $T(m)$ = the number of divisors of m ;
 $\pi(x)$ = the number of primes $\leq x$;
 $\vartheta(x)$ = the sum of the logarithms of all primes $\leq x$;
 $[x]$ = the greatest integer $\leq x$;
 $\{x\} = \min(x - [x], [x] + 1 - x)$;
 $M(p^t, n)$ = the number of solutions of the congruence $\sum_{i=1}^s h_i^k \equiv n \pmod{p^t}$;
 $N(p^t, n)$ = the number of solutions of the same congruence in which not every h_i is divisible by p (primitive solutions);
 k = an integer ≥ 6 ; $a = 1/k$; $K = 2^{k-1}$; $A = 1/K$;
 ϵ_i = a small positive number, $i = 1, 2, 3$; $\eta_i = 1/\epsilon_i$, $i = 1, 2, 3$;
 $s = [(H + (1 + (1-a)^{s-2})k2^{k-2}\epsilon_1)(1 - D\epsilon_1)^{-1}] + 3$;
 (2) $s_2 = [((k-2)\log 2 - \log k + \log(k-2))(\log k - \log(k-1))^{-1}] + 4$;
 $\lambda = 2A + (1-a)^{s-2}\epsilon_1$;
 Θ = the highest power of a prime p which divides k ;
 $\gamma = \begin{cases} \Theta + 2 & \text{if } p = 2, \\ \Theta + 1 & \text{if } p > 2; \end{cases} \quad p^\gamma = P$;
 (a, b) = the greatest common divisor of a and b ;
 $r = ((P-1)/(p-1))(k, p-1)$;
 $r(n) = r_{k,s}(n)$ = the number of solutions of $\sum_{i=1}^s h_i^k = n$, $h_i \geq 0$;
 $\rho = e^{2\pi i b/q}$, $(b, q) = 1$;
 $S_\rho = \sum_{i=1}^s \rho^{h_i^k}$;
 $A(q) = A_{k,s}(q, j) = q^{-r} \sum_{\rho} S_\rho^j \rho^{-j}$, where ρ ranges over all primitive q th roots of unity;
 $\chi_p = \sum_{i=0}^\infty A(p^i)$;
 $\mathfrak{S}(j, k, s, w) = \sum_{q=1}^w A(q)$;
 $f(x) = \sum_{h=0}^s x^{hk}$;
 $\psi_\rho(x) = \Gamma(1+a)q^{-1}S_\rho(1 + \sum_{j=1}^s (a(a+1) \cdots (a+j-1)/j!)(x/\rho)^j)$;
 $\phi_\rho(x) = \Gamma(1+a)q^{-1}S_\rho \sum_{j=n+1}^\infty (a(a+1) \cdots (a+j-1)/j!)(x/\rho)^j$;

$$(2) \quad \begin{aligned} \Psi_p(x) &= \psi_p(x) + \phi_p(x) = \Gamma(1+a)q^{-1}S_p(1-x/\rho)^{-a}; \\ \sigma(j) &= r(j) - (\Gamma^*(1+a)/\Gamma(sa)) (\Gamma(sa+j)/j!) \Theta(j, k, s, n^*). \end{aligned}$$

The letters A, α, b, B, c, C are numbered in the same way as the corresponding letters in paper L, while the letters G correspond to the C of paper G.

3. Preliminary theorems. We shall not repeat the proof of a known theorem if the constants involved are explicitly given in the original proof.

THEOREM 1.* For every $\epsilon_1 > 0$

$$T(m) \leq A_1 m^n,$$

where

$$A_1 = \frac{2^{*(2n)} \cdot (3/2)^{*(3/2)n} \cdot (4/3)^{*(4/3)n} \cdot \dots}{\exp(\epsilon_1 \vartheta(2^n) + \vartheta((3/2)^n) + \vartheta((4/3)^n) + \dots)}$$

THEOREM 2. (L, Theorem 112.) For $\xi \geq 2$,

$$\alpha_1 \xi / \log \xi < \pi(\xi) < \alpha_2 \xi / \log \xi,$$

where $8\alpha_1 \geq \log 2$ and $\alpha_2 \leq 7 \log 2$.

Since

$$[\eta] - 2[\eta/2] < \eta - 2(\eta/2 - 1) = 2$$

and the left side is an integer, it follows that

$$(3) \quad [\eta] - 2[\eta/2] \leq 1.$$

Let $n \geq 2$. For every prime $p \leq 2n$ let f denote the greatest integer such that $p^f \leq 2n$ (i.e., $f = [\log(2n)/\log p]$). We show first that

$$(4) \quad \prod_{n < p \leq 2n} p \left| \frac{(2n)!}{n!n!} \right| \prod_{p \leq 2n} p^f.$$

The first part of (4) follows at once since every p for which $n < p \leq 2n$ divides $(2n)!$ but not $n!n!$. Also, since the highest power of a prime p which divides $x!$ is

$$\sum_{1 \leq m \leq \log x / \log p} [x/p^m]$$

(see, for example, L, Theorem 27), the highest power of p which divides $(2n)!/(n!n!)$ is

$$\sum_{m=1}^f ([2n/p^m] - 2[n/p^m]) \leq \sum_{m=1}^f 1 = f$$

* S. Ramanujan, *Highly composite numbers*, Proceedings of the London Mathematical Society, (2), vol. 14 (1915), p. 392.

by (3). This proves the second part of (4). Next, the left side of (4) has $\pi(2n) - \pi(n)$ factors each $> n$, and the right side has $\pi(2n)$ factors each $\leq 2n$. Hence

$$n^{\pi(2n) - \pi(n)} < \prod_{n < p \leq 2n} p \leq \frac{(2n)!}{n!n!} \leq \prod_{p \leq 2n} p^f \leq (2n)^{\pi(2n)},$$

$$(\pi(2n) - \pi(n)) \log n < \log ((2n)!/(n!n!)) \leq \pi(2n) \log (2n).$$

Therefore

$$(\pi(2n) - \pi(n)) \log n < \log \left(\binom{2n}{n} \right) \leq \log \left(\sum_{j=1}^{2n} \binom{2n}{j} \right) = \log 2^{2n} = 2n \log 2,$$

$$(5) \quad \pi(2n) - \pi(n) < 2(\log 2)n / \log n = \alpha_3 n / \log n;$$

and

$$\pi(2n) \log (2n) \geq \log \left(\binom{2n}{n} \right) = \log \left(\prod_{j=1}^n \frac{n+j}{j} \right)$$

$$\geq \log \left(\prod_{j=1}^n 2 \right) = \log 2^n = n \log 2,$$

$$(6) \quad \pi(2n) \geq n \log 2 / \log (2n) = n \log 2 / (\log n + \log 2)$$

$$\geq n \log 2 / (2 \log n) = \alpha_4 n / \log n.$$

From (6) when $\xi \geq 4$

$$\pi(\xi) \geq \pi(2[\xi/2]) \geq \alpha_4 [\xi/2] / \log [\xi/2] \geq \alpha_4 \xi / (4 \log \xi) = \alpha_5 \xi / \log \xi.$$

When $2 \leq \xi \leq 4$ we have

$$\pi(\xi) \geq 1 = ((\log 2)/4)(4/\log 2) \geq ((\log 2)/4)(\xi/\log \xi).$$

Hence for all $\xi \geq 2$,

$$\pi(\xi) > \alpha_1 \xi / \log \xi,$$

where $8\alpha_1 \geq 8 \min(\alpha_5, (\log 2)/4) = 2\alpha_4 = \log 2$. This proves the first inequality of the theorem.

Now, since $\eta = 2 + 2(\eta/2 - 1) < 2 + 2[\eta/2]$ it follows from (5) when $\eta \geq 8$ that

$$\begin{aligned} \pi(\eta) - \pi(\eta/2) &= \pi(\eta) - \pi([\eta/2]) \leq 2 + \pi(2[\eta/2]) - \pi([\eta/2]) \\ &< 2 + \alpha_3 [\eta/2] / \log [\eta/2] < 2 + \alpha_3 \eta / (2 \log (\eta/2 - 1)) \\ &= 2 + \alpha_3 \eta / (2(\log (\eta - 2) - \log 2)) \\ &= 2 + (\alpha_3 \log \eta / (2(\log (\eta - 2) - \log 2))) (\eta / \log \eta) \\ &\leq (\log 8/4)(8/\log 8) + (\alpha_3 \log 8 / (2(\log 6 - \log 2))) (\eta / \log \eta) \\ &\leq ((\log 8/4) + (\alpha_3 \log 8 / (2 \log 3))) (\eta / \log \eta). \end{aligned}$$

When $2 \leq \eta \leq 8$ we have

$$\pi(\eta) - \pi(\eta/2) \leq 2 = (\log 8)2/\log 8 \leq (\log 8)\eta/\log \eta.$$

Therefore for all $\eta \geq 2$

$$\pi(\eta) - \pi(\eta/2) \leq \alpha_7 \eta / \log \eta$$

where $\alpha_7 = \max((\log 8/4) + (\alpha_3 \log 8 / (2 \log 3)), \log 8) = \log 8$. Then

$$\begin{aligned} \pi(\eta) \log \eta - \pi(\eta/2) \log (\eta/2) &= (\pi(\eta) - \pi(\eta/2)) \log \eta + \pi(\eta/2) \log 2 \\ &< \alpha_7 (\log \eta) \eta / \log \eta + \eta (\log 2) / 2 = (\alpha_7 + (\log 2)/2) \eta = \alpha_8 \eta. \end{aligned}$$

For $\xi \geq 2$ we have

$$\begin{aligned} \pi(\xi/2^m) \log (\xi/2^m) - \pi(\xi/2^{m+1}) \log (\xi/2^{m+1}) &< \alpha_8 \xi / 2^m, \\ \pi(\xi) \log \xi &= \sum_{m=0}^{\infty} (\pi(\xi/2^m) \log (\xi/2^m) - \pi(\xi/2^{m+1}) \log (\xi/2^{m+1})) \\ &< \alpha_8 \sum_{m=0}^{\infty} (\xi/2^m) = 2\alpha_8 \xi; \end{aligned}$$

that is,

$$\pi(\xi) < \alpha_8 \xi / \log \xi,$$

where $\alpha_8 \leq 2\alpha_8 = 2(\alpha_7 + (\log 2)/2) = 2(\log 8 + (\log 2)/2) = 7 \log 2$.

THEOREM 3.* We have

$$\vartheta(x) < 6cx/5 + 3 \log^2 x + 8 \log x + 5,$$

$$\vartheta(x) > cx - 12cx^{1/2}/5 - 3 \log^2 x/2 - 13 \log x - 15$$

where $c = 2^{1/2} \cdot 3^{1/3} \cdot 5^{1/5} / 30^{1/30} = 0.92129 \dots$

THEOREM 4. (L, Theorem 264.) Let t be an integer, $m > 0$, $z \geq 0$, and

$$S = \sum_{h=t+1}^{t+m} e^{2\pi i z h^k}.$$

Then

$$|S|^K < 4^K \left(m^{K-1} + m^{K-k} \sum_{h_1, \dots, h_{k-1}=1}^m \min(m, 1/\{zk!h_1 \cdots h_{k-1}\}) \right).$$

THEOREM 5. The number of solutions of the equation

$$(7) \quad h_1 h_2 \cdots h_{k-1} = v, \quad 0 \leq h_i \leq m,$$

is at most $A_2 m^{\epsilon_2}$ where $A_2 = A_1^{k-2}$ and ϵ_2 is given by

$$(8) \quad \begin{aligned} \epsilon_2(2) &= 0; & \epsilon_2(k) &= (k-1)\epsilon_1 + \epsilon_2(k/2) + \epsilon_2(k/2+1), & k \text{ even } \geq 4; \\ \epsilon_2(3) &= 2\epsilon_1; & \epsilon_2(k) &= (k-1)\epsilon_1 + 2\epsilon_2((k+1)/2), & k \text{ odd } \geq 5. \end{aligned}$$

* E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, §22.

- (i) Let $k=2$. Then $h_1=v$ has at most $1=A_1^{2-2}m^0=A_2m^{\epsilon_2(2)}$ solutions.
 (ii) Let $k=3$. Then $h_1h_2=v$ has at most $T(v)$ solutions since h_1 must divide v .
 By Theorem 1,

$$T(v) \leq A_1v^{\epsilon_1} \leq A_1^{3-2}m^{2\epsilon_1} = A_2m^{\epsilon_2(3)}$$

since $v=h_1h_2 \leq m^2$. (iii) Let k be even ≥ 4 and assume that the theorem is true for all integers $< k$. In equation (7) write

$$(9) \quad h_1h_2 \cdots h_{(k-2)/2} = v_1,$$

$$(10) \quad h_{k/2} \cdots h_{k-1} = v_2.$$

There are at most $A_1^{k/2-2}m^{\epsilon_1(k/2)}$ solutions of (9) and at most $A_1^{k/2+1-2}m^{\epsilon_1(k/2+1)}$ solutions of (10). The equation $v=v_1v_2$ has at most $T(v)$ solutions. Hence the number of solutions of (7) is

$$\begin{aligned} &\leq T(v) \cdot A_1^{k/2-2}m^{\epsilon_2(k/2)} \cdot A_1^{k/2+1-2}m^{\epsilon_2(k/2+1)} \\ &\leq A_1v^{\epsilon_1}A_1^{k-3}m^{\epsilon_2(k/2)+\epsilon_2(k/2+1)} \\ &\leq A_1m^{(k-1)\epsilon_1}A_1^{k-3}m^{\epsilon_2(k/2)+\epsilon_2(k/2+1)} = A_2m^{\epsilon_2(k)}. \end{aligned}$$

- (iv) Let k be odd ≥ 5 and assume that the theorem is true for all integers $< k$. As in the proof of (iii) with

$$h_1h_2 \cdots h_{(k-1)/2} = v_1,$$

$$h_{(k+1)/2} \cdots h_{k-1} = v_2,$$

the number of solutions of (7) is

$$\begin{aligned} &\leq T(v) \cdot A_1^{(k+1)/2-2}m^{\epsilon_2((k+1)/2)} \cdot A_1^{(k+1)/2-2}m^{\epsilon_2((k+1)/2)} \\ &\leq A_1m^{(k-1)\epsilon_1} \cdot A_1^{k-3}m^{2\epsilon_2((k+1)/2)} = A_2m^{\epsilon_2(k)}. \end{aligned}$$

COROLLARY. Let $d = [\log(k-1)/\log 2]$. Then

$$(11) \quad \epsilon_2(k) = ((d+2)(k-1) - 2^{d+1})\epsilon_1.$$

- (i) Let $k=2$ and hence $d=0$. Then by Theorem 5

$$\epsilon_2(2) = 0 = ((d+2)(k-1) - 2^{d+1})\epsilon_1.$$

- (ii) Let $k>2$ and hence $d>0$. We have $2^d+1 \leq k \leq 2^{d+1}$. Assume that (11) is true for all integers $< 2^d+1$. If k is even and $< 2^{d+1}$ we have

$$\begin{aligned} 2^{d-1} + 1 &\leq k/2 \leq 2^d - 1, \\ 2^{d-1} + 2 &\leq (k+2)/2 \leq 2^d. \end{aligned}$$

By (8)

$$\begin{aligned}\epsilon_2(k) &= (k-1)\epsilon_1 + ((d+1)(k/2-1) - 2^d)\epsilon_1 + ((d+1)(k/2) - 2^d)\epsilon_1 \\ &= ((d+2)(k-1) - 2^{d+1})\epsilon_1.\end{aligned}$$

If $k=2^{d+1}$ we have $k/2=2^d$, $(k+4)/4=2^{d-1}+1$ and

$$\begin{aligned}\epsilon_2(k) &= (k-1)\epsilon_1 + \epsilon_2(k/2) + \epsilon_2(k/2+1) \\ &= (k-1)\epsilon_1 + \epsilon_2(k/2) + (k/2)\epsilon_1 + 2\epsilon_2((k+4)/4) \\ &= (k-1)\epsilon_1 + ((d+1)(k/2-1) - 2^d)\epsilon_1 + 2^d\epsilon_1 + 2((d+1)(k/4) - 2^d)\epsilon_1 \\ &= ((d+2)(k-1) - 2^{d+1})\epsilon_1.\end{aligned}$$

If k is odd then $2^{d-1}+1 \leq (k+1)/2 \leq 2^d$ and we have

$$\begin{aligned}\epsilon_2(k) &= (k-1)\epsilon_1 + 2\epsilon_2((k+1)/2) \\ &= (k-1)\epsilon_1 + 2((d+1)((k+1)/2-1) - 2^d)\epsilon_1 \\ &= ((d+2)(k-1) - 2^{d+1})\epsilon_1.\end{aligned}$$

THEOREM 6. (L, Theorem 266.) Under the hypotheses of Theorem 4,

$$|S|^K < C_{15} m^{\epsilon_2} \left(m^{K-1} + m^{K-k} \sum_{v=1}^{k!m^{k-1}} \min(m, 1/\{zv\}) \right),$$

where $C_{15} = 4^K A_2$.

In the summation in Theorem 4 write $k!h_1h_2 \cdots h_{k-1} = v$. By Theorem 5 each v appears at most $A_2 m^{\epsilon_2}$ times. Hence

$$\begin{aligned}\sum_{h_1, \dots, h_{k-1}=1}^m \min(m, 1/\{zk!h_1 \cdots h_{k-1}\}) &\leq A_2 m^{\epsilon_2} \sum_{v=1}^{k!m^{k-1}} \min(m, 1/\{zv\}), \\ |S|^K &< 4^K \left(m^{K-1} + m^{K-k} A_2 m^{\epsilon_2} \sum_{v=1}^{k!m^{k-1}} \min(m, 1/\{zv\}) \right) \\ &\leq 4^K A_2 m^{\epsilon_2} \left(m^{K-1} + m^{K-k} \sum_{v=1}^{k!m^{k-1}} \min(m, 1/\{zv\}) \right).\end{aligned}$$

THEOREM 7. Let $x \geq 1$, $b > 0$. Then for every $\epsilon_3 > 0$,

$$b + \log x \leq A_3 x^{\epsilon_3},$$

where $A_3 = 1/(\epsilon_3 e^{1-b\epsilon_3})$.

Consider the function

$$\begin{aligned}y &= (b + \log x)x^{-\epsilon_3}, & y' &= (1 - \epsilon_3(b + \log x))x^{-1-\epsilon_3}, \\ y'' &= -(\epsilon_3 + \epsilon_3(1 + \epsilon_3)(b + \log x))x^{-2-\epsilon_3}.\end{aligned}$$

We have $y' = 0$ when $x = \infty$ or $b + \log x = 1/\epsilon_3$. The second value gives a maximum. Hence

$$\begin{aligned}\max y &= 1/(\epsilon_3 e^{1-b\epsilon_3}), \\ b + \log x &\leq x^{\epsilon_3}/(\epsilon_3 e^{1-b\epsilon_3}) = A_3 x^{\epsilon_3}.\end{aligned}$$

4. The singular series. The series

$$\mathfrak{S} = \mathfrak{S}(j, k, s, \infty) = \sum_{q=1}^{\infty} A(q)$$

is called the singular series. In this section we show that $\mathfrak{S} = \prod_p \chi_p$ and from this that $\mathfrak{S} > b_4$. We shall follow closely part 6, chapter 2, of paper L.

THEOREM 8. (L, Theorem 293.) Let $n = n_0 p^{\beta k + \sigma} \neq 0$, where $\beta \geq 0$, $0 \leq \sigma < k$, $(n_0, p) = 1$. Let $t_0 = \max(\beta k + \sigma + 1, \beta k + \gamma)$. Then

$$(12) \quad A(p^t) = 0 \text{ when } t > t_0,$$

$$(13) \quad \chi_p = P^{1-s} N(P, 0) \sum_{\alpha=0}^{\beta-1} p^{\alpha(k-s)} + p^{\beta(k-s)} P^{1-s} N(P, n/p^{\beta k}),$$

where the summation is omitted if $\beta = 0$.

Remark. This theorem shows that the terms of the series $\chi_p = \sum_{t=0}^{\infty} A(p^t)$ are all zero after a certain one. Also, since everything on the right side of (13) is positive or zero it follows that $\chi_p \geq 0$. If $p \nmid 2kn$ we have $\gamma = 1$, $\beta = \sigma = 0$, $t_0 = 1$ and thus $\chi_p = 1 + A(p)$.

THEOREM 9. (L, Theorem 301.) For $s \geq r$ and every $n \not\equiv 0 \pmod{p}$,

$$N(P, n) > 0;$$

for $s \geq r+1$ and every n ,

$$N(P, n) > 0.$$

COROLLARY. For $s \geq r+1$ and every n ,

$$N(P, n) \geq P^{s-r-1}.$$

In the congruence

$$(14) \quad h_1^k + h_2^k + \cdots + h_s^k \equiv n \pmod{P}$$

write $n_1 = n - h_{r+1}^k - \cdots - h_s^k$. By Theorem 9 the congruence

$$h_1^k + h_2^k + \cdots + h_{r+1}^k \equiv n \pmod{P}$$

has at least one primitive solution. Since h_{r+1}, \dots, h_s may be chosen arbitrarily mod P it follows that (14) has at least P^{s-r-1} primitive solutions.

THEOREM 10. (L, Theorem 302.) If $k \geq 5$, $s \geq 4k$, then

$$\chi_p \geq P^{-r} = b(p).$$

For $k \geq 5$,

$$r = \frac{p^r - 1}{p - 1} \left(\frac{k}{p^\theta}, p - 1 \right) \leq \begin{cases} (2^{\theta+2} - 1) \frac{k}{2^\theta} < 4k, p = 2, \\ \frac{p^{\theta+1} - 1}{p - 1} \cdot \frac{k}{p^\theta} < \frac{p}{p - 1} k < 2k, p > 2. \end{cases}$$

Hence $s \geq 4k$ implies $s \geq r + 1$ and from the Corollary to Theorem 9 we get $N(P, n) \geq P^{s-r-1}$. By Theorem 8 either $\chi_p = P^{1-s} N(P, n)$ ($\beta = 0$) or $\chi_p \geq P^{1-s} N(P, 0)$ ($\beta > 0$). In both cases it follows that

$$\chi_p \geq P^{1-s} \cdot P^{s-r-1} = P^{-r} = b(p).$$

THEOREM 11. (L, Theorem 307.) If $q = p^t$, $p \nmid k$, $2 \leq t \leq k$, then

$$S_p = p^{t-1}.$$

THEOREM 12. (L, Theorem 311.) If $q = p$, then

$$|S_p| \leq (k - 1)p^{1/2}.$$

THEOREM 13. (L, Theorem 313.) Let $T_p = q^{a-1} S_p$. Then if $q = p^t$, $t > k$,

$$T_p = T_{p^{p^k}}.$$

THEOREM 14. (L, Theorem 314.) If $q = p^t$, $t \geq 1$, then

$$|T_p| \leq \begin{cases} 1 & \text{if } p > c_{38}, \\ c_{39} & \text{always,} \end{cases}$$

where $c_{38} = k^{2k/(k-2)}$, $c_{39} = k$.

If $\rho = e^{2\pi i b/p^t}$, $(b, p) = 1$ is a primitive p^t th root of unity, then $\rho^{p^k} = e^{2\pi i b/p^{t-k}}$ is a primitive p^{t-k} th root of unity. Hence in view of Theorem 13 we may assume $1 \leq t \leq k$.

(i) If $p \nmid k$, $2 \leq t \leq k$, then by Theorem 11

$$|T_p| = p^{at-t} |S_p| = p^{at-t} p^{t-1} = p^{at-1} \leq 1.$$

(ii) If $p \nmid k$, $t = 1$, Theorem 12 gives

$$|T_p| = p^{a-1} |S_p| \leq p^{a-1} (k - 1) p^{1/2} < k p^{a-1/2}.$$

(iii) If $p \mid k$, then

$$|T_p| = p^{at-t} |S_p| \leq p^{at-t} \cdot p^t \leq p^{ak} = p \leq k.$$

It follows from (i), (ii), and (iii) that $|T_p| \leq 1$ when $p > \max(k^{2k/(k-2)}, k) = c_{33}$ and for all p we have $|T_p| \leq \max(1, k, k) = k = c_{30}$.

THEOREM 15. (L, Theorem 315.) *We have*

$$|T_p| < c_{15} \text{ and hence } |S_p| < c_{15} q^{1-\alpha},$$

where $\log c_{15} = (k-1) \log k + \alpha_2(k-2)k^{2k/(k-2)}/(2k)$.

When $(q_1, q_2) = 1$ we have $S_{\rho_1 \rho_2} = S_{\rho_1} S_{\rho_2}$ (L, Theorem 281). Therefore

$$T_{\rho_1 \rho_2} = (q_1 q_2)^{\alpha-1} S_{\rho_1 \rho_2} = q_1^{\alpha-1} S_{\rho_1} q_2^{\alpha-1} S_{\rho_2} = T_{\rho_1} T_{\rho_2}.$$

For $q > 1$ write $q = p_1^{i_1} p_2^{i_2} \cdots p_m^{i_m} = \prod p_i^{i_i}$. Then

$$|T_q| = \prod |T_{p_i}| = \prod_1 |T_{p_i}| \cdot \prod_2 |T_{p_i}| \cdot \prod_3 |T_{p_i}|,$$

where Π_1 contains those p_i for which $p_i | k$ and $p_i \leq c_{33}$, Π_2 contains those for which $p_i \nmid k$ and $p_i \leq c_{33}$, and Π_3 contains those for which $p_i > c_{33}$. By the proof of Theorem 14 we have

$$\prod_1 |T_{p_i}| \leq \prod_{p \leq k} k \leq k^{k-1};$$

$$\prod_3 |T_{p_i}| \leq \prod_{p > c_{33}} 1 = 1;$$

$$\prod_2 |T_{p_i}| \leq \prod_{p \leq c_{33}} k p^{\alpha-1/2},$$

$$\begin{aligned} \log \prod_2 &< \sum_{p \leq c_{33}} \log k = \pi(c_{33}) \log k < \alpha_2 c_{33} \log k / \log c_{33} \quad (\text{Theorem 2}) \\ &= \alpha_2 (k-2) k^{2k/(k-2)} / (2k). \end{aligned}$$

Hence when $q > 1$

$$\begin{aligned} \log |T_q| &= \log \prod_1 + \log \prod_2 + \log \prod_3 \\ &< (k-1) \log k + \alpha_2 (k-2) k^{2k/(k-2)} / (2k) = \log c_{15}. \end{aligned}$$

If $q = 1$ then $|T_p| = 1 < c_{15}$.

THEOREM 16. (L, Theorem 316.) *We have*

$$|A(q)| < b_{19} q^{1-\alpha},$$

where $b_{19} = c_{15}^*$.

We use Theorem 15:

$$|A(q)| \leq \sum_p |q^{-1} S_p|^s < q (c_{15} q^{-\alpha})^s = b_{19} q^{1-\alpha s}.$$

THEOREM 17. (L, Theorem 317.) *If $p \nmid n$ then*

$$|A(p)| < b_{20} p^{1/2-\alpha/2},$$

where $b_{20} = k^*$.

THEOREM 18. (L, Theorem 318.) *We have*

$$|A(p)| < b_{21}p^{1-s/2},$$

where $b_{21} = \max(b_{20}, b_{22}) = k^s$.

If $p \nmid n$ Theorem 17 gives

$$|A(p)| < b_{20}p^{1/2-s/2} < b_{20}p^{1-s/2}.$$

If $p \mid n$ it follows from Theorem 12 that

$$\begin{aligned} |A(p)| &< p |p^{-1}S_p|^s \leq p(p^{-1}(k-1)p^{1/2})^s \\ &< k^s p^{1-s/2} = b_{22}p^{1-s/2}. \end{aligned}$$

THEOREM 19. (L, Theorem 319.) *For $s \geq 4$,*

$$\mathfrak{S} = \sum_{q=1}^{\infty} A(q)$$

converges absolutely and

$$\mathfrak{S} = \prod_p \chi_p.$$

THEOREM 20. (L, Theorem 320.) *If $p \nmid n$ then*

$$|\chi_p - 1| < b_{23}p^{1/2-s/2},$$

where $b_{23} = \max(b_{20}, b_{24}) = b_{20} = k^s$.

(i) Let $p \nmid (2k)$. By the remark after Theorem 8 and by Theorem 17,

$$|\chi_p - 1| = |A(p)| < b_{20}p^{1/2-s/2}.$$

(ii) Let $p \mid (2k)$. Then since $p \nmid n$ we have $\beta = 0$ and $t_0 = \max(1, \gamma) \leq k$. By (12)

$$\begin{aligned} |\chi_p - 1| &= \left| \sum_{i=1}^{t_0} A(p^i) \right| \leq \sum_{i=1}^k p^i \leq \sum_{i=1}^k (2k)^i \\ &= 2k \frac{(2k)^k - 1}{2k - 1} < 2^{k+1} k^k \\ &= \frac{2^{k+1} k^k}{(2k)^{1/2-s/2}} (2k)^{1/2-s/2} \leq \frac{2^{k+1} k^k}{(2k)^{1/2-s/2}} p^{1/2-s/2} \\ &= b_{24}p^{1/2-s/2}. \end{aligned}$$

THEOREM 21. (L, Theorem 220.) Let $D_q(m)$ denote the sum of the m th powers of all primitive q th roots of unity. Then

$$D_q(m) = \sum_{d|(q,m)} d \cdot \mu(q/d),$$

where

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^j & \text{if } n \text{ is the product of } j \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

COROLLARY. If p is a prime then

$$D_{p^t}(m) = \begin{cases} p^t - p^{t-1} & \text{if } p^t | m, \\ -p^{t-1} & \text{if } p^{t-1} | m, p^t \nmid m, \\ 0 & \text{if } p^{t-1} \nmid m. \end{cases}$$

Proof: $D_{p^t}(m) = \sum_{d|(p^t, m)} d \cdot \mu(p^t/d)$ and $\mu(p^t/d) = 1$ if $d = p^t$, $\mu(p^t/d) = -1$ if $d = p^{t-1}$, and $\mu(p^t/d) = 0$ in all other cases.

THEOREM 22. (L, Theorem 321.) If $p^k \nmid n$ then

$$|\chi_p - 1| < b_{25} p^{1-s/2},$$

where $b_{25} = \max(b_{25}, b_{26}, b_{27}) = 1 + k^2$.

If $p \nmid n$ Theorem 20 gives

$$|\chi_p - 1| < b_{23} p^{1/2-s/2} < b_{23} p^{1-s/2}.$$

Hence we may assume $p | n$ and since $p^k \nmid n$ we have $\beta = 0$, $1 \leq \sigma \leq k-1$.

(i) Let $p \nmid (2k)$. Then $\gamma = 1$, $t_0 = \max(\sigma+1, 1) = \sigma+1$ and

$$(15) \quad \chi_p - 1 = A(p) + \sum_{t=2}^{\sigma+1} A(p^t).$$

Also, by Theorem 11, $S_p = p^{t-1}$ since $2 \leq t \leq \sigma+1 \leq k$ and then from the Corollary to Theorem 21 we obtain

$$\begin{aligned} A(p^t) &= \sum_p (p^{-t} S_p)^s p^{-n} = \sum (p^{-1})^s p^{-n} \\ &= p^{-s} D_{p^t}(-n) \\ (16) \quad &= p^{-s} \begin{cases} p^t - p^{t-1} & \text{when } p^t | n, \text{ that is, when } 2 \leq t \leq \sigma, \\ -p^{t-1} & \text{when } p^{t-1} | n, p^t \nmid n, \text{ that is, when } t = \sigma+1, \\ 0 & \text{when } p^{t-1} \nmid n, \text{ which does not occur.} \end{cases} \end{aligned}$$

Therefore from (15), (16), and Theorem 18 we get

$$\begin{aligned}
 |\chi_p - 1| &= \left| A(p) + \sum_{t=2}^{\sigma+1} A(p^t) \right| \\
 &= \left| A(p) + p^{-s} \left(\sum_{t=2}^{\sigma} (-p^{t-1} + p^t) - p^{\sigma} \right) \right| \\
 &= |A(p) - p^{1-s}| < b_{21}p^{1-s/2} + p^{1-s} \\
 &< (b_{21} + 1)p^{1-s/2} = b_{26}p^{1-s/2}.
 \end{aligned}$$

(ii) Let $p \mid (2k)$. Then $t_0 = \max(\sigma+1, \gamma) \leq k$ and

$$|\chi_p - 1| \leq \sum_{t=1}^k (2k)^t < b_{24}p^{1/2-s/2} < b_{24}p^{1-s/2}.$$

THEOREM 23. (L, Theorem 322.) *We have*

$$\chi_p > 1 - b_{28}p^{1-s/2},$$

where $b_{28} = \max(b_{25}, b_{29}) = 1 + k^s$.

If $p^k \nmid n$, Theorem 22 gives $\chi_p > 1 - b_{25}p^{1-s/2}$. Hence let $p^k \mid n$, that is, $\beta > 0$. (i) Let $p > k$ so that $\gamma = 1, P = p$. Applying Theorem 8 twice we get

$$\chi_p \geq p^{1-s}N(p, 0) = p^{1-s}N(p, p) = \chi_p(p).$$

Since $p^k \nmid p$ we have

$$\chi_p \geq \chi_p(p) > 1 - b_{26}p^{1-s/2}$$

by Theorem 22. (ii) Let $p \leq k$. Then

$$\chi_p \geq 0 = 1 - k^{s/2-1}k^{1-s/2} \geq 1 - k^{s/2-1}p^{1-s/2} = 1 - b_{29}p^{1-s/2}.$$

THEOREM 24. (L, Theorem 324.) *If $p > (1+k^s)^{2/(s-5)} = b_{18}$ then*

$$\chi_p > 1 - p^{-3/2}.$$

By Theorem 23

$$\chi_p > 1 - b_{28}p^{1-s/2} = 1 - (1+k^s)p^{5/2-s/2}p^{-3/2} \geq 1 - p^{-3/2}$$

when $p > (1+k^s)^{2/(s-5)} = b_{18}$.

THEOREM 25. (L, Theorems 325-326.) *We have*

$$\mathfrak{S} > b_4$$

where

$$b_4 = \prod_{p \leq b_{18}} b(p) \cdot \prod_{p > b_{18}} (1 - p^{-3/2}).$$

We use Theorems 19, 10, and 24:

$$\mathfrak{S} = \prod_p \chi_p = \prod_{p \leq b_{18}} \chi_p \cdot \prod_{p > b_{18}} \chi_p > \prod_{p \leq b_{18}} b(p) \cdot \prod_{p > b_{18}} (1 - p^{-3/2}) = b_4.$$

5. The main lemma for the third Hardy-Littlewood theorem. In this section we follow the methods of paper G. We shall assume that

$$(17) \quad \begin{aligned} \eta_1 &\geq 17 \text{ when } k = 6, \\ \eta_1 &\geq D + 2^{k-3} \text{ when } k \geq 7, \end{aligned}$$

so that from (2)

$$(18) \quad (k-2)2^{k-2} + k + 2 \leq s < 4(k-2)2^{k-2} + 4k.$$

As we shall see later our final choice of η_1 satisfies the conditions (17). It follows from the second part of (18) that

$$\begin{aligned} \frac{K-1}{K} - \frac{2s-2K}{2s-K} &= \frac{K^2+K-2s}{K(2s-K)} \geq \frac{1}{K^2}, \\ \frac{2s-2k}{2s-k} - \frac{2s-2K}{2s-K} &= \frac{2s(K-k)}{(2s-k)(2s-K)} \geq \frac{K-k}{2s-K} \geq \frac{1}{K^2}. \end{aligned}$$

We may then choose $\theta = \theta(k, s, \epsilon)$ so that

$$(19) \quad \frac{2s-2k}{2s-k} - \theta \geq \frac{1}{2K^2},$$

$$(20) \quad \frac{K-1}{K} - \theta \geq \frac{1}{2K^2},$$

$$(21) \quad \theta - \frac{2s-2K}{2s-K} \geq \frac{1}{2K^2}.$$

The purpose of this section is to find an approximation for

$$\int_c \left| f'(x) - \sum_{q,p} \psi_p'(x) \right|^2 |dx|$$

taken around the unit circle $|x|=1$. We divide the circumference into sub-arcs in the following manner. On the circle we take the points $\rho = e^{2\pi i b/q}$ which correspond to the Farey fractions* with denominators $q \leq n^{1-\epsilon}$. The mediants between two neighboring Farey fractions form the end points of our sub-arcs. It is known that if x is any point of an arc which contains the point ρ then

$$x = \rho e^{2\pi i v} = e^{2\pi i (b/q + v)},$$

* For the definition and properties of Farey fractions see L, pp. 98-100.

where

$$-y_1 \leq y \leq y_2,$$

$$1/(2qn^{1-a}) \leq y_1 < 1/(qn^{1-a}), \quad 1/(2qn^{1-a}) \leq y_2 < 1/(qn^{1-a}).$$

The arcs for which $n^a < q \leq n^{1-a}$ are called *minor arcs* and are denoted by m ; those for which $1 \leq q \leq n^a$ are called *major arcs* and denoted by M . Each major arc is further divided into two sub-arcs denoted by M_1 when $|y| \leq 1/(2q^{\theta}n^{1-a\theta})$, and by M_2 when $|y| > 1/(2q^{\theta}n^{1-a\theta})$.

THEOREM 26. (G, Theorem 1; L, Theorem 140.) Let $|y| \leq \frac{1}{2}$, $a_0 \geq a_1 \geq \dots \geq 0$. Then

$$\left| \sum_{j=0}^N a_j e^{2\pi i y j} \right| \leq a_0 / \sin \pi |y|.$$

THEOREM 27. (G, Theorem 2; L, Theorem 223.)

$$\int_{-1/2}^{1/2} \left| \sum_{j=0}^N a_j e^{2\pi i y j} \right|^2 dy = \sum_{j=0}^N |a_j|^2.$$

THEOREM 28. (G, Theorem 3; L, Theorem 262.)

$$\sum_{j=0}^N r_{k,2}^2(j) < G_3 N^{2a+a_1},$$

where $G_3 = 4(k-1)A_1$.

If k is even $r_{k,2}(j)$ is at most equal to the number of solutions of $h_1^k + h_2^k = j$ and this is

$$4 \sum_{\substack{u|j \\ u \text{ odd}}} (-1)^{(u-1)/2} \leq 4T(j) \leq 4A_1 j^{a_1}$$

by Theorem 1. If k is odd,

$$j = h_1^k + h_2^k = (h_1 + h_2)(h_1^{k-1} + \dots + h_2^{k-1})$$

implies that $h_1 + h_2$ divides j . For each positive divisor d of j the two equations

$$h_1 + h_2 = d, \quad h_1^k + h_2^k = j$$

have at most $k-1$ solutions in common since the elimination of h_2 between them yields an equation of degree $k-1$. Therefore when k is odd

$$r_{k,2}(j) \leq (k-1)T(j) \leq (k-1)A_1 j^{a_1}.$$

Thus for all $k \geq 6$

$$r_{k,2}(j) \leq \max(4, k-1)A_1 j^{a_1} = (k-1)A_1 j^{a_1}.$$

Next, $\sum_{j=0}^N r_{k,2}(j)$ is the number of solutions of $h_1^k + h_2^k \leq N$, $h_1 \geq 0$, $h_2 \geq 0$. Since $h_1 \leq N^a$, $h_2 \leq N^a$, this is at most $(1+N^a)(1+N^a) \leq 4N^{2a}$. Finally

$$\sum_{j=0}^N r_{k,2}^2(j) \leq \max_{0 \leq j \leq N} r_{k,2}(j) \cdot \sum_{j=0}^N r_{k,2}(j) \leq (k-1)A_1 N^{a_1} \cdot 4N^{2a} = G_2 N^{3a+a_1}.$$

THEOREM 29. (G, Theorem 4; L, Theorem 277.) For $\beta > 0$ and j a positive integer,

$$\left| \frac{\Gamma(\beta+1+j)}{j!} - j^\beta \right| < \gamma(\beta) j^{\beta-1},$$

where $\gamma(\beta) = 4\beta(2^\beta+1)e^\beta \Gamma(\beta+1)$ is independent of j .

It is known that*

$$\lim_{j \rightarrow \infty} \Gamma(\beta+1+j)/(j!j^\beta) = 1.$$

Let $\Phi(j) = \Gamma(\beta+1+j)/(j!j^\beta)$. Then

$$\Phi(j) = 1 + \sum_{v=j+1}^{\infty} (\Phi(v-1) - \Phi(v)),$$

$$(22) \quad \Phi(v-1) - \Phi(v) = \Phi(v-1)(1 - (1 + \beta/v)(1 - 1/v)^\beta).$$

Also,

$$\begin{aligned} & \left| 1 - \left(1 + \frac{\beta}{v}\right) \left(1 - \frac{1}{v}\right)^\beta \right| \\ &= \left| 1 - \left(1 + \frac{\beta}{v}\right) \left(1 - \binom{\beta}{1} \frac{1}{v} + \binom{\beta}{2} \frac{1}{v^2} - \dots \right) \right| \\ &= \left| 1 - \left(1 - \left(\beta \binom{\beta}{1} - \binom{\beta}{2}\right) \frac{1}{v^2} \right. \right. \\ &\quad \left. \left. + \left(\beta \binom{\beta}{2} - \binom{\beta}{3}\right) \frac{1}{v^3} - \dots \right) \right| \\ &= \left| \binom{\beta+1}{2} \frac{1}{v^2} - \binom{\beta+1}{3} \frac{2}{v^3} + \dots \right|. \end{aligned}$$

If β is an integer this expression is

$$\leq \sum_{t=1}^{\beta} \binom{\beta+1}{t+1} \frac{t}{v^{t+1}} < \frac{\beta}{v^2} \sum_{t=1}^{\beta} \binom{\beta+1}{t+1} = \frac{\beta \cdot 2^{\beta+1}}{v^2}.$$

Next, suppose that β is not an integer. Since

* See, for example, Whittaker and Watson, *Modern Analysis*, Chapter XII.

$$\left| \binom{\beta+1}{t+2} (t+1) \right| \leq \left| \binom{\beta+1}{t+1} t \right|$$

when $t \geq [\beta] + 1$, and since

$$1 \geq \binom{\beta+1}{[\beta]+2} = \frac{(\beta+1)\beta \cdots (\beta - [\beta])}{([\beta]+2)!} > 0,$$

we have

$$\begin{aligned} & \left| \binom{\beta+1}{2} \frac{1}{v^2} - \binom{\beta+1}{3} \frac{2}{v^3} + \cdots \right| \\ & \leq \sum_{t=1}^{[\beta]} \binom{\beta+1}{t+1} \frac{t}{v^{t+1}} + \sum_{t=[\beta]+1}^{\infty} \binom{\beta+1}{[\beta]+2} \frac{[\beta]+1}{v^{t+1}} \\ & \leq \frac{2\beta}{v^2} \sum_{t=1}^{[\beta]} \binom{[\beta]+1}{t+1} + \frac{[\beta]+1}{v^2} \sum_{t=[\beta]}^{\infty} \frac{1}{v^t} \\ & < \left(2\beta \cdot 2^{\beta+1} + 2\beta \frac{v}{v-1} \right) \frac{1}{v^2} \leq (2\beta \cdot 2^{\beta+1} + 4\beta) \frac{1}{v^2}. \end{aligned}$$

Hence

$$(23) \quad |1 - (1 + \beta/v)(1 - 1/v)^{\beta}| < \max(\beta \cdot 2^{\beta+1}, 4\beta(2^{\beta} + 1))v^{-2} = \gamma_1(\beta)v^{-2}.$$

Also,

$$\begin{aligned} \log \Phi(v-1) &= \log \Gamma(\beta+v) - \log((v-1)!) - \beta \log(v-1) \\ &= \log \left((\beta+v-1) \cdots (\beta+1) \Gamma(\beta+1) \right) - \sum_{n=1}^{v-1} \log n - \beta \log(v-1) \\ &= \sum_{n=1}^{v-1} (\log(\beta+n) - \log n) + \log \Gamma(\beta+1) - \beta \log(v-1) \\ &\leq \sum_{n=1}^{v-1} (\beta n^{-1}) + \log \Gamma(\beta+1) - \beta \log(v-1) \\ &\leq \beta + \beta \int_1^{v-1} u^{-1} du + \log \Gamma(\beta+1) - \beta \log(v-1) \\ &= \beta + \beta \log(v-1) + \log \Gamma(\beta+1) - \beta \log(v-1) \\ &= \beta + \log \Gamma(\beta+1). \end{aligned}$$

Therefore

$$(24) \quad \Phi(v-1) \leq e^{\beta} \Gamma(\beta+1) = \gamma_2(\beta).$$

From (22), (23), and (24) we get

$$\begin{aligned} |\Phi(v-1) - \Phi(v)| &\leq \gamma_1(\beta) \cdot \gamma_2(\beta) v^{-2} = \gamma(\beta) v^{-2}; \\ |\Gamma(\beta+1+j)/(j!j^\beta) - 1| &= |\Phi(j) - 1| = \left| \sum_{v=j+1}^{\infty} (\Phi(v-1) - \Phi(v)) \right| \\ &< \gamma(\beta) \sum_{v=j+1}^{\infty} v^{-2} \leq \gamma(\beta) \left((j+1)^{-2} + \int_{j+1}^{\infty} u^{-2} du \right) < \gamma(\beta) j^{-1}. \end{aligned}$$

COROLLARY. For $\beta > 0$ and j an integer ≥ 1 ,

$$\Gamma(\beta+1+j)/j! > \gamma_3 j^\beta,$$

where $\gamma_3 = (1+\gamma(1))^{-1} = (1+12e)^{-1}$.

Since $\Gamma(\beta+2+j) = (\beta+1+j) \Gamma(\beta+1+j)$ it follows that $\Gamma(\beta+2+j)/j! > \gamma_3 j^{\beta+1}$ if $\Gamma(\beta+1+j)/j! > \gamma_3 j^\beta$. Hence we may assume $0 < \beta \leq 1$.

(i) Let $j \geq 1 + \gamma(1)$. Then

$$\begin{aligned} \Gamma(\beta+1+j)/j! &> j^\beta - \gamma(\beta) j^{\beta-1} > j^\beta - \gamma(1) j^{\beta-1} \\ &= j^\beta (1 + \gamma(1))^{-1} + \gamma(1) j^\beta (1 + \gamma(1))^{-1} - \gamma(1) j^{\beta-1} \\ &\geq j^\beta (1 + \gamma(1))^{-1} = \gamma_3 j^\beta. \end{aligned}$$

(ii) Let $1 \leq j < 1 + \gamma(1)$. Then

$$\begin{aligned} \Gamma(\beta+1+j)/j! &= (\beta+j) \cdots (\beta+2) \Gamma(\beta+2)/j! > j! \Gamma(2)/j! = 1 \\ &= (1 + \gamma(1))(1 + \gamma(1))^{-1} > j(1 + \gamma(1))^{-1} \\ &\geq j^\beta (1 + \gamma(1))^{-1} = \gamma_3 j^\beta. \end{aligned}$$

THEOREM 30. (G, Theorem 5; L, Theorem 267.) Let t be an integer, $m > 0$, and

$$S = \sum_{k=t+1}^{t+m} \rho^{k\lambda}.$$

Then for every $\epsilon_2 > 0$, $\epsilon_3 > 0$,

$$|S|^K < G_4 m^{\epsilon_2 q^{\epsilon_3}} (m^{K-1} + m^K q^{-1} + m^{K-k} q),$$

where $G_4 = C_{18} = 8k! C_{15} A_3$.

By Theorem 6 with $z = b/q$ we have

$$(25) \quad |S|^K < C_{15} m^{\epsilon_2} \left(m^{K-1} + m^{K-k} \sum_{v=1}^{k \lfloor m^{k-1} \rfloor} \min(m, 1/\{bv/q\}) \right).$$

Divide the summation into partial sums according to the j in

$$bv \equiv j \pmod{q}, \quad 0 \leq j \leq q-1.$$

Since $\{b_1/q\} = \{b_2/q\}$ when $b_1 \equiv b_2 \pmod{q}$ we have

$$\min(m, 1/\{bv/q\}) = \min(m, 1/\{j/q\}).$$

Each partial sum has at most $k! m^{k-1} q^{-1} + 1$ terms and thus

$$\sum_{v=1}^{k!m^{k-1}} \min(m, 1/\{bv/q\}) \leq (k! m^{k-1} q^{-1} + 1) \sum_{j=0}^{q-1} \min(m, 1/\{j/q\}).$$

Also

$$\begin{aligned} \sum_{j=0}^{q-1} \min(m, 1/\{j/q\}) &\leq m + \sum_{j=1}^{q-1} (1/\{j/q\}) \leq m + 2 \sum_{1 \leq j \leq q/2} (1/\{j/q\}) \\ &= m + 2 \sum_{1 \leq j \leq q/2} (1/(j/q)) < m + 2 \sum_{j=1}^q qj^{-1} \leq m + 2q(1 + \log q) \\ &\leq m + 2qA_3q^a \quad (\text{Theorem 7}). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{v=1}^{k!m^{k-1}} \min(m, 1/\{bv/q\}) &< k!(m^{k-1}q^{-1} + 1)(m + 2A_3q^{1+a}) \\ &\leq 2k!A_3q^a(m^{k-1}q^{-1} + 1)(m + q) \\ &< 4k!A_3q^a(m^{k-1}q^{-1} + m^{k-1} + q). \end{aligned}$$

Combining this result with (25) we obtain

$$\begin{aligned} |S|^K &< C_{15}m^{a_2}(m^{K-1} + m^{K-k} \cdot 4k!A_3q^a(m^{k-1}q^{-1} + m^{k-1} + q)) \\ &\leq 4k!C_{15}A_3m^{a_2}q^a(m^{K-1} + m^Kq^{-1} + m^{K-1} + m^{K-k}q) \\ &< 8k!C_{15}A_3m^{a_2}q^a(m^{K-1} + m^Kq^{-1} + m^{K-k}q). \end{aligned}$$

We choose $\epsilon_3 = \epsilon_1/(10k)$ so that $\epsilon_2 + k\epsilon_3 = D\epsilon_1$.

THEOREM 31. (G, Theorem 6.) *On the entire circle $|x| = 1$*

$$|\psi_\rho(x)| < G_5 n^a q^{-a},$$

where $G_5 = c_{15}(\Gamma(1+a) + 2a\gamma(a)(1-a^2)^{-1} + (2a+1)(a+1)^{-1})$.

We use Theorems 15 and 29:

$$\begin{aligned} |\psi_\rho(x)| &= |q^{-1}S_\rho| \cdot |\Gamma(1+a) + a \sum_{j=1}^n (\Gamma(1+a+j)/j!)(a+j)^{-1}(x/\rho)^j| \\ &< c_{15}q^{-a} \left(\Gamma(1+a) + a \sum_{j=1}^n (j^a + \gamma(a)j^{a-1})(a+j)^{-1} \right) \end{aligned}$$

$$\begin{aligned}
&\leq c_{15}q^{-a} \left(\Gamma(1+a) + a \left((1+\gamma(a))(a+1)^{-1} \right. \right. \\
&\quad \left. \left. + \int_1^n (j^{a-1} + \gamma(a)j^{a-2})dj \right) \right) \\
&< c_{15}q^{-a} \left(\Gamma(1+a) + a((1+\gamma(a))(a+1)^{-1} + n^a a^{-1} + \gamma(a)(1-a)^{-1}) \right) \\
&\leq c_{15}q^{-a} \left(\Gamma(1+a) + a((1+\gamma(a))(a+1)^{-1} + k + \gamma(a)(1-a)^{-1}) \right) n^a \\
&= G_5 n^a q^{-a}.
\end{aligned}$$

THEOREM 32. (G, Theorem 7.) *We have*

$$|\phi_p(x)| < G_7 n^{a-1} q^{-a} |y|^{-1}; \quad |\Psi_p(x)| < G_8 q^{-a} |y|^{-a};$$

where $G_7 = \frac{1}{2}ac_{15}(1+\gamma(a))$, $G_8 = \Gamma(1+a)c_{15}$.

(i) By Theorems 15, 26, and 29,

$$\begin{aligned}
|\phi_p(x)| &< c_{15}q^{-a}\Gamma(1+a) \left| \sum_{j=n+1}^{\infty} (a(a+1) \cdots (a+j-1)/j!)(x/\rho)^j \right| \\
&\leq c_{15}q^{-a}\Gamma(1+a) \cdot a(a+1) \cdots (a+n)/((n+1)! \sin \pi |y|) \\
&= c_{15}q^{-a}a(n+1)^{-1}(\Gamma(1+a+n)/n!)(\sin \pi |y|)^{-1} \\
&< c_{15}q^{-a}n^{-1}a(n^a + \gamma(a)n^{a-1})(2|y|)^{-1} \\
&\leq \frac{1}{2}ac_{15}(1+\gamma(a))n^{a-1}q^{-a}|y|^{-1}.
\end{aligned}$$

(ii) From Theorem 15 it follows that

$$\begin{aligned}
|\Psi_p(x)| &= |\Gamma(1+a)q^{-1}S_p(1-x/\rho)^{-a}| \\
&= |\Gamma(1+a)| \cdot |q^{-1}S_p| \cdot |1 - e^{2\pi i y}|^{-a} \\
&= |\Gamma(1+a)| \cdot |q^{-1}S_p| \cdot |e^{\pi i y} - e^{-\pi i y}|^{-a} \\
&< \Gamma(1+a)c_{15}q^{-a} |2 \sin \pi y|^{-a} \\
&\leq \Gamma(1+a)c_{15}q^{-a}(4|y|)^{-a} < G_8 q^{-a} |y|^{-a}.
\end{aligned}$$

THEOREM 33. (G, Theorem 8.) *We have*

$$|\psi_p(x)| < G_9 q^{-a} \min(n^a, |y|^{-a}),$$

where $G_9 = \max(G_5, G_7 + G_8)$.

(i) Let $|y| \leq 1/n$. By Theorem 31

$$|\psi_p(x)| < G_5 n^a q^{-a} = G_9 q^{-a} \min(n^a, |y|^{-a}).$$

(ii) Let $|y| \geq 1/n$. From Theorem 32 we get

$$\begin{aligned} |\psi_p(x)| &\leq |\phi_p(x)| + |\Psi_p(x)| < G_7 n^{a-1} q^{-a} |y|^{-1} + G_8 q^{-a} |y|^{-a} \\ &= G_7 n^{a-1} |y|^{a-1} q^{-a} |y|^{-a} + G_8 q^{-a} |y|^{-a} \\ &\leq G_7 q^{-a} |y|^{-a} + G_8 q^{-a} |y|^{-a} \\ &= (G_7 + G_8) q^{-a} \min(n^a, |y|^{-a}). \end{aligned}$$

THEOREM 34. (G, Theorem 9.) *We have*

$$\sum_{M_1} \int_{C-M_1} |\psi_p^*(x)|^2 |dx| < G_{10} n^{2sa-1-\lambda-2a},$$

where

$$G_{10} = \frac{2^{2sa}}{2sa-1} \frac{2sa-1-(2sa-1)\theta}{2sa-2-(2sa-1)\theta} G_9^{2a}.$$

The integral is taken around the entire circle with the exception of the arc M_1 itself and the summation extends over all M_1 -arcs.

From Theorem 33 we get

$$\begin{aligned} \int_{C-M_1} |\psi_p^*(x)|^2 |dx| &< 2G_9^{2a} q^{-2sa} \int_{1/(2q^n n^{1-\theta a})}^{\infty} y^{-2sa} dy \\ &= 2^{2sa} G_9^{2a} (2sa-1)^{-1} n^{2sa-1-(2sa-1)\theta a} q^{-2sa+(2sa-1)\theta}. \end{aligned}$$

The exponent of q is < -2 since $\theta < (2sa-2)/(2sa-1)$ by (19). Also, for each q there are at most q arcs. Hence

$$\begin{aligned} \sum_{M_1} \int_{C-M_1} |\psi_p^*(x)|^2 |dx| &< 2^{2sa} G_9^{2a} (2sa-1)^{-1} \cdot n^{2sa-1-(2sa-1)\theta a} \sum_{1 \leq q \leq n^a} q^{1-2sa+(2sa-1)\theta} \\ &\leq 2^{2sa} G_9^{2a} (2sa-1)^{-1} \cdot n^{2sa-1-(2sa-1)\theta a} \left(1 + \int_1^{n^a} q^{1-2sa+(2sa-1)\theta} dq \right) \\ &< 2^{2sa} G_9^{2a} (2sa-1)^{-1} \cdot n^{2sa-1-(2sa-1)\theta a} (1 + (2sa-2-(2sa-1)\theta)^{-1}) \\ &= G_{10} n^{2sa-1-(2sa-1)\theta a}. \end{aligned}$$

In the exponent of n we have

$$(2sa-1)\theta a > ((2s-k)/k^2)((2s-2K)/(2s-K)) \quad (\text{by (21)})$$

$$\geq ((2s-k)/k^2)((k-2)K-2K)/((k-2)K-K) \quad (\text{by (2)})$$

$$= ((2s-k)/k^2)((k-4)/(k-3))$$

$$\begin{aligned}
&> \frac{((k-2)K + 2k + (1 + (1-a)^{s-2})kK\epsilon_1)(k-4)}{k^2(k-3)} \\
&> 2a + 2A + (1-a)^{s-2}\epsilon_1 = 2a + \lambda.
\end{aligned}$$

Therefore $G_{10}n^{2sa-1-(2sa-1)\theta a} < G_{10}n^{2sa-1-\lambda-2a}$.

THEOREM 35. (G, Theorem 10.) *On m we have*

$$|f(x)| < G_{13}n^{s-aA+aAD\epsilon_1},$$

where $G_{13} = (2\pi + 1)2^{1+A}G_4^A$.

Let $\tau(j) = \sum_{h=0}^j \rho^{hk}$, $j \geq 0$, $\tau(-1) = 0$. Then

$$\begin{aligned}
(26) \quad f(x) &= \sum_{h=0}^{na} \rho^{hk} (x/\rho)^{hk} = \sum_{j=0}^n (\tau(j) - \tau(j-1))(x/\rho)^j \\
&= \sum_{j=0}^{n-1} \tau(j)((x/\rho)^j - (x/\rho)^{j+1}) + \tau(n)(x/\rho)^n \\
&= (1 - x/\rho) \sum_{j=0}^{n-1} \tau(j)(x/\rho)^j + \tau(n)(x/\rho)^n.
\end{aligned}$$

By Theorem 30 with $m = [j^a] + 1$, $t = -1$,

$$\begin{aligned}
|\tau(j)|^K &= \left| \sum_{h=0}^j \rho^{hk} \right|^K < G_4([j^a] + 1)^{\epsilon_2 q \epsilon_1} \left(([j^a] + 1)^{K-1} \right. \\
&\quad \left. + ([j^a] + 1)^{Kq-1} + ([j^a] + 1)^{K-kq} \right) \\
&\leq G_4(2j^a)^{\epsilon_2 q \epsilon_1} ((2j^a)^{K-1} + (2j^a)^{Kq-1} + (2j^a)^{K-kq}) \\
&\leq G_4(2n^a)^{\epsilon_2 q \epsilon_1} ((2n^a)^{K-1} + (2n^a)^{Kq-1} + (2n^a)^{K-kq}) \\
&\leq G_4(2n^a)^{\epsilon_2 n^{(1-a)\epsilon_1}} (2^{K-1} + 2^K + 2^{K-k}) n^{aK-s} \quad (n^a < q \leq n^{1-s} \text{ on } m) \\
&< 2^{K+1} G_4(n^a)^{K-1+D\epsilon_1} \quad (\epsilon_2 + (k-1)\epsilon_3 < \epsilon_2 + k\epsilon_3 = D\epsilon_1);
\end{aligned}$$

$$(27) \quad |\tau(j)| < 2^{1+A} G_4^A n^{s-aA+aAD\epsilon_1}.$$

Also,

$$|1 - x/\rho| = |1 - e^{2\pi i y}| = |e^{\pi i y} - e^{-\pi i y}| = 2|\sin \pi y| \leq 2\pi|y| < 2\pi/(qn^{1-s}).$$

Therefore from (26) and (27)

$$\begin{aligned}
|f(x)| &\leq |1 - x/\rho| \sum_{j=0}^{n-1} |\tau(j)| + |\tau(n)| \\
&< (2\pi n/(qn^{1-s}) + 1)(2^{1+A} G_4^A) n^{s-aA+aAD\epsilon_1} \\
&\leq (2\pi + 1) 2^{1+A} G_4^A n^{s-aA+aAD\epsilon_1}.
\end{aligned}$$

THEOREM 36. (G, Theorem 11.) *We have*

$$\sum_m \int_m |f^s(x)|^2 |dx| < G_{18} n^{2s-1-\lambda},$$

where $G_{18} = 2^{2a+\epsilon_1} G_3 G_{13}^{2s-4}$.

By Theorem 35,

$$\begin{aligned} \sum_m \int_m |f^s(x)|^2 |dx| &\leq \max |f(x)|^{2s-4} \cdot \sum_m \int_m |f^2(x)|^2 |dx| \\ (28) \qquad \qquad \qquad &< G_{13}^{2s-4} n^{(2s-4)a - (2s-4)(1-D\epsilon_1)A} \int_{-1/2}^{1/2} |f^2(e^{2\pi i y})|^2 dy. \end{aligned}$$

But $f^2(x) = \sum_{j=0}^{2n} R(j)x^j$, where

$$\begin{aligned} R(j) &= r_{k,2}(j), & 0 \leq j \leq n; \\ 0 \leq R(j) &\leq r_{k,2}(j), & n < j \leq 2n; \end{aligned}$$

and by Theorems 27 and 28,

$$\int_{-1/2}^{1/2} |f^2(e^{2\pi i y})|^2 dy = \sum_{j=0}^{2n} R^2(j) \leq \sum_{j=0}^{2n} r_{k,2}^2(j) < G_3 (2n)^{2a+\epsilon_1}.$$

Combining this with (28) we get

$$\sum_m \int_m |f^s(x)|^2 |dx| < G_{13}^{2s-4} \cdot G_3 \cdot 2^{2a+\epsilon_1} n^{(2s-4)a - (2s-4)(1-D\epsilon_1)A + 2a+\epsilon_1}.$$

The exponent of n equals

$$\begin{aligned} 2sa - 1 - ((2s-4)(1-D\epsilon_1)A - 1 - 2a) + \epsilon_1 \\ = 2sa - 1 - \left((2s-4)(1-D\epsilon_1) - (k-2)K - 2k \right. \\ \left. - (1 + (1-a)^{s-2})kK\epsilon_1 \right) A - 2A - (1-a)^{s-2}\epsilon_1 \\ \leq 2sa - 1 - 2A - (1-a)^{s-2}\epsilon_1 \qquad \text{(by (2))} \\ = 2sa - 1 - \lambda \end{aligned}$$

and the theorem follows.

THEOREM 37. (G, Theorem 12.) *We have*

$$|f(x) - \psi_\sigma(x)| < G_{18} q^{1-A+AD\epsilon_1} \cdot \max(n|y|, 1),$$

where $G_{18} = (2\pi+1) (2(3G_4)^A + \gamma(a))$.

As before

$$(29) \quad f(x) = (1 - x/\rho) \sum_{j=0}^{n-1} \tau(j)(x/\rho)^j + \tau(n)(x/\rho)^n;$$

$$\begin{aligned} \psi_p(x) &= q^{-1}S_p \left((1 - x/\rho) \sum_{j=0}^{n-1} (\Gamma(1 + a + j)/j!)(x/\rho)^j \right. \\ &\quad \left. + (\Gamma(1 + a + n)/n!)(x/\rho)^n \right) \end{aligned}$$

$$(30) \quad = (1 - x/\rho) \sum_{j=0}^{n-1} v(j)(x/\rho)^j + v(n)(x/\rho)^n.$$

Each $\tau(j)$ has $[j^a] + 1$ terms and so may be written as $[j^a/q]$ partial sums each equal to S_p and $[j^a] + 1 - [j^a/q]q \leq q$ further terms. Then by Theorem 30 with $t=0$, $m \leq q$,

$$\begin{aligned} |\tau(j) - [j^a/q]S_p| &\leq G_4^A q^{(\epsilon_2 + \epsilon_3)A} (q^{K-1} + q^{K-1} + q^{K-k}q)^A \\ &\quad < (3G_4)^A q^{AD_{\epsilon_1+1}-A} \quad (\epsilon_2 + \epsilon_3 < \epsilon_2 + k\epsilon_3 = D_{\epsilon_1}); \\ |\tau(j) - (j^a/q)S_p| &\leq |\tau(j) - [j^a/q]S_p| + |S_p| \\ (31) \quad &< 2(3G_4)^A q^{1-A+AD_{\epsilon_1}}. \end{aligned}$$

Since by Theorem 29

$$|\Gamma(1 + a + j)/j! - j^a| < \gamma(a)j^{a-1} \leq \gamma(a),$$

we have

$$(32) \quad |v(j) - j^a q^{-1}S_p| = |q^{-1}S_p| |\Gamma(1 + a + j)/j! - j^a| < \gamma(a).$$

From (31) and (32) it follows that

$$\begin{aligned} |\tau(j) - v(j)| &= |\tau(j) - j^a q^{-1}S_p + j^a q^{-1}S_p - v(j)| \\ &\leq |\tau(j) - j^a q^{-1}S_p| + |v(j) - j^a q^{-1}S_p| \\ &< 2(3G_4)^A q^{1-A+AD_{\epsilon_1}} + \gamma(a) \\ &\leq (2(3G_4)^A + \gamma(a))q^{1-A+AD_{\epsilon_1}}. \end{aligned}$$

Then from (29) and (30)

$$\begin{aligned} |f(x) - \psi_p(x)| &= \left| (1 - x/\rho) \sum_{j=0}^{n-1} (\tau(j) - v(j))(x/\rho)^j + (\tau(n) - v(n))(x/\rho)^n \right| \\ &\leq |1 - x/\rho| \sum_{j=0}^{n-1} |\tau(j) - v(j)| + |\tau(n) - v(n)| \\ &< (2\pi n|y| + 1)(2(3G_4)^A + \gamma(a))q^{1-A+AD_{\epsilon_1}} \\ &< (2\pi + 1)(2(3G_4)^A + \gamma(a))q^{1-A+AD_{\epsilon_1}} \cdot \max(n|y|, 1). \end{aligned}$$

THEOREM 38. (G, Theorem 13.) *We have*

$$\sum_{M_1} \int_{M_1} |f^s(x) - \psi_p^s(x)|^2 dx < (G_{24} + G_{25}) n^{2sa-1-\lambda},$$

where

$$G_{24} = 2^{2s+1} G_{18}^{2s} \frac{(2s+2)(2s+3 - (2s+1)\theta - 2s(1-D\epsilon_1)A)}{(2s+1)(2s+2 - (2s+1)\theta - 2s(1-D\epsilon_1)A)},$$

$$G_{25} = 2^{2s+1} G_{18}^2 G_9^{2s} \frac{(2sa-2a-2)(2sa-3+2(1-D\epsilon_1)A)}{(2sa-2a-3)(2sa-4+2(1-D\epsilon_1)A)}.$$

Write $\Phi_p(x) = f(x) - \psi_p(x)$. Then

$$\begin{aligned} |f^s(x) - \psi_p^s(x)|^2 &= |(\Phi_p(x) + \psi_p(x))^s - \psi_p^s(x)|^2 \\ &= |\Phi_p(x)|^2 |\Phi_p^{s-1}(x) + \dots + s\psi_p^{s-1}(x)|^2 \\ &< |\Phi_p(x)|^2 \cdot 2^{2s} (|\Phi_p(x)|^{2s-2} + |\psi_p(x)|^{2s-2}) \\ &= 2^{2s} (|\Phi_p(x)|^{2s} + |\Phi_p(x)|^2 |\psi_p(x)|^{2s-2}), \end{aligned}$$

where by Theorem 37

$$|\Phi_p(x)| < G_{18} q^{1-A+AD\epsilon_1} \max(n|y|, 1).$$

Hence from Theorems 37 and 33 we get

$$\begin{aligned} &\int_{M_1} |f^s(x) - \psi_p^s(x)|^2 dx \\ &< 2^{2s} \left(\int_{M_1} |\Phi_p(x)|^{2s} dx + \int_{M_1} |\Phi_p(x)|^2 |\psi_p(x)|^{2s-2} dx \right) \\ &< 2^{2s+1} G_{18}^{2s} q^{2s-2s(1-D\epsilon_1)A} \left(\int_0^{1/n} dy + n^{2s} \int_{1/n}^{1/(q^\theta n^{1-\theta a})} y^{2s} dy \right) \\ &\quad + 2^{2s+1} G_{18}^2 G_9^{2s} q^{2s-2(1-D\epsilon_1)A-(2s-2)a} \left(n^{2sa-2a} \int_0^{1/n} dy \right. \\ &\quad \left. + n^2 \int_{1/n}^{1/(q^\theta n^{1-\theta a})} y^{2-(2s-2)a} dy \right) \\ &< 2^{2s+1} G_{18}^{2s} q^{2s-2s(1-D\epsilon_1)A} (n^{-1} + n^{2s-(2s+1)(1-\theta a)} q^{-(2s+1)\theta} (2s+1)^{-1}) \\ &\quad + 2^{2s+1} G_{18}^2 G_9^{2s} q^{2s-2a-2sa-2(1-D\epsilon_1)A} (n^{2sa-1-2a} + n^{2+(2s-2)a-3} ((2s-2)a-3)^{-1}) \\ &< G_{24} n^{(2s+1)\theta a-1} q^{2s-(2s+1)\theta-2s(1-D\epsilon_1)A} + G_{25} n^{2sa-1-2a} q^{2s+2a-2sa-2(1-D\epsilon_1)A}, \end{aligned}$$

where

$$G_{29} = 2^{2s+1} G_{18}^{2s} (2s+2)(2s+1)^{-1},$$

$$G_{30} = 2^{2s+1} G_{18}^{2s} G_9^{2s-2} (2sa-2a-2)(2sa-2a-3)^{-1}.$$

The exponent of q in the first term is

$$> 2s - (2s+1)\theta - 2sA = (1-\theta-A)2s - \theta > -\theta > -2 \quad (\text{by (20)}).$$

The exponent of q in the second term is $< 2+2a-2sa < -2$ since $2s > 4k+2$. Thus

$$\begin{aligned} \sum_{M_1} \int_{M_1} |f^s(x) - \psi_{\rho^s}(x)|^2 dx &< G_{29} n^{(2s+1)\theta a-1} \sum_{q=1}^{n^a} q^{1+2s-(2s+1)\theta-2s(1-D_{\epsilon_1})A} \\ &+ G_{30} n^{2sa-1-2a} \sum_{q=1}^{n^a} q^{3+2a-2sa-2(1-D_{\epsilon_1})A} \\ &\leq G_{29} n^{(2s+1)\theta a-1} \left(1 + \int_1^{n^a} q^{1+2s-(2s+1)\theta-2s(1-D_{\epsilon_1})A} dq \right) \\ &+ G_{30} n^{2sa-1-2a} \left(1 + \int_1^{n^a} q^{3+2a-2sa-2(1-D_{\epsilon_1})A} dq \right) \\ &< G_{29} n^{(2s+1)\theta a-1} \cdot n^{2a+2sa-(2s+1)\theta a-2s(1-D_{\epsilon_1})A} \\ &\quad \times (1 + (2s+2 - (2s+1)\theta - 2s(1-D_{\epsilon_1})A)^{-1}) \\ &\quad + G_{30} n^{2sa-1-2a} (1 + (2sa+2a+4 - 2(1-D_{\epsilon_1})A)^{-1}) \\ &\leq G_{24} n^{2sa-1-(2s(1-D_{\epsilon_1})A-2a)} + G_{25} n^{2sa-1-2a}. \end{aligned}$$

In the exponents of n we have

$$2s(1-D_{\epsilon_1})A - 2a > (2s-4)(1-D_{\epsilon_1})A - 1 + 2a \geq 2A + (1-a)^{s-2}\epsilon_1$$

as in the proof of Theorem 36; and

$$2sa-1-2a \leq 2sa-1-2A - (1-a)^{s-2}\epsilon_1 = 2sa-1-\lambda.$$

This completes the proof.

THEOREM 39. (G, Theorem 14.) On M_2

$$|f(x)| < G_{31} n^{a-(1-aD_{\epsilon_1})A} q^{-A} |y|^{-A},$$

where $G_{31} = (2\pi+1)2^{1+2A+(s+2s)A} G_4^A$.

On M_2 we have

$$x = e^{2\pi i(b/q+y)}, \quad 1 \leq q \leq n^a;$$

$$1/(2q^s n^{1-\theta a}) \leq |y| < 1/(qn^{1-a}), \quad 2/(q|y|) > 1.$$

From the theory of Farey fractions* it is known that there exist integers b_1 and q_1 and a number y_1 such that

$$b_1/q_1 + y_1 = b/q + y, \quad 0 \leq b_1 \leq q_1 \leq 2/(q|y|), \quad (b_1, q_1) = 1; \quad (33)$$

$$|y_1| < q|y|/(2q_1).$$

It follows from (33) that $|y_1| < 1/(2q_1n^{1-a})$ and so we cannot have $n^a < q_1 \leq n^{1-a}$, for if this were the case x would be a point of a minor arc m . Also, $1 \leq q_1 \leq n^a$ is impossible when $n > 2^k$, since otherwise

$$|bq_1 - b_1q| = qq_1|y_1 - y| \leq qq_1|y| + qq_1|y_1| < qq_1|y| + q^2|y| \\ = (q + q_1)|y| < 2n^aq/(qn^{1-a}) = 2n^{2a-1} \leq 2n^{-a} < 1,$$

and therefore $bq_1 - b_1q = 0$. Since $(b, q) = 1$, $(b_1, q_1) = 1$, we have $q_1 = q$, $b_1 = b$, $y_1 = y$, $|y_1| = q|y|/q_1 > q|y|/(2q_1)$, which contradicts (33). Hence, $q_1 > n^{1-a}$. Write $\rho_1 = e^{2\pi i b_1/q_1}$, $\tau_1(j) = \sum_{h=0}^{j-1} \rho_1^{h^2}$,

$$(34) \quad f(x) = (1 - x/\rho_1) \sum_{j=0}^{n-1} \tau_1(j)(x/\rho_1)^j + \tau_1(n)(x/\rho_1)^n.$$

By Theorem 30 with $q = q_1$, $m = [j^a] + 1$, $l = -1$,

$$|\tau_1(j)|^K < G_4([j^a] + 1)^{a_2} q_1^{a_2} (([j^a] + 1)^{K-1} + ([j^a] + 1)^K q^{-1} + ([j^a] + 1)^{K-k} q_1) \\ \leq G_4(2j^a)^{a_2} q_1^{a_2} ((2j^a)^{K-1} + (2j^a)^K q^{-1} + (2j^a)^{K-k} q_1) \\ \leq G_4(2n^a)^{a_2} q_1^{a_2} ((2n^a)^{K-1} + (2n^a)^K q^{-1} + (2n^a)^{K-k} q_1) \\ < G_4 2^{a_2} n^{a_2 a_2} q_1^{a_2} (2^{K-1} + 2^K + 2^{K-k}) n^{aK-1} q_1 \quad (q_1 > n^{1-a}) \\ \leq 2^{K+1+a_2+2a_2} G_4 n^{aK-1+aD_{a_1}} \cdot 2q^{-1} |y|^{-1};$$

$$(35) \quad |\tau_1(j)| < 2^{1+2A+(a_2+2a_2)A} G_4^A n^{a-(1-aD_{a_1})A} q^{-A} |y|^{-A}.$$

Also,

$$|1 - x/\rho_1| = |e^{\pi i y_1} - e^{-\pi i y_1}| = 2 \sin \pi |y_1| \leq 2\pi |y_1| < 2\pi q_1^{-1} < 2\pi n^{2a-1}.$$

Then from (34) and (35) we obtain

$$|f(x)| \leq |1 - x/\rho_1| \sum_{j=0}^{n-1} |\tau_1(j)| + |\tau_1(n)| \\ < (2\pi n \cdot n^{2a-2} + 1) 2^{1+2A+(a_2+2a_2)A} G_4^A n^{a-(1-aD_{a_1})A} q^{-A} |y|^{-A} \\ < G_{31} n^{a-(1-aD_{a_1})A} q^{-A} |y|^A.$$

This proves the theorem when $n \geq 2^k$. If $n < 2^k$ we have

* See the footnote at the beginning of this section.

$$\begin{aligned}
 |f(x)| &\leq n^a = n^{(1-aD_{\epsilon_1})A-(A-aA)} n^{a-(1-aD_{\epsilon_1})} \cdot n^{A-aA} \\
 &< 2^{k(1-aD_{\epsilon_1})A-k(A-aA)} n^{a-(1-aD_{\epsilon_1})} q^{-A} |y|^{-A} \\
 &< G_{31} n^{a-(1-aD_{\epsilon_1})A} q^{-A} |y|^{-A}.
 \end{aligned}$$

THEOREM 40. (G, Theorem 15.) *We have*

$$\sum_{M_2} \int_{M_2} |f^*(x)|^2 |dx| < G_{35} n^{2sa-1-\lambda},$$

where $G_{35} = 2^{2sA} G_{31}^{2s} (2sA-1)^{-1} (2sA-(2sA-1)\theta+3) (2sA-(2sA-1)\theta+2)^{-1}$.

By Theorem 39

$$\begin{aligned}
 \int_{M_2} |f^*(x)|^2 |dx| &< 2G_{31}^{2s} n^{2sa-2s(1-aD_{\epsilon_1})A} q^{-2sA} \int_{1/(3q^{\theta}n^{1-\theta a})}^{\infty} y^{-2sA} dy \\
 &= 2G_{31}^{2s} n^{2sa-2s(1-aD_{\epsilon_1})A} q^{-2sA} 2^{2sA-1} (2sA-1)^{-1} n^{(2sA-1)(1-\theta a)} q^{(2sA-1)\theta} \\
 &= 2^{2sA} (2sA-1)^{-1} G_{31}^{2s} n^{2sa-1-(2sA-1)\theta a+2saAD_{\epsilon_1}} q^{-2sA+(2sA-1)\theta}.
 \end{aligned}$$

Since the exponent of q is > -2 by (21), we have

$$\begin{aligned}
 \sum_{M_2} \int_{M_2} |f^*(x)|^2 |dx| &< 2^{2sA} (2sA-1)^{-1} G_{31}^{2s} n^{2sa-1-(2sA-1)\theta a+2saAD_{\epsilon_1}} \sum_{q=1}^{n^a} q^{1-2sA+(2sA-1)\theta} \\
 &\leq 2^{2sA} (2sA-1)^{-1} G_{31}^{2s} n^{2sa-1-(2sA-1)\theta a+2saAD_{\epsilon_1}} \left(1 + \int_1^{n^a} q^{1-2sA+(2sA-1)\theta} dq \right) \\
 &< 2^{2sA} (2sA-1)^{-1} G_{31}^{2s} n^{2sa-1-(2sA-1)\theta a+2saAD_{\epsilon_1}} \\
 &\quad \times (1 + ((2sA-1)\theta - 2sA + 2)^{-1} n^{(2sA-1)\theta a-2saAD_{\epsilon_1}+2a}) \\
 &\leq 2^{2sA} (2sA-1)^{-1} (2sA-(2sA-1)\theta+3) (2sA-(2sA-1)\theta+2)^{-1} \\
 &\quad \times G_{31}^{2s} n^{2sa-1-(2s(1-D_{\epsilon_1})aA-2a)}.
 \end{aligned}$$

As in Theorem 38 we have $2s(1-D_{\epsilon_1})aA-2a > \lambda$ and the proof is complete.

THEOREM 41. (G, Theorem 16.) *We have*

$$\int_C \left| f^*(x) - \sum_{q, \rho} \psi_{\rho}^*(x) \right|^2 |dx| < G_1 n^{2sa-1-\lambda},$$

where $G_1 = 2(G_{10} + G_{15} + G_{24} + G_{25} + G_{35})$. The summation extends over all ρ for which $1 \leq q \leq n^a$. There are $\sum_{q=1}^{n^a} \phi(q)$ terms in the sum.

We may write

$$\begin{aligned} \int_C \left| f^*(x) - \sum_{q,p} \psi_p^*(x) \right|^2 dx &= \sum_m \int_m |f^*(x) - \sum \psi_p^*(x)|^2 dx \\ &+ \sum_{M_2} \int_{M_2} |f^*(x) - \sum \psi_p^*(x)|^2 dx \\ &+ \sum_{M_1} \int_{M_1} |f^*(x) - \psi_p^*(x) - \sum' \psi_p^*(x)|^2 dx, \end{aligned}$$

the accent indicating that the term which corresponds to M_1 itself is omitted in the summation and written separately. Using the inequality

$$\left| \sum_{j=1}^N \xi_j \right|^2 \leq N \sum |\xi_j|^2$$

we obtain

$$\begin{aligned} &\int_C |f^*(x) - \sum \psi_p^*(x)|^2 dx \\ &\leq 2 \sum_m \int_m |f^*(x)|^2 dx + 2 \sum_m \int_m |\sum \psi_p^*(x)|^2 dx \\ &\quad + 2 \sum_{M_2} \int_{M_2} |f^*(x)|^2 dx + 2 \sum_{M_2} \int_{M_2} |\sum \psi_p^*(x)|^2 dx \\ &\quad + 2 \sum_{M_1} \int_{M_1} |f^*(x) - \psi_p^*(x)|^2 dx \\ &\quad + 2 \sum_{M_1} \int_{M_1} |\sum' \psi_p^*(x)|^2 dx \\ &\leq 2 \sum_m \int_m |f^*(x)|^2 dx + 2n^{2a} \sum_m \int_m |\sum \psi_p^*(x)|^2 dx \\ &\quad + 2 \sum_{M_2} \int_{M_2} |f^*(x)|^2 dx + 2n^{2a} \sum_{M_2} \int_{M_2} |\sum \psi_p^*(x)|^2 dx \\ &\quad + 2 \sum_{M_1} \int_{M_1} |f^*(x) - \psi_p^*(x)|^2 dx + 2n^{2a} \sum_{M_1} \int_{M_1} |\sum' \psi_p^*(x)|^2 dx \\ &= 2 \sum_m \int_m |f^*(x)|^2 dx + 2 \sum_{M_2} \int_{M_2} |f^*(x)|^2 dx \\ &\quad + 2 \sum_{M_1} \int_{M_1} |f^*(x) - \psi_p^*(x)|^2 dx + 2n^{2a} \sum_{M_1} \int_{C-M_1} |\psi_p^*(x)|^2 dx. \end{aligned}$$

To these terms we apply Theorems 36, 40, 35, and 34, respectively.

Then

$$\int_C \left| f^s(x) - \sum_{q, \rho} \psi_{\rho}^s(x) \right|^2 dx < 2(G_{10} + G_{15} + G_{24} + G_{25} + G_{35})n^{2s\sigma-1-\lambda}.$$

THEOREM 42. THE MAIN LEMMA. (G, §9.) *We have*

$$\sum_{j=1}^n |\sigma(j)|^2 < G_1 n^{2s\sigma-1-\lambda}.$$

We note that

$$\begin{aligned} \psi_{\rho}^s(x) &= \Gamma^s(1+a)q^{-s}S_{\rho}^s \times \text{the first } n+1 \text{ terms of } (1-x/\rho)^{sa} \\ &\quad + \text{a finite number of terms with higher powers of } x/\rho \\ &= \Gamma^s(1+a)q^{-s}S_{\rho}^s \left(1 + \sum_{j=1}^n (sa(sa+1) \cdots (sa+j-1)/j!)(x/\rho)^j \right) \\ &\quad + \text{higher powers} \\ &= (\Gamma^s(1+a)/\Gamma(sa))q^{-s}S_{\rho}^s \sum_{j=0}^n (\Gamma(sa+j)/j!)(x/\rho)^j \\ &\quad + \text{higher powers;} \\ \sum_{q, \rho} \psi_{\rho}^s(x) &= (\Gamma^s(1+a)/\Gamma(sa)) \sum_{j=0}^n (\Gamma(sa+j)/j!) \sum_q \sum_{\rho} q^{-s}S_{\rho}^s \rho^{-j} x^j \\ &\quad + \text{higher powers} \\ &= (\Gamma^s(1+a)/\Gamma(sa)) \sum_{j=0}^n (\Gamma(sa+j)/j!) \mathfrak{S}(j, k, s, n^s) x^j \\ &\quad + \text{higher powers.} \end{aligned}$$

Also,

$$f^s(x) = 1 + \sum_{j=1}^n r_{k,s}(j)x^j + \text{higher powers.}$$

Hence

$$f^s(x) - \sum_{q, \rho} \psi_{\rho}^s(x) = \sigma(0) + \sum_{j=1}^n \sigma(j)x^j + \text{higher powers.}$$

By Theorem 27,

$$\int_{-1/2}^{1/2} |f^s(e^{2\pi i y}) - \sum_{q, \rho} \psi_{\rho}^s(e^{2\pi i y})|^2 dy = \sum_{j=1}^n |\sigma(j)|^2 + \text{a positive quantity.}$$

Therefore

$$\sum_{j=1}^n |\sigma(j)|^2 < \int_{-1/2}^{1/2} |f^s(e^{2\pi i y}) - \sum_{q, \rho} \psi_{\rho}^s(e^{2\pi i y})|^2 dy < G_1 n^{2s\sigma-1-\lambda}$$

by Theorem 41.

COROLLARY. *Let*

$$\sigma_0(j) = (\Gamma^s(1+a)/\Gamma(sa))(\Gamma(sa+j)/j!) \mathfrak{E}(j, k, s, \infty).$$

Then

$$\sum_{j=1}^n |\sigma_0(j)|^2 < A_4 n^{2sa-1-\lambda},$$

where $A_4 = 2G_1 + 2(\Gamma^{2s}(1+a)/\Gamma^2(sa))(sa/(sa-1))^2(2sa/(2sa-1))b_{10}^2\gamma_2^2(sa-1)$.

By the proof of Theorem 4 we have

$$\begin{aligned} |\sigma_0(j) - \sigma(j)| &= (\Gamma^s(1+a)/\Gamma(sa))(\Gamma(sa+j)/j!) |\mathfrak{E}(j, k, s, \infty) - \mathfrak{E}(j, k, s, n^a)|, \\ |\sigma_0(j)| &< |\sigma(j)| + (\Gamma^s(1+a)/\Gamma(sa))\gamma_2(sa-1)j^{sa-1} \sum_{q>na} |A(q)|. \end{aligned}$$

Also, by Theorem 16,

$$\begin{aligned} \sum_{q>na} |A(q)| &< b_{10} \sum_{q>na} q^{1-sa} \leq b_{10} \left(n^{a(1-sa)} + \int_{na}^{\infty} q^{1-sa} dq \right) \\ &\leq b_{10}(sa/(sa-1))n^{a(2-sa)}. \end{aligned}$$

Hence

$$\begin{aligned} |\sigma_0(j)|^2 &< 2|\sigma(j)|^2 \\ &\quad + 2(\Gamma^{2s}(1+a)/\Gamma^2(sa))(sa/(sa-1))^2 b_{10}^2 \gamma_2^2 (sa-1) n^{2a(2-sa)} j^{2sa-2}, \\ \sum_{j=1}^n |\sigma_0(j)|^2 &< 2G_1 n^{2sa-1-\lambda} \\ &\quad + 2(\Gamma^{2s}(1+a)/\Gamma^2(sa))(sa/(sa-1))^2 b_{10}^2 \gamma_2^2 (sa-1) n^{2a(2-sa)} \\ &\quad \times \left(\int_1^n j^{2sa-2} dj + n^{2sa-2} \right) \\ &< 2G_1 n^{2sa-1-\lambda} + 2(\Gamma^{2s}(1+a)/\Gamma^2(sa)) \\ &\quad \times (sa/(sa-1))^2 b_{10}^2 \gamma_2^2 (sa-1) (2sa/(2sa-1)) n^{2sa-1-2a(sa-2)} \\ &\leq A_4 n^{2sa-1-\lambda}. \end{aligned}$$

6. The third and fourth Hardy-Littlewood theorems. In this section we again follow paper L in the proof of the third and fourth Hardy-Littlewood theorems which are here Theorems 43 and 45, respectively.

THEOREM 43. (L, Theorem 346.) *Let $H(\xi)$ denote the number of positive integers $j \leq \xi$ for which the equation*

$$(36) \quad j = \sum_{i=1}^s h_i^k, \quad h_i \geq 0,$$

is not solvable. Then

$$H(\xi) < C_{66}\xi^{1-\lambda},$$

where $C_{66} = 3A_4/c_{105}$, $c_{105} = (\Gamma^{2a}(1+a)/\Gamma^2(sa))2^{2-2sa}\gamma_3^2b_4$.

By the Corollary to Theorem 42 we have for $\xi \geq 2$

$$(37) \quad \sum_{\xi/2 < j \leq \xi} |\sigma_0(j)|^2 \leq \sum_{j=1}^{\xi} |\sigma_0(j)|^2 < A_4 \xi^{2sa-1-\lambda}.$$

In the summation $\sum_{\xi/2 < j \leq \xi} |\sigma_0(j)|^2$ there are $H(\xi) - H(\xi/2)$ terms in which $r_{k,s}(j) = 0$. For these terms

$$|\sigma_0(j)|^2 = |(\Gamma^a(1+a)/\Gamma(sa))(\Gamma(sa+j)/j!) \mathfrak{S}|^2 > (\Gamma^{2a}(1+a)/\Gamma^2(sa))\gamma_3^2 b_4^2 j^{2sa-2}$$

(by Theorem 25 and the Corollary to Theorem 29)

$$= c_{104} j^{2sa-2} > c_{104} (\xi/2)^{2sa-2} = c_{105} \xi^{2sa-2}.$$

In the remaining terms $|\sigma_0(j)|^2 \geq 0$. From (37) we get

$$(H(\xi) - H(\xi/2))c_{105}\xi^{2sa-2} < \sum_{\xi/2 < j \leq \xi} |\sigma_0(j)|^2 < A_4 \xi^{2sa-1-\lambda},$$

$$H(\xi) - H(\xi/2) < (A_4/c_{105})\xi^{1-\lambda} = C_{66}\xi^{1-\lambda}.$$

This holds also when $0 < \xi < 2$ since (36) is solvable for $j=1$ and then $H(\xi) - H(\xi/2) = 0$. Hence for $\xi > 0$ and every integer $v \geq 0$

$$\begin{aligned} H(\xi/2^v) - H(\xi/2^{v+1}) &< C_{68}(\xi/2^v)^{1-\lambda} = C_{68}\xi^{1-\lambda}(2^v)^{-1+\lambda} \\ &< C_{68}\xi^{1-\lambda}2^{-2v/3} \quad (-1+\lambda < -2/3), \end{aligned}$$

$$\begin{aligned} H(\xi) &= \sum_{v=0}^{\infty} (H(\xi/2^v) - H(\xi/2^{v+1})) < C_{68}\xi^{1-\lambda} \sum_{v=0}^{\infty} 2^{-2v/3} \\ &< 3C_{68}\xi^{1-\lambda} = C_{66}\xi^{1-\lambda}. \end{aligned}$$

THEOREM 44. (L, Theorem 348.) Let $L_{s_1}(n)$ denote the number of positive integers $j \leq n$ for which equation (36) with $s=s_2$ is solvable. Then

$$L_{s_2}(n) > B_{19}n^{1-(1-2a)(1-a)s_1-2-(1-a)s_1-2s_1},$$

where $B_{19} = 2^{2-s_1}c_{111}^{s_1-2}C_{71}$, $c_{111} = 2^{-1-a}(2^a-1)$, $C_{71} = 2^{-1-2a}/((k-1)A_1)$.

(i) Let $s_2 = 2$. The number of solutions of the inequalities

$$(38) \quad 1 \leq h_1^k + h_2^k \leq n, \quad h_1 \geq 0, h_2 \geq 0,$$

is at least equal to the number of solutions of

$$(39) \quad 0 \leq h_1 \leq (n/2)^a, \quad 0 \leq h_2 \leq (n/2)^a, \quad h_1 + h_2 > 0.$$

The number of solutions of (39) is

$$([(n/2)^a] + 1)^2 - 1 > (n/2)^{2a} - 1 > n^{2a} \cdot 2^{-1-2a} = c_{109} n^{2a},$$

when $n > 2^{(2a+1)/(2a)} = c_{108}$. For each positive integer $j \leq n$ the equation $j = h_1^k + h_2^k$ has at most $(k-1)A_1 j^{2-1/k} \leq (k-1)A_1 n^{2-1/k}$ solutions by the proof of Theorem 28. Therefore

$$\begin{aligned} L_2(n) &> c_{109} n^{2a} / ((k-1)A_1 n^{2-1/k}) = C_{71} n^{2a-1} \\ &= 2^{2-2a} c_{111}^{2-2a} C_{71} n^{1-(1-2a)(1-a)s_1-2-(1-a)s_1-2s_1}. \end{aligned}$$

(ii) Let $s_2 > 2$ and assume that the theorem is true for $s_2 - 1$, i.e.,

$$(40) \quad L_{s_2-1}(n) > B_{20} n^{1-(1-2a)(1-a)s_1-2-(1-a)s_1-2s_1},$$

where $B_{20} = 2^{3-s_2} c_{111}^{s_2-3} C_{71}$. Consider all integers

$$(41) \quad h^k + z$$

such that

h is an integer > 0 , z is an integer,

$$(42) \quad n/2 < h^k < (h+1)^k < n, \quad 0 < z \leq n^{1-a},$$

z is representable in the form $z = \sum_{i=1}^{s_2-1} h_i^k$, $h_i \geq 0$.

Since $(h+1)^k - h^k > kh^{k-1} > 2h^{k-1} > 2(n/2)^{a(k-1)} = 2^a n^{1-a} > z$, we have $h^k < h^k + z < (h+1)^k$. This shows that to distinct pairs of values h, z of (42) correspond distinct integers (41). For suppose $h_1^k + z_1 = h_2^k + z_2$. Then

$$h_1^k < h_1^k + z_1 = h_2^k + z_2 < (h_2 + 1)^k,$$

$$h_2^k < h_2^k + z_2 = h_1^k + z_1 < (h_1 + 1)^k$$

is impossible unless $h_1 = h_2$ and then $z_1 = z_2$. Moreover, each of the integers (41) is > 0 and $< (h+1)^k < n$. Therefore $L_{s_2}(n)$ is at least equal to the number of pairs of values of h, z of (42). Since $(n/2)^a < h < n^a - 1$ by (42), the number of values h takes is

$$\begin{aligned} [n^a] - 1 - [(n/2)^a] - 1 &> n^a - 2 - (n/2)^a - 1 = 2^{-a}(2^a - 1)n^a - 3 \\ &\geq 2^{-1-a}(2^a - 1)n^a = c_{111} n^a \end{aligned}$$

when $n > (3 \cdot 2^{1+a}/(2^a - 1))^k = c_{110}$. The number of values z takes is $L_{s_2-1}([n^{1-a}])$. Hence from (40)

$$\begin{aligned} L_{s_2}(n) &> c_{111} n^a \cdot L_{s_2-1}([n^{1-a}]) > c_{111} n^a B_{20} (n^{1-a}/2)^{1-(1-2a)(1-a)s_1-2-(1-a)s_1-2s_1} \\ &> 2^{-1} c_{111}^{2-2a} \cdot 2^{3-s_2} c_{111}^{s_2-3} C_{71} n^{1-(1-2a)(1-a)s_1-2-(1-a)s_1-2s_1} \\ &= B_{19} n^{1-(1-2a)(1-a)s_1-2-(1-a)s_1-2s_1}. \end{aligned}$$

THEOREM 45. (L, Theorem 350.) For s_2 as defined in §2 we have

$$(2 - 2a)A > (1 - 2a)(1 - a)^{s_2-2}.$$

THEOREM 46. (L, Theorem 349.) For s and s_2 as defined in §2 and every integer $n > C = \max(c_{108}, c_{110}, (C_{66}/B_{19})^{k2^{k-2}})$, the equation

$$(43) \quad \sum_{i=1}^{s+s_2} h_i^k = n, \quad h_i \geq 0,$$

has at least one solution. That is, every integer $> C$ is a sum of s_0 k th powers ≥ 0 when

$$s_0 \geq s + s_2 = g_1(k, \epsilon_1).$$

Let n be an integer for which (43) is not solvable and write $n = n_1 + n_2$. Then, since there are $L_{s_1}(n)$ integers $n_2 \leq n$ for which $\sum_{i=s+1}^{s+s_2} h_i^k = n_2$, $h_i \geq 0$, is solvable, there must be $L_{s_2}(n)$ integers $n_1 = n - n_2 \leq n$ for which

$$(44) \quad \sum_{i=1}^s h_i^k = n_1, \quad h_i \geq 0,$$

is not solvable. For if (44) were solvable for one of the $L_{s_1}(n)$ integers n_1 , then (43) would be solvable for n contrary to our assumption. By Theorem 43 the number of positive integers $\leq n$ for which (44) is not solvable is $< C_{66}n^{1-\lambda}$. Hence

$$L_{s_2}(n) < C_{66}n^{1-\lambda}.$$

By Theorem 44, when $n > \max(c_{108}, c_{110})$

$$L_{s_1}(n) > B_{19}n^{1-(1-2a)(1-a)^{s_1-2}-(1-a)^{s_1-2}\epsilon_1}.$$

Therefore

$$B_{19}n^{1-(1-2a)(1-a)^{s_1-2}-(1-a)^{s_1-2}\epsilon_1} < C_{66}n^{1-\lambda},$$

$$n^{\lambda-(1-2a)(1-a)^{s_1-2}-(1-a)^{s_1-2}\epsilon_1} < C_{66}/B_{19},$$

$$n^{\lambda-(2-2a)A-(1-a)^{s_1-2}\epsilon_1} < C_{66}/B_{19} \quad (\text{Theorem 45}),$$

$$n^{2aA} < C_{66}/B_{19} \quad (\lambda = 2A + (1-a)^{s_1-2}\epsilon_1).$$

It follows that (43) is always solvable when

$$n > \max(c_{108}, c_{110}, (C_{66}/B_{19})^{k2^{k-2}}).$$

7. The solution of (43) for integers $< C$. The following theorem is well known:

THEOREM 47.* If every integer n for which $f < n \leq h$ is a sum of $s-1$ k th powers ≥ 0 and if m is the greatest integer such that

$$(45) \quad (m+1)^k - m^k < h - f,$$

then every integer n for which $f < n \leq h + (m+1)^k$ is a sum of s k th powers ≥ 0 .

THEOREM 48. For $L = (k+1)^k - k^k > k^k$ we have

$$s_3 < 2^k + \left(\frac{3}{2}\right)^k + 2\left(\frac{4}{3}\right)^k + 2\left(\frac{2}{3}\right)^k + 2\left(\frac{1}{2}\right)^k + \frac{k(2k+7)}{9} - 9.$$

Consider any integer n such that $0 < n \leq 2^{k+1} - 2$. If $n \leq 2^k - 1$ it is obviously the sum of $2^k - 1$ k th powers, 0 or 1. If $2^k \leq n \leq 2^{k+1} - 2$ we write $n = 2^k + x$, $0 \leq x \leq 2^k - 2$, and again n is a sum of $2^k - 1$ k th powers since x is a sum of $2^k - 2$ k th powers, 0 or 1. Hence every integer in the interval

$$0 < n \leq 2^{k+1} - 2 = h_1$$

is a sum of $2^k - 1 = m_1$ k th powers ≥ 0 . Since $2^k - 1^k < h_1 < 3^k - 2^k$, it follows from Theorem 47 with $m = 1$ that every integer in the interval $0 < n \leq h_1 + 2^k$ is a sum of $m_1 + 1$ k th powers ≥ 0 . We repeat this step m_2 times so that every integer in the interval $0 < n \leq h_1 + m_2 2^k$ is a sum of $m_1 + m_2$ k th powers ≥ 0 , where

$$(46) \quad h_1 + (m_2 - 1)2^k \leq 3^k - 2^k < h_1 + m_2 2^k.$$

We now apply Theorem 47 m_3 times with $m = 2$ and conclude that every integer in the interval $0 < n \leq h_1 + m_2 2^k + m_3 3^k$ is a sum of $m_1 + m_2 + m_3$ k th powers ≥ 0 , where

$$(47) \quad h_1 + m_2 2^k + (m_3 - 1)3^k \leq 4^k - 3^k < h_1 + m_2 2^k + m_3 3^k.$$

In general every integer n such that

$$0 < n \leq h_1 + \sum_{j=2}^t m_j j^k$$

is a sum of $\sum_{j=1}^t m_j$ k th powers ≥ 0 , where

$$(48) \quad h_1 + \sum_{j=2}^{t-1} m_j j^k + (m_t - 1)t^k \leq (t+1)^k - t^k < h_1 + \sum_{j=2}^t m_j j^k.$$

From (47) and (46) we get

$$m_3 3^k \leq 4^k - m_2 2^k - h_1 < 4^k - 3^k + 2^k,$$

and in general from (48) when $t \geq 3$,

* L. E. Dickson, loc. cit.

$$m_i t^k \leq (t+1)^k - \sum_{j=2}^{t-1} m_{ij}^k - h_1 < (t+1)^k - t^k + (t-1)^k,$$

$$(49) \quad m_t < (1+t^{-1})^k - 1 + (1-t^{-1})^k.$$

Hence

$$\begin{aligned} \sum_{j=1}^t m_j &< m_1 + m_2 + \sum_{j=3}^t ((1+j^{-1})^k - 1 + (1-j^{-1})^k) \\ &= m_1 + m_2 - (t-2) + 2 \sum_{j=3}^t \left(1 + \binom{k}{2} \frac{1}{j^2} + \binom{k}{4} \frac{1}{j^4} + \cdots \right) \\ &\leq m_1 + m_2 - (t-2) + \left(\frac{4}{3} \right)^k + \left(\frac{2}{3} \right)^k \\ &\quad + 2 \int_3^t \left(1 + \binom{k}{2} \frac{1}{j^2} + \cdots \right) dj \\ &< m_1 + m_2 - (t-2) + \left(\frac{4}{3} \right)^k + \left(\frac{2}{3} \right)^k \\ &\quad + 2(t-3) + 2 \left(\binom{k}{2} \frac{1}{3} + \binom{k}{4} \frac{1}{3 \cdot 3^3} + \cdots \right) \\ &< m_1 + m_2 + t - 4 + \left(\frac{4}{3} \right)^k + \left(\frac{2}{3} \right)^k \\ &\quad + 2 \left(\binom{k}{2} \frac{2}{3^2} - 1 + 1 + \binom{k}{2} \frac{1}{3^2} + \binom{k}{4} \frac{1}{3^4} + \cdots \right) \\ &= m_1 + m_2 + t - 4 + \left(\frac{4}{3} \right)^k + \left(\frac{2}{3} \right)^k \\ &\quad + \frac{2k(k-1)}{9} - 2 + \left(\frac{4}{3} \right)^k + \left(\frac{2}{3} \right)^k. \end{aligned}$$

From (46), $m_2 2^k \leq 3^k - h_1 = 3^k - 2^{k+1} + 2$, and hence when $t=k$ we get $L = (k+1)^k - k^k$ and

$$\begin{aligned} s_3 &= \sum_{j=1}^k m_j < 2^k - 1 + \left(\frac{3}{2} \right)^k - 2 + 2 \left(\frac{1}{2} \right)^k \\ &\quad + k - 4 + 2 \left(\frac{4}{3} \right)^k + 2 \left(\frac{2}{3} \right)^k + \frac{2k(k-1)}{9} - 2 \\ &= 2^k + \left(\frac{3}{2} \right)^k + 2 \left(\frac{4}{3} \right)^k + 2 \left(\frac{2}{3} \right)^k + 2 \left(\frac{1}{2} \right)^k + \frac{k(2k+7)}{9} - 9. \end{aligned}$$

THEOREM 49. *If every positive integer $\leq L$ is a sum of $s-1$ k th powers ≥ 0 , then every positive integer $\leq (L/k)^{k/(k-1)}$ is a sum of s k th powers ≥ 0 .*

Since $(L/k)^{k/(k-1)} - ((L/k)^{1/(k-1)} - 1)^k \leq k(L/k) = L$, we may apply Theorem 47 with $m+1 = [(L/k)^{1/(k-1)}]$. Thus every positive integer $\leq L + [(L/k)^{1/(k-1)}]^k$ is a sum of s k th powers ≥ 0 , and $L + [(L/k)^{1/(k-1)}]^k \geq L + ((L/k)^{1/(k-1)} - 1)^k \geq (L/k)^{k/(k-1)}$.

THEOREM 50. *If every positive integer $\leq L$, where $L > k^k$, is a sum of s_3 k th powers ≥ 0 , then every positive integer $\leq C$ is a sum of $s_3 + s_4$ k th powers ≥ 0 , where*

$$s_4 = \left\lceil \frac{\log \log C - \log (\log L - k \log k)}{\log k - \log (k-1)} \right\rceil + 1.$$

That is, every integer $\leq C$ is a sum of s_0 k th powers ≥ 0 when

$$s_0 \geq s_3 + s_4 = g_2(k, \epsilon_1).$$

By Theorem 49 every positive integer $\leq (L/k)^{k/(k-1)}$ is a sum of $s_3 + 1$ k th powers ≥ 0 . Write $L_1 = (L/k)^{k/(k-1)}$ and apply Theorem 49 again. Thus every positive integer $\leq (L_1/k)^{k/(k-1)} = L_2$ is a sum of $s_3 + 2$ k th powers ≥ 0 . Also,

$$L_2 = \left(\frac{L_1}{k} \right)^{k/(k-1)} = \left(\frac{L^{k/(k-1)}}{k^{k/(k-1)+1}} \right)^{k/(k-1)},$$

$$\log L_2 = (k/(k-1))^2 \log L - (k/(k-1) + (k/(k-1))^2) \log k.$$

In general, every positive integer $\leq L_{s_4}$ is a sum of $s_3 + s_4$ k th powers ≥ 0 , where

$$\begin{aligned} \log L_{s_4} &= (k/(k-1))^{s_4} \log L - (k/(k-1) + \dots + (k/(k-1))^{s_4}) \log k \\ &= (k/(k-1))^{s_4} \log L - k((k/(k-1))^{s_4} - 1) \log k \\ &> (k/(k-1))^{s_4} (\log L - k \log k). \end{aligned}$$

This expression is $\geq \log C$ when

$$s_4 \geq \frac{\log \log C - \log (\log L - k \log k)}{\log k - \log (k-1)}.$$

8. Evaluation of the constants. We first prove three lemmas.

LEMMA 1. *For $w \geq 5$ we have*

$$\sum_{j=1}^n (1+j^{-1})^w < 2^{w+1} + n - 1 + (w+1) \log n.$$

Let $t = [w] + 1$. Then

$$\begin{aligned} \sum_{j=1}^n (1+j^{-1})^w &\leq 2^w + \int_1^n (1+j^{-1})^w dj < 2^w + \int_1^n (1+j^{-1})^t dj \\ &= 2^w + \int_1^n \left(1 + \binom{t}{1} \frac{1}{j} + \binom{t}{2} \frac{1}{j^2} + \cdots + \frac{1}{j^t}\right) dj \\ &= 2^w + \left(j + t \log j - \binom{t}{2} \frac{1}{j} - \binom{t}{3} \frac{1}{2j^2} - \cdots - \frac{1}{(t-1)j^{t-1}} \right) \Big|_1^n \\ &< 2^w + n - 1 + t \log n + \binom{t}{2} + \binom{t}{3} \frac{1}{2} + \binom{t}{4} \frac{1}{3} + \cdots + \frac{1}{t-1} \\ &= 2^w + n - 1 + t \log n + \frac{1}{2} + \binom{t}{1} \frac{1}{2} + \binom{t}{2} \frac{1}{2} \\ &\quad + \binom{t}{3} \frac{1}{2} + \binom{t}{4} \frac{1}{2} + \cdots + \frac{1}{2} \\ &\quad - \frac{1}{2} - \binom{t}{1} \frac{1}{2} + \binom{t}{2} \frac{1}{2} - \binom{t}{4} \frac{1}{6} \\ &\quad - \binom{t}{5} \frac{1}{4} - \cdots - \left(\frac{1}{2} - \frac{1}{t-1} \right). \end{aligned}$$

Since

$$\frac{1}{2} + \binom{t}{1} \frac{1}{2} - \binom{t}{2} \frac{1}{2} + \binom{t}{4} \frac{1}{6} + \binom{t}{5} \frac{1}{4} > 0$$

when $t = [w] + 1 \geq 6$, we have

$$\begin{aligned} \sum_{j=1}^n (1+j^{-1})^w &< 2^w + n - 1 + t \log n + 2^{t-1} \\ &= 2^w + n - 1 + ([w] + 1) \log n + 2^{[w]} \\ &\leq 2^{w+1} + n - 1 + (w + 1) \log n. \end{aligned}$$

LEMMA 2. For $x \geq 0$ we have

$$(2^x - 1)^{-1} \leq (x \log 2)^{-1}.$$

Consider the function

$$y = x(2^x - 1)^{-1}, \quad y' = (2^x - 1 - x2^x \log 2)(2^x - 1)^{-2}.$$

We have $y' \leq 0$ when $x \geq 0$. Hence y attains its maximum value when $x = 0$. That is, $\max y = 1/\log 2$ and the desired result follows.

LEMMA 3. Let t be an integer ≥ 0 . Then

$$\log(t!) \leq (t+1) \log t - t + 1.$$

We have

$$\log(t!) = \sum_{n=1}^t \log n \leq \int_1^t \log x \, dx + \log t = (t+1) \log t - t + 1.$$

The constants are now evaluated as follows.

$$(\alpha_2) \quad \alpha_2 \leq 7 \log 2 \quad (\text{Theorem 2}).$$

$$(\gamma(\beta)) \quad \gamma(\beta) = 4\beta(2^\beta + 1)e^\beta \Gamma(\beta + 1) \quad (\text{Theorem 29}).$$

$$(\gamma_2(\beta)) \quad \gamma_2(\beta) = e^\beta \Gamma(\beta + 1) \quad (\text{Theorem 29}).$$

$$(\gamma_3) \quad \gamma_3 = (1 + 12e)^{-1} \quad (\text{Theorem 29}).$$

$$(c_{15}) \quad \log c_{15} = (k-1) \log k + \alpha_2(k-2)k^{2k/(k-2)}/(2k) \quad (\text{Theorem 15}).$$

$$(b_{18}) \quad b_{18} = (1 + k^*)^{2/(s-5)} \quad (\text{Theorem 24}).$$

$$(b(p)) \quad b(p) > \begin{cases} 2^{-k(4k-1)}, & p = 2, \\ p^{-k(2k-1)}, & p > 2 \end{cases} \quad (\text{Theorem 10}).$$

Proof: By the proof of Theorem 10, $r \leq 4k-1$ when $p=2$ and $r \leq 2k-1$ when $p > 2$. Also, $\gamma \leq k$, since for $p > 2$

$$\gamma = \Theta + 1 \leq 2^\Theta \leq p^\Theta \leq k;$$

for $p=2$, $\Theta > 1$,

$$\gamma = \Theta + 2 \leq 2^\Theta \leq k;$$

and for $p=2$, $\Theta \leq 1$,

$$\gamma = \Theta + 2 \leq 3 < k.$$

Hence

$$b(p) = P^{-r} = p^{-\gamma r} > \begin{cases} 2^{-k(4k-1)}, & p = 2, \\ p^{-k(2k-1)}, & p > 2. \end{cases}$$

$$(b_4) \quad \log(1/b_4) < 2k^2 \quad (\text{Theorem 25}).$$

Proof: From Theorem 25

$$b_4 = \prod_{p \leq b_{18}} b(p) \prod_{p > b_{18}} (1 - p^{-2/3}).$$

Let

$$\Pi_1 = 1/\left(\prod_{p \leq b_{18}} b(p)\right), \quad \Pi_2 = 1/\left(\prod_{p > b_{18}} (1 - p^{-2/3})\right).$$

Then

$$\begin{aligned}
 \prod_1 &\leq 2^{k(4k-1)} \prod_{3 \leq p \leq b_{18}} p^{k(2k-1)}, \\
 \log \prod_1 &\leq k(4k-1) \log 2 + k(2k-1) \vartheta(b_{18}) - k(2k-1) \log 2 \\
 &< 2k^3 \log 2 + k(2k-1) \left(\frac{6}{5} c b_{18} + 3 \log^2 b_{18} + 8 \log b_{18} + 5 \right) \\
 &\quad \text{(Theorem 3)} \\
 &= 2k^3 \log 2 + k(2k-1) \left(\frac{6}{5} c(1+k^s)^{2/(s-5)} \right. \\
 &\quad \left. + 3(2/(s-5))^2 \log^2(1+k^s) + 8(2/(s-5)) \log(1+k^s) + 5 \right) \\
 &\leq 2k^3 \log 2 + k(2k-1) \left(\frac{6}{5} c 2^{3/65} k^{28/13} + 3(2/65)^2 \log^2(2k^{70}) \right. \\
 &\quad \left. + 8(2/65) \log(2k^{70}) + 5 \right) \quad \text{(since } s \geq 70 \text{ by (2))} \\
 &< 2k^5 - \log 3 \quad (k \geq 6).
 \end{aligned}$$

Also,

$$\prod_2 \leq 1 / \left(\prod_p (1 - p^{-3/2}) \right) = \sum_{n=1}^{\infty} n^{-3/2} \leq 1 + \int_1^{\infty} n^{-3/2} dn = 3.$$

Therefore

$$\log(1/b_4) = \log \prod_1 + \log \prod_2 < 2k^5 - \log 3 + \log 3 = 2k^5.$$

$$(A_1) \quad \log A_1 < 9\epsilon_1 2^{n_1} \quad \text{(Theorem 1).}$$

Proof: We have

$$\begin{aligned}
 \log A_1 &= \pi(2^{n_1}) \log 2 + \pi((3/2)^{n_1}) \log(3/2) \\
 &\quad + \dots - \epsilon_1(\vartheta(2^{n_1}) + \vartheta((3/2)^{n_1}) + \dots).
 \end{aligned}$$

Since $\pi((1+j^{-1})^{n_1}) = 0$ and $\vartheta((1+j^{-1})^{n_1}) = 0$ when $(1+j^{-1})^{n_1} < 2$, that is, when $j > 1/(2^{n_1}-1)$, we may write

$$\log A_1 = \sum_{j=1}^n \pi((1+j^{-1})^{n_1}) \log(1+j^{-1}) - \epsilon_1 \sum_{j=1}^n \vartheta((1+j^{-1})^{n_1}),$$

where $n = [1/(2^{n_1}-1)] + 1$. By Theorems 2 and 3

$$\begin{aligned}
\log A_1 &< \alpha_2 \sum_{j=1}^n \frac{(1+j^{-1})^{\eta_1}}{\eta_1 \log(1+j^{-1})} \log(1+j^{-1}) - \epsilon_1 \left(c \sum_{j=1}^n (1+j^{-1})^{\eta_1} \right. \\
&\quad \left. - \frac{12}{5} c \sum_{j=1}^n (1+j^{-1})^{\eta_1/2} - \frac{3}{2} \sum_{j=1}^n \log^2(1+j^{-1}) \right. \\
&\quad \left. - 13 \sum_{j=1}^n \log(1+j^{-1}) - 15(n-1) \right) \\
&< (\alpha_2 - c) \epsilon_1 (2^{\eta_1+1} + n - 1 + (\eta_1 + 1) \log n) \\
&\quad + \frac{12}{5} c \epsilon_1 (2^{(\eta_1+2)/2} + n - 1 + (\eta_1/2 + 1) \log n) \\
&\quad + \frac{3}{2} \epsilon_1 (n - 1) \log^2 2 + 13 \epsilon_1 \log(n+1) + 15 \epsilon_1 (n - 1) \quad (\text{Lemma 1}) \\
&< (\alpha_2 - c) (\epsilon_1 2^{\eta_1+1} + \epsilon_1 (2^{\eta_1} - 1)^{-1} + (1 + \epsilon_1) \log((2^{\eta_1} - 1)^{-1} + 1)) \\
&\quad + \frac{12}{5} (\epsilon_1 2^{(\eta_1+2)/2} + \epsilon_1 (2^{\eta_1} - 1)^{-1} + (1 + \epsilon_1) \log((2^{\eta_1} - 1)^{-1} + 1)) \\
&\quad + \frac{3}{2} \epsilon_1 (2^{\eta_1} - 1)^{-1} + 13 \epsilon_1 \log((2^{\eta_1} - 1)^{-1} + 2) + 15 \epsilon_1 (2^{\eta_1} - 1)^{-1} \\
&\leq (\alpha_2 - c) (\epsilon_1 2^{\eta_1+1} + 1/\log 2 + (1 + \epsilon_1) \log((\epsilon_1 \log 2)^{-1} + 1)) \\
&\quad + \frac{12}{5} (\epsilon_1 2^{(\eta_1+2)/2} + 1/\log 2 + (1 + \epsilon_1) \log((\epsilon_1 \log 2)^{-1} + 1)) \\
&\quad + 17 \epsilon_1 (\epsilon_1 \log 2)^{-1} + 13 \epsilon_1 \log((\epsilon_1 \log 2)^{-1} + 1) \quad (\text{Lemma 2}) \\
&< (\alpha_2 - c) (\epsilon_1 2^{\eta_1+1} + 1/\log 2 + (1 + \epsilon_1) \log 2 \eta_1) \\
&\quad + \frac{12}{5} (\epsilon_1 2^{(\eta_1+2)/2} + 1/\log 2 + (1 + \epsilon_1) \log 2 \eta_1) \\
&\quad + 17/\log 2 + 13 \epsilon_1 \log 3 \eta_1.
\end{aligned}$$

Since $\alpha_2 - c < 4$ this expression is $< 9 \epsilon_1 2^{\eta_1}$ when $\eta_1 \geq 12$. By (12)

$$\eta_1 \geq 17 > 12 \text{ when } k = 6,$$

$$\eta_1 \geq D + 2^{k-3} > 20 > 12 \text{ when } k > 6,$$

and hence $\log A_1 < 9 \epsilon_1 2^{\eta_1}$.

$$(A_2) \quad \log A_2 = (k - 2) \log A_1 \quad (\text{Theorem 5}).$$

$$(A_3) \quad \log A_3 < \log \eta_1 + \log 10k \quad (\text{Theorem 7}).$$

Proof:

$$\log A_3 = \log (1/(\epsilon_3 e^{1-a})) < \log (1/\epsilon_3) = \log (10k\eta_1)$$

since $10k\epsilon_3 = \epsilon_1$.

$$(C_{15}) \quad \log C_{15} = (k-2) \log A_1 + K \log 4 \quad (\text{Theorem 6}).$$

$$(G_3) \quad \log G_3 = \log A_1 + \log (4k-4) \quad (\text{Theorem 28}).$$

$$(G_4) \quad \begin{aligned} \log G_4 &< (k-2) \log A_1 + \log \eta_1 + K \log 4 \\ &\quad + (k+2) \log k - k + 1 + \log 80 \end{aligned} \quad (\text{Theorem 30}).$$

Proof:

$$\begin{aligned} \log G_4 &= \log (8k!C_{15}A_3) = \log 8 + \log (k!) + \log C_{15} + \log A_3 \\ &< \log 8 + (k+1) \log k - k + 1 + (k-2) \log A_1 + K \log 4 \\ &\quad + \log \eta_1 + \log 10k \end{aligned} \quad (\text{Lemma 3}).$$

$$(G_5) \quad G_5 < 3c_{15} \quad (\text{Theorem 31}).$$

Proof:

$$G_5 = c_{15}(\Gamma(1+a) + 2a\gamma(a)(1-a^2)^{-1} + (2a+1)(a+1)^{-1}) < 3c_{15}.$$

$$(G_7) \quad G_7 < c_{15} \quad (\text{Theorem 32}).$$

$$(G_8) \quad G_8 < c_{15} \quad (\text{Theorem 32}).$$

$$(G_9) \quad G_9 = \max (G_5, G_7 + G_8) < 3c_{15} \quad (\text{Theorem 33}).$$

$$(G_{10}) \quad G_{10} < 3 \cdot 2^{2sa+k-1} G_9^{2s} \quad (\text{Theorem 34}).$$

Proof: We have

$$\begin{aligned} \frac{2sa-1-(2sa-1)\theta}{2sa-2-(2sa-1)\theta} &\leq \frac{2sa-1+2K^2}{2sa-1} \quad (\text{by (19)}) \\ &\leq \frac{(k-2)K+k+2kK^2}{(k-2)K+k} < 3K \quad (\text{by (2)}). \end{aligned}$$

Hence

$$\begin{aligned} G_{10} &= 2^{2sa}(2sa-1)^{-1}(2sa-1-(2sa-1)\theta)(2sa-2-(2sa-1)\theta)^{-1} G_9^{2s} \\ &< 3 \cdot 2^{2sa+k-1} G_9^{2s}. \end{aligned}$$

$$(G_{13}) \quad G_{13} < 15G_4^A \quad (\text{Theorem 35}).$$

Proof:

$$G_{13} = (2\pi+1)2^{1+A}G_4^A \leq (2\pi+1)2^{33/32}G_4^A < 15G_4^A.$$

$$(G_{15}) \quad \log G_{15} < ((2s-4)(k-3/2)A+1) \log A_1 \quad (\text{Theorem 36}).$$

Proof:

$$\begin{aligned}
 \log G_{18} &= (2s-4) \log G_{13} + \log G_3 + (2a + \epsilon_1) \log 2 \\
 &= (2s-4)A \log G_4 + (2s-4) \log 15 + \log G_3 + (2a + \epsilon_1) \log 2 \\
 &< ((2s-4)(k-2)A + 1) \log A_1 + (2s-4)A(\log \eta_1 + K \log 4) \\
 &\quad + (k+2) \log 2 - k + 1 + \log 80 \\
 &\quad + \log (4k-4) + (2s-4) \log 15 + (2a + \epsilon_1) \log 2 \\
 &= ((2s-4)(k-2)A + 1) \log A_1 + (s-2)A \log A_1 \left(\frac{2 \log \eta_1}{\log A_1} + \frac{2K \log 4}{\log A_1} \right. \\
 &\quad \left. + \frac{2(k+2) \log k}{\log A_1} - \frac{2k-2}{\log A_1} + \frac{2 \log 80}{\log A_1} + \frac{\log (4k-4)}{(s-2)A \log A_1} \right. \\
 &\quad \left. + \frac{2 \log 15}{A \log A_1} + \frac{(2a + \epsilon_1) \log 2}{(s-2)A \log A_1} \right).
 \end{aligned}$$

Each of the positive terms in the coefficient of $(s-2)A \log A_1$ is $< 1/7$ when $\eta_1 \geq 2k$ and thus

$$\log G_{18} < ((2s-4)(k-2)A + 1) \log A_1 + (s-2)A \log A_1$$

since $\eta_1 > 17 > 12 = 2k$ when $k=6$ and $\eta_1 > 2^{k-3} > 2k$ when $k \geq 7$.

$$(G_{18}) \quad G_{18} < 25G_4^A \quad (\text{Theorem 37}).$$

Proof:

$$\begin{aligned}
 G_{18} &= (2\pi + 1)(2(3G_4)^A + \gamma(a)) \leq (2\pi + 1)(2 \cdot 3^{1/32} G_4^A + \gamma(1/6)) \\
 &\leq (2\pi + 1)(2 \cdot 3^{1/32} + \gamma(1/6)) G_4^A < 25G_4^A.
 \end{aligned}$$

$$(G_{24}) \quad \log G_{24} < 2s(k-3/2)A \log A_1 \quad (\text{Theorem 38}).$$

Proof: We have

$$\begin{aligned}
 \frac{2s+2}{2s+1} \frac{2s+3-(2s+1)\theta-2s(1-D\epsilon_1)A}{2s+2-(2s+1)\theta-2s(1-D\epsilon_1)A} &< \frac{2s+2}{2s+1} \frac{2s(1-A-\theta)+3-\theta}{2s(1-A-\theta)+2-\theta} \\
 &< \frac{2s+2}{2s+1} \frac{3-\theta}{2-\theta} < 2 \quad (\text{by (20) and (2)}).
 \end{aligned}$$

Hence

$$G_{24} = 2^{2s+1} \frac{(2s+2)(2s+3-(2s+1)\theta-2s(1-D\epsilon_1)A)}{(2s+1)(2s+2-(2s+1)\theta-2s(1-D\epsilon_1)A)} G_{18}^{2s} < 2^{2s+2} G_{18}^{2s},$$

$$\begin{aligned}
 \log G_{24} &< 2s \log G_{18} + (2s+2) \log 2 < 2sA \log G_4 + 2s \log 25 + (2s+2) \log 2 \\
 &= 2s(k-2)A \log A_1 + 2sA(\log \eta_1 + K \log 4 + (k+2) \log k \\
 &\quad - k + 1 + \log 80) + (2s+2) \log 2 + 2s \log 25 \\
 &= 2s(k-2)A \log A_1 + sA \log A_1 \left(\frac{2 \log \eta_1}{\log A_1} + \frac{2K \log 4}{\log A_1} + \frac{2(k+2) \log k}{\log A_1} \right. \\
 &\quad \left. - \frac{2k-2}{\log A_1} + \frac{2 \log 80}{\log A_1} + \frac{(2s+2) \log 2}{sA \log A_1} + \frac{2 \log 25}{A \log A_1} \right).
 \end{aligned}$$

As before the coefficient of $sA \log A_1$ is < 1 and thus

$$\log G_{24} < 2s(k-2)A \log A_1 + sA \log A_1.$$

$$(G_{24}) \quad \log G_{24} < 2s(k-3/2)A \log A_1 \quad (\text{Theorem 38}).$$

Proof:

$$\frac{(2sa-2-2a)(2sa-3)}{(2sa-3-2a)(2sa-4)} < \frac{((k-2)K-2)((k-2)K-k)}{((k-2)K-k-2)((k-2)K-2k)} < 2 \text{ (by (2))}.$$

Also

$$\begin{aligned}
 \log G_9 &< \log c_{15} + \log 3 = (k-1) \log k + \alpha_2(k-2)k^{2k/(k-2)}/(2k) + \log 3 \\
 &< (k-1) \log k + 7(\log 2)k^3/3 + \log 3 \\
 &< (k-2)A \log A_1 \quad (\eta_1 \geq 2k) \\
 &< A \log G_4 < \log G_{18}.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \log G_{24} &< (2s+2) \log 2 + (2s-2) \log G_9 + 2 \log G_{18} \\
 &< (2s+2) \log 2 + 2s \log G_{18} \\
 &< 2s(k-3/2)A \log A_1
 \end{aligned}$$

as in (G_{24}) .

$$(G_{31}) \quad G_{31} < 25G_4^A \quad (\text{Theorem 39}).$$

Proof:

$$G_{31} = (2\pi+1)^{2^{1+2A+(s+2s)A}} G_4^A \leq (2\pi+1)^{2^{25/32}} G_4^A < 25G_4^A.$$

$$(G_{36}) \quad \log G_{36} < 2s(k-3/2)A \log A_1 \quad (\text{Theorem 40}).$$

Proof:

$$\begin{aligned} & \frac{(2s-K)\theta - (2s-3K)}{(2s-K)\theta - (2s-2K)} \leq \frac{2s-K+2K^3}{2s-K} \quad (\text{by (21)}) \\ & \leq \frac{(k-2)K+2k-K+2K^3}{(k-2)K+2k-K} < \frac{(k-3)K+2K^3}{(k-3)K} < \frac{3+2K^2}{3} < 2K^2 = 2^{2k-1}. \end{aligned}$$

Hence

$$\begin{aligned} G_{35} &< 2^{2sA+1+2k-1} G_{31}^{2s}, \\ \log G_{35} &< 2s \log G_{31} + (2sA+2k) \log 2 \\ &< 2sA \log G_4 + 2s \log G_{25} + (2sA+2k) \log 2 \\ &< 2s(k-3/2)A \log A_1, \end{aligned}$$

as in (G_{34}) .

$$(G_1) \quad \log G_1 < ((2s-4)(k-3/2)A+1) \log A_1 + \log 10 \quad (\text{Theorem 41}).$$

Proof: Consider first G_{10} . We have

$$\begin{aligned} \log G_{10} &< 2s \log G_9 + (2sa+k-1) \log 2 + \log 3 \\ &< 2sA \log G_4 + (2sa+k-1) \log 2 + \log 3 \end{aligned}$$

by the proof for (G_{25}) , and then as before

$$\log G_{10} < 2s(k-3/2)A \log A_1.$$

Also,

$$(2s-4)(k-3/2)A+1 = 2s(k-3/2)A+1-4A(k-3/2) > 2s(k-3/2)A.$$

Hence

$$\begin{aligned} G_1 &= 2(G_{10} + G_{15} + G_{24} + G_{25} + G_{35}) \\ &\leq 10 \max(G_{10}, G_{15}, G_{24}, G_{25}, G_{35}), \\ \log G_1 &< ((2s-4)(k-3/2)A+1) \log A_1 + \log 10. \end{aligned}$$

$$(A_4) \quad \log A_4 < ((2s-4)(k-3/2)A+1) \log A_1 + \log 22 \quad (\text{Theorem 41}).$$

Proof:

$$\begin{aligned} A_4 &= 2G_1 + 2(\Gamma^{2s}(1+a)/\Gamma^2(sa))(sa/(sa-1))^2(2sa/(2sa-1))^{2s} e^{2sa-2} \Gamma^2(sa) \\ &< 2G_1 + 8e^{2sa-2} c_{15}^{2s} < 2G_1 + 8e^{2sa-2} (G_{15}/3)^{2s} \quad (\text{by } (G_{25})) \\ &< 2(G_1 + G_{15}^2) < 2(G_1 + G_{24}) \leq 2(10 \max(G_{10}, \dots, G_{35}) + G_{24}) \\ &< 22 \max(G_{10}, \dots, G_{35}), \end{aligned}$$

$$\log A_4 < ((2s-4)(k-3/2)A+1) \log A_1 + \log 22.$$

$$(c_{104}) \quad c_{104} = (\Gamma^{2s}(1+a)/\Gamma^2(sa))b_4^2\gamma_3^2 \quad (\text{Theorem 43}).$$

$$(c_{105}) \quad c_{105} = c_{104}2^{2-2sa} \quad (\text{Theorem 43}).$$

$$(C_{68}) \quad \log C_{68} = \log A_4 + \log (1/c_{105}) \quad (\text{Theorem 43}).$$

$$(C_{66}) \quad \log C_{66} < ((2s-4)(k-1)A+1) \log A_1 \quad (\text{Theorem 43}).$$

Proof:

$$\begin{aligned} \log C_{66} &= \log C_{68} + \log 3 = \log A_4 + \log (1/c_{105}) + \log 3 \\ &< ((2s-4)(k-3/2)A+1) \log A_1 + (2sa-2) \log 2 + 2 \log \Gamma(sa) \\ &\quad - 2s \log \Gamma(1+a) + 2 \log (1/b_4) + 2 \log (1/\gamma_3) + \log 66 \\ &< ((2s-4)(k-3/2)A+1) \log A_1 + (s-2)A \log A_1 \left(\frac{(2sa-2) \log 2}{(s-2)A \log A_1} \right. \\ &\quad \left. + \frac{2 \log \Gamma(sa)}{(s-2)A \log A_1} + \frac{2s \log 2}{(s-2)A \log A_1} \right. \\ &\quad \left. + \frac{2 \log (1+12e)}{(s-2)A \log A_1} + \frac{\log 66}{(s-2)A \log A_1} \right). \end{aligned}$$

As before each of the terms of the coefficient of $(s-2)A \log A_1$ is $< 1/6$ when $\eta_1 > 2k+5$. Since

$$\eta_1 > 17 = 2k+5 \quad \text{when } k=6, \quad \eta_1 > 33 > 19 = 2k+5 \quad \text{when } k=7,$$

$$\eta_1 > 2^{k-3} > 2k+5 \quad \text{when } k \geq 8,$$

we have

$$\begin{aligned} \log C_{66} &< ((2s-4)(k-3/2)A+1) \log A_1 + (s-2)A \log A_1. \\ (c_{109}) \quad c_{109} &= 2^{-2a-1} \quad (\text{Theorem 44}). \\ (c_{110}) \quad c_{110} &= 3^k \cdot 2^{k+1}(2^a-1)^{-k} \quad (\text{Theorem 44}). \\ (c_{111}) \quad c_{111} &= 2^{-1-a}(2^a-1) \quad (\text{Theorem 44}). \\ (C_{71}) \quad \log (1/C_{71}) &= (k-1) \log A_1 + \log (1/c_{109}) \quad (\text{Theorem 44}). \\ (B_{19}) \quad \log (1/B_{19}) &< (2s-4)A \log A_1 + (k-1) \log A_1 \quad (\text{Theorem 44}). \end{aligned}$$

Proof:

$$\begin{aligned} \log (1/B_{19}) &= (s-2) \log (2/c_{111}) + \log (1/C_{71}) \\ &= (k-1) \log A_1 + (s-2) \log (2^{2+a}/(2^a-1)) + (2a+1) \log 2 \\ &= (k-1) \log A_1 + (2s-4)A \log A_1 \left(\frac{(s-2) \log (2^{2+a}/(2^a-1))}{(2s-4)A \log A_1} \right. \\ &\quad \left. + \frac{(2a+1) \log 2}{(2s-4)A \log A_1} \right). \end{aligned}$$

As before the coefficient of $(2s-4)A \log A_1$ is <1 and thus

$$\log (1/B_{19}) < (2s-4)A \log A_1 + (k-1) \log A_1.$$

$$(C) \quad \log C < 20k^3 2^n \quad (\text{Theorem 46}).$$

Proof:

$$\begin{aligned} \log C &= k2^{k-2}(\log C_{66} + \log (1/B_{19})) \\ &< k2^{k-2}((2s-4)kA + k) \log A_1 \\ &< 9k2^{k-2}((2s-4)kA + k)\epsilon_1 2^n \quad (\text{by } (A_1)) \\ &< 20k^3 2^n \quad (\text{by } (2) \text{ and } (17)). \end{aligned}$$

9. Proof of the main theorem. We prove the following

THEOREM. We have

$$\begin{aligned} g(k) &\leq \left[\frac{1}{2} (H + FD + Q + E \right. \\ &\quad \left. + \left((H + FD + Q - E)^2 + 4F(ED + R) \right)^{1/2} \right] + 1, \end{aligned}$$

$$\lim_{k \rightarrow \infty} \frac{g(k)}{k \cdot 2^{k-1}} \leq \frac{1}{2}.$$

By Theorem 46 every integer $\geq C$ is a sum of s_0 k th powers when

$$\begin{aligned} s_0 &\geq s + s_2, \\ s_0 &\geq (H + (1 + (1-a)^{s-2})k2^{k-2}\epsilon_1)(1 - D\epsilon_1)^{-1} + 2 + 4 + \zeta_k \\ &= (H\eta_1 + (1 + (1-a)^{s-2})k2^{k-2} + (\eta_1 - D)(6 + \zeta_k))(\eta_1 - D)^{-1}, \\ (50) \quad s_0 &\geq ((H + Q)\eta_1 + R)(\eta_1 - D)^{-1}. \end{aligned}$$

Also, by Theorem 50 every integer $< C$ is a sum of s_0 k th powers if

$$\begin{aligned} s_0 &\geq s_3 + s_4, \\ s_0 &\geq s_3 + (\log \log C - \log (\log L - k \log k))(\log k - \log (k-1))^{-1} \\ &= s_3 + (3 \log k + \log 20 + \eta_1 \log 2 - \log (\log L - k \log k)) \\ &\quad \times (\log k - \log (k-1))^{-1} \quad (\text{by } (C)), \\ (51) \quad s_0 &\geq F\eta_1 + E. \end{aligned}$$

The right members of (50) and (51) are equal when

$$(52) \quad \eta_1 = \left(H + FD + Q - E + \left((H + FD + Q - E)^2 + 4F(ED + R) \right)^{1/2} \right) (2F)^{-1}$$

and then every integer is a sum of s_0 k th powers ≥ 0 when

$$s_0 > \frac{1}{2} \left(H + FD + Q + E + \left((H + FD + Q - E)^2 + 4F(ED + R) \right)^{1/2} \right).$$

It remains to show that this choice of η_1 satisfies the condition (17). Since $E > 2^k$ we have $ED + R > 0$ and thus

$$\begin{aligned} \eta_1 &> (H + FD + Q - E)F^{-1}, \\ (53) \quad \eta_1 - D &> (H + Q - E)F^{-1}. \end{aligned}$$

Also,

$$\begin{aligned} H + Q - E &= (k - 2)2^{k-2} + k + 6 + \zeta_k - s_3 \\ &\quad - (3 \log k + \log 20 - \log (\log L - k \log k))(\log k - \log (k - 1))^{-1} \\ &> (k - 2)2^{k-2} + k + 6 + \zeta_k - 2^k \\ &\quad - \left(\left(\frac{3}{2} \right)^k + 2 \left(\frac{4}{3} \right)^k + 2 \left(\frac{2}{3} \right)^k + 2 \left(\frac{1}{2} \right)^k + \frac{k(2k + 7)}{9} - 9 \right) \\ &\quad - (3 \log k + \log 20 - \log (\log L - k \log k))(\log k - \log (k - 1))^{-1} \\ &= (k - 6)2^{k-2} + k + 6 + \zeta_k \\ &\quad - \left(\left(\frac{3}{2} \right)^k + 2 \left(\frac{4}{3} \right)^k + 2 \left(\frac{2}{3} \right)^k + 2 \left(\frac{1}{2} \right)^k + \frac{k(2k + 7)}{9} - 9 \right) \\ &\quad - (3 \log k + \log 20 - \log (\log L - k \log k))(\log k - \log (k - 1))^{-1}; \end{aligned}$$

and

$$F = \log 2(\log k - \log (k - 1))^{-1}.$$

Hence $(H + Q - E)F^{-1} - 2^{k-3}$ is an increasing function of k which is positive when $k = 7$ so that

$$(H + Q - E)F^{-1} - 2^{k-3} \geq 0$$

for all $k \geq 7$. Then from (53)

$$\eta_1 - D \geq 2^{k-3}$$

for all $k \geq 7$. When $k = 6$ direct substitution in (52) yields $\eta_1 > 17$.

To obtain the values of $g(k)$ which are given in the introduction we require the following:

| Every integer from | to | is a sum of |
|------------------------------|------------------------------|-----------------|
| $11 \cdot 2^6$ | $12 \cdot 2^6$ | 39 6th powers |
| $25 \cdot 2^7 + 6 \cdot 3^7$ | $26 \cdot 2^7 + 6 \cdot 3^7$ | 58 7th powers |
| $25 \cdot 2^8 + 9 \cdot 3^8$ | $26 \cdot 2^8 + 9 \cdot 3^8$ | 120 8th powers |
| $38 \cdot 2^9$ | $39 \cdot 2^9$ | 285 9th powers |
| $57 \cdot 2^{10}$ | $58 \cdot 2^{10}$ | 737 10th powers |

By repeated application of Theorem 47 as indicated in the proof of Theorem 48 we obtain the following values for L and s_n .

| Every positive integer \leq | is a sum of |
|-------------------------------|------------------|
| $10^{17.98}$ | 73 6th powers |
| 10^{1948} | 143 7th powers |
| $10^{3920000}$ | 279 8th powers |
| $10^{10^{9.7}}$ | 548 9th powers |
| $10^{10^{11.1}}$ | 1079 10th powers |

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ALMOST PERIODIC FUNCTIONS IN A GROUP. I*

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INTRODUCTION

1. The object of the present paper is to extend H. Bohr's famous theory of almost periodic functions [4, I]† to arbitrary groups, and to show that it gives just the maximum range over which the fundamental results of Frobenius-Schur representation theory [21; 22; 30] and its extensions by Peter and Weyl [32] hold. We shall see in particular that all bounded linear representations of a group are equivalent to unitary representations and belong to this class. Another point of importance is that we free ourselves completely from all topological assumptions (such as continuity, etc.) by the use of a definition of almost periodicity due to Bochner [2]. Thus we find that the general theory, which applies to every group \mathfrak{G} whatsoever, is completely free from topological assumptions, but all of its results (for example, all series expansions) have a property of closure; if applied to functions which are continuous in a certain topology, they will lead only to functions of the same kind. It is remarkable that we find in the classical case of Bohr new almost periodic functions in addition to the known ones; even the elementary functions $f(a) = e^{2\pi\lambda a}$ can be generalized (this connects with results of Ursell [28]). On the other hand, in some groups (for example in all semi-simple Lie groups) almost periodicity automatically implies continuity (this will be proved with the aid of a theorem of van der Waerden [29]).

2. The principal difficulty in building up a general theory of almost periodic functions lies in finding a generalization of the Bohr integral mean

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(x) dx$$

if the real numbers x and T are replaced by the elements of an arbitrary group \mathfrak{G} which need not be even topological; also, the function $f(x)$ may be discontinuous. We meet this difficulty by finding an entirely new definition (cf. Definitions 4 and 5) which may be proved to be fit for the role of a "mean" under all conditions. The direct discussion of our mean is very simple and is given in Part I.

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† The numbers in brackets refer to the bibliography at the end of the paper.

This mean is an extension of an integral in compact groups previously defined by the author [19]. It is defined by means entirely different from those employed in Haar's integral [11] with which it coincides for compact groups, but from which it differs widely for non-compact groups, first, because for such groups it is an integral-mean and not an integral; second, because it is free from topological restrictions, while Haar's integral applies only to locally compact and separable groups; third, because it is defined for almost periodic functions while Haar's integral is defined for measurable functions, and in general neither of these two classes contains the other.

3. The content of Parts I-V is as follows: Part I gives our general theory of the mean. Part II applies this theory (by using the powerful method of Weyl [31]) to prove the fundamental theorems of the Bohr theory, Parseval's formula and the approximation theorem. As we have to combine the devices contained in two papers of Weyl [30; 31] we find it advisable to give the proofs in full, even though the repetition is often almost literal. Part III repeats the main results of the Frobenius-Schur and Peter-Weyl theory of representations, and connects them with the theory of almost periodic functions. It provides a basis for the statement that the present general theory of almost periodic functions is the widest range over which this theory of representations holds without any loss of strength. Part IV connects our theory with topological and other restrictive conditions. By investigating the details of eight examples we illustrate the principal types of combinations of these notions which are likely to occur. Finally, we discuss the question as to how many almost periodic functions exist in a given group. Part V is entirely devoted to the proof that the maximal amount exists in Abelian groups (subject, however, to certain topological restrictions). Here the integral of Haar is used in combination with certain theorems of the author on operators and functions of operators [17]. The extension of some results of Haar on countably infinite Abelian groups [10] is of great importance for these investigations.

4. It is probable that most of the further developments of the Bohr theory will also apply to our general theory. Among these developments are finer convergence theorems, summability theorems, and Stepanoff's generalizations (where some topological restrictions will be necessary, as the Haar integral must be applied). In this connection it may be of interest to point out a needed generalization of an important notion of the Bohr theory, namely, the fact that the product of two elementary almost periodic functions is a function of the same kind: $e^{2\pi\lambda ai}e^{2\pi\mu ai} = e^{2\pi(\lambda+\mu)ai}$. This is unchanged for Abelian groups and leads to the important character-group; but in non-Abelian groups the corresponding situation is that the direct product of two irreducible representations (the elements $D_{\rho\sigma}(a; \mathfrak{G})$ of which are the analogues

of $e^{2\pi\lambda\alpha i}$, cf. Definitions 11 and 12, and Theorems 24 and 28) is a sum of a finite number of irreducible representations, that is, there is the so-called composition formula

$$(*) \quad D_{\rho\sigma}(a; \mathfrak{G})D_{\tau\nu}(a; \mathfrak{D}) = \sum_{\xi, \eta} \Gamma_{\xi, \eta}(\rho, \sigma; \mathfrak{G} | \tau, \nu; \mathfrak{D})D_{\xi\eta}(a; \mathfrak{G}).$$

Another important notion in Bohr's theory is the independence of the expansion functions $e^{2\pi\lambda\alpha i}$, $e^{2\pi\mu\alpha i}$, \dots (that is, the linear independence of their exponents with integral coefficients), since almost periodic functions with such expansions possess particularly simple convergence properties. The corresponding requirement in our general theory is probably that the right-hand member of (*) should contain no term originating from the representation $D(a; \mathfrak{G}) \equiv 1$ if the left-hand member is any product of powers of $D(a; \mathfrak{G})$, $D(a; \mathfrak{D})$, \dots .

I. EXISTENCE OF THE MEAN, GENERAL PROPERTIES

5. Let \mathfrak{G} be a group, that is, a set in which the operations ab and a^{-1} are defined and satisfy the group postulates. While \mathfrak{G} may be topological* this property is not needed in Parts I-III and we do not yet make this assumption concerning \mathfrak{G} . Elements of \mathfrak{G} will be denoted by a, b, c, x, y, z, \dots , real or complex numbers by $m, n, u, v, \alpha, \beta, \xi, \eta, \dots$, and functions defined in \mathfrak{G} with complex numbers as values by $f(x), g(x), \dots$.

For such functions $f(x)$ and $g(x)$ we define distance† by

$$D(f, g) = \text{l.u.b.}_x |f(x) - g(x)|.$$

A set \mathfrak{M} of such functions is called conditionally compact (c.c.) if every sequence f_1, f_2, \dots extracted from it contains a subsequence f_{n_1}, f_{n_2}, \dots such that $D(f_{n_\mu}, f_{n_\nu}) \rightarrow 0$ as $\mu, \nu \rightarrow \infty$ (that is, a "fundamental" subsequence [13, p. 107]); this means that there exists a function f (not necessarily belonging to \mathfrak{M}) such that $D(f_{n_\mu}, f) \rightarrow 0$ as $\mu \rightarrow \infty$.

We now extend Bohr's notion of almost periodic functions [4; 2, §5] to all $f(x)$ in \mathfrak{G} , but we prefer to generalize the definition given by S. Bochner [2], as it allows us to rid ourselves completely of topological conditions on $f(x)$ (continuity, etc.).

* That is, a topological set in the sense of Hausdorff [13, pp. 226-230]. One may take his topological system based on the notion of a neighborhood by means of Axioms 1, 2, 3 (or A, B, C) and one of the "separation" Axioms 4-8, such as 5. Furthermore, certain continuity assumptions have to be made concerning ab and a^{-1} . In Parts I-III we shall need no topology at all, in Part IV we must assume that ab is continuous in a for fixed b and in b for fixed a , and in Part V we must assume that ab is continuous in (a, b) and that a^{-1} is continuous in a .

† We shall consider only bounded functions. l.u.b. denotes the least upper bound for all x 's in \mathfrak{G} .

DEFINITION 1. A function $f(x)$ in \mathfrak{G} (with complex values) is called *right almost periodic* (r.a.p.) if the set R_f of all functions $f(xa)$ (x is the variable, a is a parameter running over \mathfrak{G}) is c.c.; it is called *left almost periodic* (l.a.p.) if the set L_f of all functions $f(ax)$ is c.c.; it is called *almost periodic* (a.p.) if it is r.a.p. and l.a.p.

The equivalence of this definition to the obvious generalization of the Bohr definition is shown in the usual way if $f(x)$ is continuous; similarly, the uniform continuity of $f(x)$ follows in this case. But as we do not wish now to assume any topology in \mathfrak{G} , we shall not go into the details of this matter. On the other hand, the following theorems are of major importance:

THEOREM 1. Each of the three notions r.a.p., l.a.p. and a.p. is invariant under the following operations: $f(xa)$, $f(ax)$, $\overline{f(x)}$, $\alpha f(x)$ (α any complex number), $f(x) \pm g(x)$, $f(x)g(x)$, and the operation of passing from $f_1(x), f_2(x), \dots$ to $f(x)$ if $f_n(x)$ converges uniformly to $f(x)$ as $n \rightarrow \infty$. Passing from $f(x)$ to $f(x^{-1})$ interchanges r.a.p. and l.a.p. and leaves a.p. invariant.

The statement concerning $f(x^{-1})$ is obvious. In the other cases we need to consider only r.a.p., as l.a.p. results, for example, by replacing ab by ba when defining \mathfrak{G} , and a.p. results by combining r.a.p. and l.a.p. That $f(xa)$ is r.a.p. is seen by replacing a_1, a_2, \dots (Definition 1) by a_1a, a_2a, \dots ; that $f(ax)$ is r.a.p. results from replacing x by ax ; the situation concerning $\overline{f(x)}$ and $\alpha f(x)$ is obvious; the r.a.p. of $f(x) \pm g(x)$ and of $f(x)g(x)$ is proved by applying Definition 1 first to $f(x)$ and a_1, a_2, \dots , and then to $g(x)$ and the subsequence which has been selected. An obvious and simple application of the diagonal process shows the invariance of r.a.p. under the operation of passing from $f_n(x)$ to $f(x)$.

THEOREM 2. Every r.a.p. or l.a.p. function $f(x)$ is bounded.

Again it is sufficient to consider r.a.p. If $f(x)$ were not bounded, we could select a sequence a_1, a_2, \dots such that $|f(a_n)| \rightarrow \infty$ as $n \rightarrow \infty$, and then no subsequence of $f(xa_1), f(xa_2), \dots$ could have a finite limit at $x=1$.

DEFINITION 2. If \mathfrak{M} is a set of functions in \mathfrak{G} , we call the set of all functions $\alpha_1 f_1(x) + \dots + \alpha_n f_n(x)$ ($n=1, 2, \dots$; $\alpha_1, \dots, \alpha_n$ non-negative real numbers such that $\alpha_1 + \dots + \alpha_n = 1$; f_1, \dots, f_n any elements of \mathfrak{M}) the *convex* of \mathfrak{M} and denote it by $\text{Co}(\mathfrak{M})$.

6. We prove

THEOREM 3. If either of the sets \mathfrak{M} and $\text{Co}(\mathfrak{M})$ is c.c., the other is also c.c.

If $\text{Co}(\mathfrak{M})$ is c.c., its subset \mathfrak{M} is c.c. Conversely, suppose that \mathfrak{M} is c.c. The c.c. property of a set \mathfrak{M} is equivalent to the following condition: for every

$\epsilon > 0$ there exists a finite number of functions $\bar{f}_1, \dots, \bar{f}_m$ of \mathfrak{N} ($m = m(\epsilon)$) such that, for each $f \in \mathfrak{N}$, some $D(f, \bar{f}_\mu) \leq \epsilon$, $\mu = 1, \dots, m$ [13, pp. 108-109]. Now if an $\epsilon > 0$ is given, choose the functions $\bar{f}_1, \dots, \bar{f}_m$ for \mathfrak{N} and ϵ , put

$$\max_{\mu} \text{l.u.b.}_x |\bar{f}_\mu(x)| = C,$$

and select an integer $N \geq C m \epsilon^{-1}$. Then $\beta_1 \bar{f}_1 + \dots + \beta_m \bar{f}_m$ (where β_1, \dots, β_m are non-negative rational numbers with denominators N such that $\beta_1 + \dots + \beta_m = 1$) can be written as a finite sequence $\bar{g}_1, \dots, \bar{g}_M$ and have the property described above for $\text{Co}(\mathfrak{N})$ and 2ϵ .

DEFINITION 3. If $f(x)$ is a real bounded function in \mathfrak{G} , we call

$$\text{l.u.b.}_{x,y} |f(x) - f(y)| \quad (x \text{ and } y \text{ vary independently over } \mathfrak{G})$$

the oscillation of $f(x)$ and denote it by $\text{Osc}_x f(x)$. If $f(x)$ is not a constant, $\text{Osc}_x f(x) > 0$; otherwise, $\text{Osc}_x f(x)$ is zero.

THEOREM 4. For every real $g \in \text{Co} R_f$ we have $\text{Osc}_x g(x) \leq \text{Osc}_x f(x)$. If the relation $\text{Osc}_x g(x) < \text{Osc}_x f(x)$ never occurs, and if $f(x)$ is l.a.p., $f(x)$ is necessarily a constant.

The first statement is obvious. Suppose that the assumptions of the second statement are valid. Let a'_1, \dots, a'_n be any elements of \mathfrak{G} ; then

$$\frac{f(xa'_1) + \dots + f(xa'_n)}{n} \in \text{Co } R_f,$$

and thus

$$\text{Osc}_x \frac{f(xa'_1) + \dots + f(xa'_n)}{n} = \text{Osc}_x f(x).$$

This implies that

$$\text{l.u.b.}_x \frac{f(xa'_1) + \dots + f(xa'_n)}{n} = \text{l.u.b.}_x f(x).$$

Put $\text{l.u.b.}_x f(x) = C$; then for every $\epsilon > 0$ there exists an x' such that

$$\frac{f(x'a'_1) + \dots + f(x'a'_n)}{n} \geq C - \epsilon.$$

As all $f(x'a'_i) \leq C$, they all must also be $\geq C - n\epsilon$.

Now choose a $\delta > 0$ and find a finite number of elements of L_f such that each element of L_f has a distance $\leq \delta$ from one of them (cf. the proof of The-

orem 3). That is, find a finite number of elements a_1, \dots, a_n of \mathfrak{G} such that for every a of \mathfrak{G} there exists a $\mu = 1, 2, \dots, n$ for which $|f(a_\mu x) - f(ax)| \leq \delta$ identically. Now choose a b of \mathfrak{G} and repeat the argument just described in the case where $\epsilon = \delta/n$, $a'_1 = a_1^{-1}b, \dots, a'_n = a_n^{-1}b$. Thus an x' exists for which all $f(x'a_\nu^{-1}b) \geq C - \delta$, $\nu = 1, 2, \dots, n$, and therefore, for a properly chosen $\mu = 1, 2, \dots, n$, all $f(a_\mu a_\nu^{-1}b) \geq C - 2\delta$. If $\nu = \mu$, then $f(b) \geq C - 2\delta$.

On the other hand, $f(b) \leq C$ and, as δ was arbitrary, it follows that $f(b) = C$. Finally, $f(x)$ is constant since b was arbitrary. This completes the proof of the theorem.

THEOREM 5. *If $f(x)$ is a.p., there exists a constant A toward which a certain sequence extracted from $\text{Co}R_f$ converges uniformly.*

Since $f(x)$ is r.a.p., R_f and $\text{Co}R_f$ are c.c. Denote real and imaginary parts by \Re and \Im respectively, consider the non-negative numbers $\text{Osc}_x \Re g(x) + \text{Osc}_x \Im g(x)$, $g \in \text{Co}R_f$ and call their greatest lower bound ω . We can extract a sequence $g_1(x), g_2(x), \dots$ from $\text{Co}R_f$ such that $\text{Osc}_x \Re g_n(x) + \text{Osc}_x \Im g_n(x) \rightarrow \omega$ as $n \rightarrow \infty$, and from this a subsequence $g_{n_1}(x), g_{n_2}(x), \dots$, which converges uniformly to a function $g(x)$. Hence $\text{Osc}_x \Re g(x) + \text{Osc}_x \Im g(x) = \omega$. It is obvious that, $f(x)$ being l.a.p., every element $f(xa)$ of R_f is l.a.p. Therefore every element of $\text{Co}R_f$ is l.a.p., and the uniform limit $g(x)$ as well as the real functions $\Re g(x)$ and $\Im g(x)$ are l.a.p. If we show that $\text{Osc}_x \Re g(x) = \text{Osc}_x \Im g(x) = 0$, we have $\Re g(x) = \text{constant}$, $\Im g(x) = \text{constant}$, that is, $g(x) = \text{constant}$, which proves our statement.

Suppose that $\text{Osc}_x \Re g(x) > 0$. Then Theorem 4 shows that an $h \in \text{Co}R_{\Re g}$ exists such that $\text{Osc}_x h(x) < \text{Osc}_x \Re g(x)$. Here $h(x) = \alpha_1 \Re g(xa_1) + \dots + \alpha_n \Re g(xa_n)$ ($\alpha_1, \dots, \alpha_n$ each ≥ 0 , $\alpha_1 + \dots + \alpha_n = 1$). Putting $k(x) = \alpha_1 g(xa_1) + \dots + \alpha_n g(xa_n)$, we have $h(x) = \Re k(x)$, so that $\text{Osc}_x \Re k(x) < \text{Osc}_x \Re g(x)$. But it is obvious that $\text{Osc}_x \Im k(x) \leq \text{Osc}_x \Im g(x)$. Therefore $\text{Osc}_x \Re k(x) + \text{Osc}_x \Im k(x) < \omega$. Now $g(x)$ can be uniformly approximated by functions $l \in \text{Co}R_f$, that is, $l(x) = \beta_1 f(xb_1) + \dots + \beta_m f(xb_m)$ (β_1, \dots, β_m each ≥ 0 , $\beta_1 + \dots + \beta_m = 1$). Hence $k(x)$ can be uniformly approximated by functions $q(x) = \alpha_1 \beta_1 f(xa_1 b_1) + \alpha_1 \beta_2 f(xa_1 b_2) + \dots + \alpha_n \beta_m f(xa_n b_m)$, that is, by functions $q \in \text{Co}R_f$. Since $\text{Osc}_x \Re k(x) + \text{Osc}_x \Im k(x) < \omega$, the relation that $\text{Osc}_x \Re q(x) + \text{Osc}_x \Im q(x) < \omega$ results. This contradicts the definition of ω . Similarly $\text{Osc}_x \Im g(x) > 0$ is disproved.

REMARK. *If a finite number of a.p. functions $f_1(x), \dots, f_t(x)$ are given, it is possible to find a set of constants A_1, \dots, A_t toward which t sequences extracted from $\text{Co}R_{f_1}, \dots, \text{Co}R_{f_t}$ respectively, with the same $\alpha_1, \dots, \alpha_n$, a_1, \dots, a_n , converge uniformly (that is, sequences of the form $\alpha_1^{(v)} f_1(xa_1^{(v)})$)*

$+\dots + \alpha_{n_p}^{(p)} f_1(x a_{n_p}^{(p)}), \dots, \alpha_{1_t}^{(p)} f(x a_{1_t}^{(p)}) + \dots + \alpha_{n_p}^{(p)} f_t(x a_{n_p}^{(p)}),$ where $p \rightarrow \infty, \alpha_1^{(p)} \geq 0, \dots, \alpha_{n_p}^{(p)} \geq 0,$ and $\alpha_1^{(p)} + \dots + \alpha_{n_p}^{(p)} = 1$.

The argument which proved Theorem 5 may be repeated here if we use $\text{Osc}_x \Re f_1(x) + \text{Osc}_x \Im f_1(x) + \dots + \text{Osc}_x \Re f_t(x) + \text{Osc}_x \Im f_t(x)$ instead of $\text{Osc}_x \Re f(x) + \text{Osc}_x \Im f(x)$.

DEFINITION 4. A real number A which may be uniformly approximated by functions from $\text{Co}R_I$ or $\text{Co}L_I$, that is, a number A such that, for every $\epsilon > 0$, there exists a number $n=1, 2, \dots$, numbers $\alpha_1, \dots, \alpha_n$ each ≥ 0 with $\alpha_1 + \dots + \alpha_n = 1$, and elements a_1, \dots, a_n of \mathfrak{G} such that the condition $|\alpha_1 f(x a_1) + \dots + \alpha_n f(x a_n) - A| \leq \epsilon$ or $|\alpha_1 f(a_1 x) + \dots + \alpha_n f(a_n x) - A| \leq \epsilon$ holds throughout \mathfrak{G} , is called a right-mean or a left-mean of $f(x)$ respectively.

THEOREM 6. If $f(x)$ is a.p., it has exactly one right-mean, exactly one left-mean, and these means are equal.

The existence of a right-mean has been proved by Theorem 5. If we change the multiplication law ab in \mathfrak{G} to ba , all notions remain unchanged except for the interchange of "right" and "left." Thus a left-mean must exist.

Now let A be a right-mean, let B be a left-mean, and let ϵ be > 0 . Choose $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m, a_1, \dots, a_n, b_1, \dots, b_m$ such that

$$\begin{aligned} |\alpha_1 f(x a_1) + \dots + \alpha_n f(x a_n) - A| &\leq \epsilon, \\ |\beta_1 f(b_1 x) + \dots + \beta_m f(b_m x) - B| &\leq \epsilon. \end{aligned}$$

If we replace x in the first equation by $b_1 x, \dots, b_m x$ in succession, and add, we obtain

$$|\alpha_1 \beta_1 f(b_1 x a_1) + \alpha_1 \beta_2 f(b_2 x a_1) + \dots + \alpha_n \beta_m f(b_m x a_n) - A| \leq \epsilon.$$

Similarly, if we replace x in the second equation by $x a_1, \dots, x a_n$ in succession, we obtain

$$|\alpha_1 \beta_1 f(b_1 x a_1) + \alpha_1 \beta_2 f(b_2 x a_1) + \dots + \alpha_n \beta_m f(b_m x a_n) - B| \leq \epsilon.$$

Therefore $|A - B| \leq 2\epsilon$ and, as ϵ may be arbitrarily small, $A = B$.

DEFINITION 5. If $f(x)$ is a.p., we call the common value of its uniquely determined right- and left-means the mean of $f(x)$, and denote it by $M_x f(x)$.*

We now state the most important properties of the mean.

* Definitions 3-5 and the argument of Theorems 3-6 are in very close analogy to the author's construction of the Haar-Lebesgue measure in compact groups [19]. It is noteworthy that for non-compact groups, where Haar proved by his method the existence of an integral [11], our method leads to an integral-mean.

THEOREM 7. If $f(x)$ and $g(x)$ are a.p. functions, all the functions $f(xa)$, $f(ax)$, $f(x^{-1})$, $\overline{f(x)}$, $\alpha f(x)$, $f(x) \pm g(x)$ (α a complex number, a an element of \mathfrak{G}) are a.p. (cf. Theorem 1). Furthermore, we have the following:

- (1) $M_x[\alpha f(x)] = \alpha M_x f(x)$.
- (2) $M_x[f(x) \pm g(x)] = M_x f(x) \pm M_x g(x)$.
- (3) $M_x 1 = 1$.
- (4) If $f(x)$ is real and ≥ 0 throughout \mathfrak{G} , then $M_x f(x) \geq 0$; and if, in addition, $f(x) \neq 0$, then $M_x f(x) > 0$.
- (5) $|M_x[f(x)]| \leq M_x[|f(x)|]$.
- (6) $M_x[\overline{f(x)}] = \overline{M_x[f(x)]}$.
- (7) $M_x f(xa) = M_x f(x)$.
- (8) $M_x f(ax) = M_x f(x)$.
- (9) $M_x f(x^{-1}) = M_x f(x)$.

The equations (1), (3), (5), (6) and the first half of (4) are obvious; as every left-mean of $f(x)$ is a left-mean of $f(xa)$ and as every right-mean of $f(x)$ is a right-mean of $f(ax)$, (7) and (8) are valid; as every right-mean of $f(x)$ is a left-mean of $f(x^{-1})$, (9) is true. Thus, only (2) and the second half of (4) remain unproved.

In order to prove (2), put $M_x f(x) = A$, $M_x g(x) = B$, let ϵ be > 0 , and choose $\alpha_1, \dots, \alpha_n$ ($\alpha_1, \dots, \alpha_n$ each ≥ 0 , $\alpha_1 + \dots + \alpha_n = 1$) and a_1, \dots, a_n such that

$$|\alpha_1 f(xa_1) + \dots + \alpha_n f(xa_n) - A| \leq \epsilon.$$

Now $\alpha_1 g(xa_1) + \dots + \alpha_n g(xa_n)$ obviously has the same left-mean as $g(x)$, i.e., B . Therefore we can choose β_1, \dots, β_m (β_1, \dots, β_m each ≥ 0 , $\beta_1 + \dots + \beta_m = 1$) and b_1, \dots, b_m such that

$$|\alpha_1 \beta_1 g(xb_1a_1) + \alpha_1 \beta_2 g(xb_2a_1) + \dots + \alpha_n \beta_m g(xb_ma_n) - B| \leq \epsilon.$$

If we replace x in the first inequality by xb_1, \dots, xb_m in succession, and add, we obtain

$$|\alpha_1 \beta_1 f(xb_1a_1) + \alpha_1 \beta_2 f(xb_2a_1) + \dots + \alpha_n \beta_m f(xb_ma_n) - A| \leq \epsilon.$$

Denote nm by p ; $\alpha_1 \beta_1, \alpha_1 \beta_2, \dots, \alpha_n \beta_m$ by $\gamma_1, \dots, \gamma_p$ ($\gamma_1, \dots, \gamma_p$ each ≥ 0 , $\gamma_1 + \dots + \gamma_p = 1$); $b_1a_1, b_2a_1, \dots, b_ma_n$ by c_1, \dots, c_p ; we get, by adding and subtracting our inequalities,

$$|\gamma_1(f(xc_1) \pm g(xc_1)) + \dots + \gamma_p(f(xc_p) \pm g(xc_p)) - (A \pm B)| \leq 2\epsilon.$$

As ϵ may be arbitrarily small, this shows that $M_x[f(x) \pm g(x)] = A \pm B$. (An-

other way to prove (2) would be to apply the Remark following Theorem 5 to $f(x)$ and $g(x)$.

In order to prove the second half of (4), assume $f(x) \geq 0$ everywhere and $f(x_0) > 0$ for one particular x_0 . For any $\epsilon > 0$ a finite number of elements of R_f exist such that each element of R_f has a distance $\leq \epsilon$ from one of them (cf. the proof of Theorem 3). Hence there is a finite number of the elements a_1, \dots, a_n such that, for every a , there exists a $\mu = 1, 2, \dots, n$ such that $|f(xa_\mu) - f(xa)| \leq \epsilon$ identically. Now take $\epsilon = f(x_0)/2$. The substitution $x = x_0 a_\mu^{-1}$ shows that $f(x_0 a_\mu^{-1} a) \geq f(x_0)/2$. Hence, for each a it follows that $f(x_0 a_\nu^{-1} a) \geq 0$ for every $\nu = 1, \dots, n$, but that $f(x_0 a_\nu^{-1} a) \geq f(x_0)/2$ for at least one ν . Thus $f(x_0 a_1^{-1} a) + \dots + f(x_0 a_n^{-1} a) \geq f(x_0)/2$, that is, the function $g(y) = f(x_0 a_1^{-1} y) + \dots + f(x_0 a_n^{-1} y) - f(x_0)/2$ is always ≥ 0 . Hence the first half of (4) leads to the result that $M_y g(y) \geq 0$, (2), (3), (6), (7), and (8) show that $M_y g(y) = n M_y f(y) - f(x_0)/2$, and it follows that

$$M_y f(y) \geq \frac{f(x_0)}{2n} > 0.$$

THEOREM 8. *The formal properties (1)–(9) determine $M_x f(x)$ uniquely; in fact, (1)–(3), the first half of (4), and (7) or (8) are sufficient.*

It is sufficient to consider (1)–(3), the first half of (4), and (7), as (8) may be obtained by replacing ab in \mathfrak{G} by ba . So assume that a functional $M_x' f(x)$, defined for all a.p. $f(x)$ and satisfying (1)–(3), the first half of (4), and (7), is given.

For every $\epsilon > 0$ we can choose $\alpha_1, \dots, \alpha_n, a_1, \dots, a_n$ ($\alpha_1, \dots, \alpha_n$ each ≥ 0 , $\alpha_1 + \dots + \alpha_n = 1$) such that $|\alpha_1 f(xa_1) + \dots + \alpha_n f(xa_n) - M_x f(x)| \leq \epsilon$, or if $f(x)$ is real,

$$M_x f(x) - \epsilon \leq \alpha_1 f(xa_1) + \dots + \alpha_n f(xa_n) \leq M_x f(x) + \epsilon.$$

Then (1)–(3), the first half of (4), and (7) show that $M_x f(x) - \epsilon \leq M_x' f(x) \leq M_x f(x) + \epsilon$, and as ϵ was arbitrary, $M_x' f(x) = M_x f(x)$. Property (1) with $\alpha = i$ shows that this holds also for pure imaginary $f(x)$, and property (2) shows that it holds for every $f(x)$.

Theorems 6–8 show that, for a.p. functions $f(x)$, there is exactly one way to define a notion $M_x f(x)$ possessing the essential formal properties of a mean. Our $M_x f(x)$ is the equivalent of the well known integral mean

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(x) dx$$

in Bohr's theory, when \mathfrak{G} is the addition group of all real numbers. But even

in this case the form of our definition is essentially different from the usual one (for example, it does not use continuity), and it gives a new approach to the problem.

REMARK.* The notion of the mean can be modified in the following manner. Consider the doubled group $\mathfrak{G}\mathfrak{G}'$, that is, the set of all pairs $[a, a']$. This set is a group by virtue of the definitions $[a, a'] [b, b'] = [ab, b'a']$ and $[a, a']^{-1} = [a^{-1}, a'^{-1}]$. (This is similar to the construction in the next paragraph except that here we use $b'a'$ while there we use $a'b'$.) The argument in the proof of Theorem 9 below shows that if $f(x)$ is a.p. in \mathfrak{G} , then $f_0([x, x']) = f(xx')$ is a.p. in $\mathfrak{G}\mathfrak{G}'$. By Theorem 5 it then follows that there exists a constant A such that, for every $\epsilon > 0$, there exists a number $n = 1, 2, \dots$, numbers $\alpha_1, \dots, \alpha_n$ each ≥ 0 with $\alpha_1 + \dots + \alpha_n = 1$, and elements a_1, \dots, a_n and b_1, \dots, b_n of \mathfrak{G} such that the condition

$$|\alpha_1 f_0([x, y][a_1, b_1]) + \dots + \alpha_n f_0([x, y][a_n, b_n]) - A| \leq \epsilon$$

holds for all x and y in \mathfrak{G} . If we write $c_1 = a_1 b_1, \dots, c_n = a_n b_n$, this condition assumes the form

$$|\alpha_1 f(x c_1 y) + \dots + \alpha_n f(x c_n y) - A| \leq \epsilon.$$

This mean is even easier to handle than our right- and left-means (which are special cases of it). This is due to the following fact: choose two arbitrary sets of numbers β_1, \dots, β_k and $\gamma_1, \dots, \gamma_l$, each ≥ 0 , with $\beta_1 + \dots + \beta_k = 1$ and $\gamma_1 + \dots + \gamma_l = 1$, and two arbitrary sets of elements a_1, \dots, a_k and b_1, \dots, b_l of \mathfrak{G} . In our last inequality replace x and y by $x a_\kappa$ and $b_\lambda y$, multiply by $\beta_\kappa \gamma_\lambda$, and add over all $\kappa = 1, \dots, k$; $\lambda = 1, \dots, l$. Then we obtain an inequality of the same type except that there are knl terms instead of n terms and $\beta_\kappa \alpha_\nu \gamma_\lambda$ and $a_\kappa c_\nu b_\lambda$ appear in place of α_ν and a_ν . This shows that the conditions

$$\begin{aligned} |\alpha_1 f(x c_1 y) + \dots + \alpha_m f(x c_m y) - A| &\leq \epsilon, \\ |\alpha'_1 g(x c'_1 y) + \dots + \alpha'_n g(x c'_n y) - B| &\leq \epsilon \end{aligned}$$

imply the conditions

$$\begin{aligned} |\alpha''_1 f(x c''_1 y) + \dots + \alpha''_{mn} f(x c''_{mn} y) - A| &\leq \epsilon, \\ |\alpha''_1 g(x c''_1 y) + \dots + \alpha''_{mn} g(x c''_{mn} y) - B| &\leq \epsilon \end{aligned}$$

if α''_ν and c''_ν are $\alpha_\mu \alpha'_\nu$ and $c_\mu c'_\nu$ (in some order). This gives the uniqueness, the extension to complex $f(x)$, and the additivity of our new mean at once. Of course this mean coincides with our former means.

* Added February 4, 1934.

7. The applications to be made in the next chapter necessitate our proving some facts concerning double means. We therefore pass to this subject.

The group \mathfrak{G} can be "doubled," that is, we can consider the set $\mathfrak{G}\mathfrak{G}$ of all pairs $[a, a']$, which by the definitions $[a, a'] [b, b'] = [ab, a'b']$, $[a, a']^{-1} = [a^{-1}, a'^{-1}]$ becomes a group, and we will denote functions in it by $f(x, x')$ instead of by $f([x, x'])$. All our notions apply to $\mathfrak{G}\mathfrak{G}$: we have a.p. functions $f(x, x')$ in $\mathfrak{G}\mathfrak{G}$, and a mean $M_{x, x'}f(x, x')$.

THEOREM 9. *If $f(x)$ is a.p. in \mathfrak{G} , the eight functions $f(xx')$, $f(x'x)$, $f(xx'^{-1})$, \dots , $f(x'^{-1}x^{-1})$ are all a.p. in $\mathfrak{G}\mathfrak{G}$.*

Interchange of x and x' , of ab in \mathfrak{G} with ba , and of $f(x)$ with $f(x^{-1})$ reduces our task to discussing $f(xx')$ and $f(xx'^{-1})$ alone. Their a.p. character in $\mathfrak{G}\mathfrak{G}$ means that the sets of functions $f(axa'x')$, $f(xax'a')$, $f(axx'^{-1}a'^{-1})$, $f(xaa'^{-1}x'^{-1})$ in $\mathfrak{G}\mathfrak{G}$ are c.c. or else that the sets of functions of one or of two variables, $f(axby)$, $f(xayb)$, $f(axb)$, $f(xay)$ in \mathfrak{G} are c.c. The third case arises from the first by setting $y=1$, the fourth from the second by setting $b=1$, and the second from the first by interchanging a and x with b and y , and ab in \mathfrak{G} with ba . So we need to discuss only $f(axby)$.

Choose an $\epsilon > 0$. As $f(x)$ is r.a.p., there is a finite number of elements y_1, \dots, y_n , such that for each y there is a $\nu=1, \dots, n$ for which $|f(zy) - f(zy_\nu)| \leq \epsilon$ identically. As each $f(xy_\nu)$ is r.a.p., the set of all functions $f(xby_\nu)$ is c.c. for every $\nu=1, \dots, n$, and therefore even the set of all "vector-functions" with n components $[f(xby_1), \dots, f(xby_n)]$ is c.c. Therefore a finite number of elements b_1, \dots, b_m exist such that for each b there is a $\mu=1, 2, \dots, m$ for which $|f(zby_\nu) - f(zb_\mu y_\nu)| \leq \epsilon$ for all z and for every $\nu=1, 2, \dots, n$. Our two inequalities together give the result that

$$|f(zby) - f(zb_\mu y)| \leq 3\epsilon.$$

Finally, $f(x)$ is l.a.p., so that there exists a finite set of elements a_1, \dots, a_l such that for every a there is a $\lambda=1, \dots, l$ for which $|f(au) - f(a_\lambda u)| \leq \epsilon$ identically. This, together with our last inequality, implies that $|f(axby) - f(a_\lambda x b_\mu y)| \leq 5\epsilon$ identically.

As this holds for every $\epsilon > 0$, the c.c. character of the set of functions $f(axby)$ is proved [13, pp. 108-109].

THEOREM 10. *If $f(x, x')$ is a.p. in $\mathfrak{G}\mathfrak{G}$, it is also a.p. in \mathfrak{G} as a function of x or as a function of x' . Thus we can form $M_x f(x, x')$ and $M_{x'} f(x, x')$ which are a.p. in \mathfrak{G} as a function of x' and as a function of x , respectively. Thus we can form $M_{x'} [M_x f(x, x')]$ and $M_x [M_{x'} f(x, x')]$. These expressions are both equal to $M_{xx'} f(x, x')$.*

The first statement is obvious. In the second and third statements it is sufficient to consider $M_x f(x, x')$ and $M_{x'} [M_x f(x, x')]$, as interchange of x

with x' and of $f(x, x')$ with $f(x', x)$ leads to the rest of the theorem.

Consider a sequence a_1, a_2, \dots of elements of \mathfrak{G} . As $f(x, x')$ is r.a.p., the sequence $f(x, x'a_1), f(x, x'a_2), \dots$ contains a uniformly convergent subsequence $f(x, x'a_{n_1}), f(x, x'a_{n_2}), \dots$ such that, for every $\epsilon > 0$ and almost all μ and ν , $|f(x, x'a_{n_\mu}) - f(x, x'a_{n_\nu})| \leq \epsilon$. This implies that $|M_x f(x, x'a_{n_\mu}) - M_x f(x, x'a_{n_\nu})| \leq \epsilon$. Thus the set of functions of x' , $M_x f(x, x'a)$, is c.c. Therefore $M_x f(x, x')$ is r.a.p. and interchange of ab in \mathfrak{G} with ba shows that it is l.a.p. Hence it is a.p.

Now it is obvious that $M'_{xx'} f(x, x') = M_{x'} [M_x f(x, x')]$ has Properties (1)–(4) and (7) enumerated in Theorem 7 if we look at it as an $[x, x']$ -mean. Therefore we may conclude from Theorem 8 that it is $M_{xx'} f(x, x')$.

Theorems 9 and 10 may be extended by iterating them m times to functions of 2^m variables; by choosing $2^m \geq n$ and taking the functions constant in the last $2^m - n$ variables these theorems may be extended to functions of n variables.

II. APPLICATION OF THE METHOD OF WEYL AND E. SCHMIDT

PROOF OF THE FUNDAMENTAL THEOREMS

8. The results of Part I enable us to apply the method of Weyl to the proof of the fundamental theorems of Bohr's theory of a.p. functions (in the addition group of real numbers) and to the discussion of the linear-orthogonal representations of continuous groups.* The present part, II, contains a proof of "Parseval's formula" (equivalent to Theorem 15), which runs exactly along the lines of Weyl's proof. It also contains the proof of the "approximation theorem" (equivalent to Theorem 18) where a different device, due to N. Wiener, has to be used because of the difficulties of constructing in our general case an a.p. function with the required properties (cf. [31, pp. 348–349], and our Theorem 17). The next part, III, contains an interpretation and application of these theorems connecting the theories of a.p. functions and of representations. In this, Weyl's method is of fundamental importance.

DEFINITION 6. If $f(x)$ and $g(x)$ are a.p., we set

$$h(x) = M_y [f(xy^{-1})g(y)] = M_y [f(y)g(y^{-1}x)] = f \times g.$$

We observe that the two expressions for $h(x)$ are equal by Theorem 7 and Properties (7) and (9), after making the substitution $y^{-1}x$ for y , and that $h(x)$ is a.p. by Theorems 9 and 10.

* Cf. H. Weyl [31], H. Weyl and F. Peter [32]. The operational methods used there are partly based on the thesis of E. Schmidt [20].

REMARK. $f \times g$ can be uniformly approximated by functions of the form $\gamma_1 f(xc_1) + \dots + \gamma_n f(xc_n)$ ($\gamma_1, \dots, \gamma_n$ are complex numbers), that is, for every $\epsilon > 0$ there exist numbers $\gamma_1, \dots, \gamma_n$ and elements c_1, \dots, c_n such that

$$|f \times g(x) - \gamma_1 f(xc_1) - \dots - \gamma_n f(xc_n)| \leq \epsilon$$

holds throughout \mathfrak{G} .

$g(x)$ is a.p. and therefore bounded (Theorem 2); suppose $g(x) \leq C$. Now choose a $\delta > 0$. According to our remark in the proof of Theorem 3, it is possible to find a finite number of elements b_1, \dots, b_k of \mathfrak{G} such that to every x there exists a $\kappa = 1, \dots, k$ for which $|f(xz) - f(b_\kappa z)| \leq \delta$ holds identically. Now consider the a.p. y -functions $f(b_\kappa y^{-1})g(y)$, $\kappa = 1, \dots, k$, and apply to them the Remark following Theorem 5 (with $\epsilon/2$): if $\epsilon > 0$, there exists a set of real numbers $\alpha_1, \dots, \alpha_n$ ($\alpha_1, \dots, \alpha_n$ each ≥ 0 , $\alpha_1 + \dots + \alpha_n = 1$) and a set of elements a_1, \dots, a_n of \mathfrak{G} such that

$$|\alpha_1 f(b_1 a_1^{-1} y^{-1})g(ya_1) + \dots + \alpha_n f(b_n a_n^{-1} y^{-1})g(ya_n) - M_y[f(b_\kappa y^{-1})g(y)]| \leq \frac{\epsilon}{2}$$

holds identically for every $\kappa = 1, \dots, k$. For every x there exists a κ such that $|f(xz) - f(b_\kappa z)| \leq \delta$ and therefore such that

$$\begin{aligned} |f(xu^{-1})g(u) - f(b_\kappa u^{-1})g(u)| &\leq C\delta \quad \text{and} \\ |M_u[f(xu^{-1})g(u)] - M_u[f(b_\kappa u^{-1})g(u)]| &\leq C\delta. \end{aligned}$$

Hence we have the result that

$$\begin{aligned} &|\alpha_1 f(xa_1^{-1} y^{-1})g(ya_1) + \dots + \alpha_n f(xa_n^{-1} y^{-1})g(ya_n) - M_y[f(xy^{-1})g(y)]| \\ &\leq 2C\delta + \frac{\epsilon}{2}. \end{aligned}$$

Our statement is proved if we put $\delta = \epsilon/(4C)$ and $y = 1$, and substitute $\alpha_1 g(a_1), \dots, \alpha_n g(a_n)$ for $\gamma_1, \dots, \gamma_n$ and $a_1^{-1}, \dots, a_n^{-1}$ for c_1, \dots, c_n .

THEOREM 11. The "multiplication" $f \times g$ is distributive (linear) in both factors, associative, and if \mathfrak{G} is Abelian, commutative.

The theorem is obvious except for associativity. Our second form for $h = f \times g$ gives

$$\begin{aligned} (f \times g) \times k(x) &= M_z[M_y[f(y)g(y^{-1}z)]k(z^{-1}x)] \\ &= M_z[M_y[f(y)g(y^{-1}z)k(z^{-1}x)]], \\ f \times (g \times k)(x) &= M_y[f(y)M_z[g(y^{-1}z)k(z^{-1}x)]] \\ &= M_y[M_z[f(y)g(y^{-1}z)k(z^{-1}x)]], \end{aligned}$$

and these expressions are equal by Theorems 9 and 10.

DEFINITION 7. If $f(x)$ is a.p., we denote $f \times f \times \cdots \times f$ (n factors) by f^n , and $\overline{f(x^{-1})}$ by $f'(x)$. Furthermore, we define

$$Nf = \{M_x[|f(x)|^2]\}^{1/2}.$$

THEOREM 12. Let $f(x)$ and $g(x)$ be a.p. functions. The following formulas hold:

- (1) If $f \neq 0$, $Nf > 0$.
- (2) $N[\alpha f] = |\alpha| Nf$, $N[f \pm g] \leq Nf + Ng$, $N[f \times g] \leq (Nf)(Ng)$.
- (3) $ff'(1) = f'f(1) = (Nf)^2$.
- (4) $|M_x f(x)| \leq Nf$.
- (5) $|f \times g(x)| \leq (Nf)(Ng)$.
- (6) $M_x[|f(x)| |g(x)|] \leq (Nf)(Ng)$.

Statements (1), the first part of (2), and (3) are obvious. The second part of (2), after being squared, means that

$$M_x[|f(x) \pm g(x)|^2] \leq M_x[|f(x)|^2] + M_x[|g(x)|^2] + 2(Nf)(Ng),$$

$$|M_x[\Re(f(x)\overline{g(x)})]| \leq (Nf)(Ng).$$

This obviously follows from (6). The third part of (2) again follows from (5) by squaring and applying M_x . (4) follows from (5) by putting $g(x) \equiv 1$, since $f \times 1(x) = M_y[f(y)]$.

Hence we need to prove only (5) and (6). Since

$$|f(y)g(y^{-1}x)| \leq \frac{1}{2}|f(y)|^2 + \frac{1}{2}|g(y^{-1}x)|^2$$

it follows that

$$|M_y[f(y)g(y^{-1}x)]| \leq \frac{1}{2}M_y[|f(y)|^2] + \frac{1}{2}M_y[|g(y^{-1}x)|^2]$$

$$\leq \frac{1}{2}M_y[|f(y)|^2] + \frac{1}{2}M_y[|g(y)|^2],$$

$$|f \times g(x)| \leq \frac{1}{2}(Nf)^2 + \frac{1}{2}(Ng)^2.$$

Starting from

$$|f(x)| |g(x)| \leq \frac{1}{2}|f(x)|^2 + \frac{1}{2}|g(x)|^2$$

we obtain similarly

$$M_x[|f(x)| |g(x)|] \leq \frac{1}{2}(Nf)^2 + \frac{1}{2}(Ng)^2.$$

If we replace f and g by γf and g/γ (γ real and >0) we see that $|f \times g(x)|$ and $M_x[|f(x)| |g(x)|]$ do not exceed

$$\frac{\gamma^2}{2}(Nf)^2 + \frac{1}{2\gamma^2}(Ng)^2.$$

The greatest lower bound of this expression is $(Nf)(Ng)$. This completes the proof of (5) and (6).

THEOREM 13. Let $f(x)$ be an a.p. function $\neq 0$. Put

$$\begin{aligned}\Gamma_n &= N[f \times f' \times \cdots]^2 = N[f' \times f \times \cdots]^2 \quad (n \text{ factors}) \\ &= (f \times f')^n(1) = (f' \times f)^n(1).\end{aligned}$$

Then

$$\Gamma_n > 0, \quad (\Gamma_n)^2 \leq \Gamma_{n-1}\Gamma_{n+1}, \quad \Gamma_{m+n} \leq \Gamma_m\Gamma_n.$$

First we prove that the four expressions above for Γ_n are equal. Indeed the first and third expressions for Γ_n are equal, and so are the second and the fourth, since $(g \times h)' = h' \times g'$. The equality of the third and fourth expressions follows from

$$(f \times f')^n = f \times (f' \times f \times \cdots \times f'), \quad (f' \times f)^n = (f' \times f \times \cdots \times f') \times f,$$

and from

$$f \times g(1) = g \times f(1) = M_y[f(y)g(y^{-1})].$$

By (5) of Theorem 12, $|f \times g(1)| \leq (Nf)(Ng)$. If we replace here f and g by $f \times f' \times \cdots$ with $n-1$ and $n+1$ factors respectively, we obtain $\Gamma_n^2 \leq \Gamma_{n-1}\Gamma_{n+1}$. And if we replace f and g in (2) of Theorem 12 by $f \times f' \times \cdots$ or $f' \times f \times \cdots$ with m and n factors respectively, we obtain $\Gamma_{m+n} \leq \Gamma_m\Gamma_n$. That $\Gamma_n \geq 0$ is obvious; but the condition $\Gamma_n = 0$ would imply that $\Gamma_{n-1} = 0$ (because $\Gamma_{n-1}^2 = \Gamma_{n-2}\Gamma_n$ provided that $n \geq 3$), so that $\Gamma_2 = 0$. This means that $N[f \times f'] = 0$, $f \times f'(x) \equiv 0$, hence $N[f]^2 = f \times f'(1) = 0$ (that is, $\Gamma_1 = 0$) and $f \equiv 0$, contrary to our assumption. Thus we have $\Gamma_n > 0$.

THEOREM 14. Let $f(x)$ be as before, and define $\Gamma_1, \Gamma_2, \dots$ as before. Then as $n \rightarrow \infty$,

$$\frac{\Gamma_{n+1}}{\Gamma_n} \rightarrow \gamma, \quad \frac{\Gamma_n}{\gamma^n} \rightarrow \kappa, \quad \frac{(f \times f')^n(x)}{\gamma^n} \rightarrow \phi(x) \quad (\text{uniformly}).$$

Furthermore, $0 < \gamma \leq \Gamma_1$, $1 \leq \kappa$, $\phi(x)$ is a.p., and $\phi' = \phi$, $\phi \times \phi = \phi$, $f \times f' \times \phi = \phi \times f \times f' = \gamma\phi$, $\phi(1) = \kappa$.

The formulas of Theorem 13 imply that

$$0 < \frac{\Gamma_2}{\Gamma_1} \leq \frac{\Gamma_3}{\Gamma_2} \leq \cdots \leq \Gamma_1,$$

and therefore Γ_{n+1}/Γ_n has a limit γ as $n \rightarrow \infty$, $0 < \gamma \leq \Gamma_1$. Furthermore,

$$\frac{\Gamma_{n+1}}{\Gamma_n} \leq \gamma, \quad \frac{\Gamma_n}{\gamma^n} \geq \frac{\Gamma_{n+1}}{\gamma^{n+1}},$$

that is,

$$\frac{\Gamma_1}{\gamma} \geq \frac{\Gamma_2}{\gamma^2} \geq \dots > 0,$$

and therefore Γ_n/γ^n has a limit κ as $n \rightarrow \infty$, $\kappa \geq 0$. Finally we have $\Gamma_{m+n} \leq \Gamma_m \Gamma_n$, that is,

$$\Gamma_n \geq \frac{\Gamma_{m+n}}{\Gamma_m} = \frac{\Gamma_{m+1}}{\Gamma_m} \frac{\Gamma_{m+2}}{\Gamma_{m+1}} \dots \frac{\Gamma_{m+n}}{\Gamma_{m+n-1}} \geq \left(\frac{\Gamma_{m+1}}{\Gamma_m} \right)^n.$$

The limiting process $m \rightarrow \infty$ shows that $\Gamma_n \geq \gamma^n$ and $\Gamma_n/\gamma^n \geq 1$, and then the limiting process $n \rightarrow \infty$ shows that $\kappa \geq 1$.

By (4) and (2) of Theorem 12,

$$\begin{aligned} & \left| \frac{(f \times f')^n(x)}{\gamma^n} - \frac{(f \times f')^m(x)}{\gamma^m} \right|^2 = \left| f \times \left(\frac{(f' \times f)^{n-1}}{\gamma^n} - \frac{(f' \times f)^{m-1}}{\gamma^m} \right) \times f'(x) \right|^2 \\ & \leq (Nf)^2 \left(N \left[\frac{(f' \times f)^{n-1}}{\gamma^n} - \frac{(f' \times f)^{m-1}}{\gamma^m} \right] \right)^2 (Nf')^2 \\ & = \Gamma_f^2 \left(\frac{(f' \times f)^{n-1}}{\gamma^n} - \frac{(f' \times f)^{m-1}}{\gamma^m} \right)^2 (1) \\ & = \Gamma_f^2 \left(\frac{(f' \times f)^{2n-2}(1)}{\gamma^{2n}} - 2 \frac{(f' \times f)^{m+n-2}(1)}{\gamma^{m+n}} + \frac{(f' \times f)^{2m-2}(1)}{\gamma^{2m}} \right) \\ & = \Gamma_f^2 \left(\frac{\Gamma_{2n-2}}{\gamma^{2n}} - 2 \frac{\Gamma_{m+n-2}}{\gamma^{m+n}} + \frac{\Gamma_{2m-2}}{\gamma^{2m}} \right). \end{aligned}$$

As m and $n \rightarrow \infty$ the last expression converges to 0; thus the first expression converges uniformly to 0, that is, as $n \rightarrow \infty$, $(f \times f')^n(x)/\gamma^n$ converges uniformly to a limiting function $\phi(x)$. As the functions $(f \times f')^n(x)/\gamma^n$ are a.p., $\phi(x)$ is also a.p.

The relations

$$\begin{aligned} \left(\frac{(f \times f')^n}{\gamma^n} \right)' &= \frac{(f \times f')^n}{\gamma^n}, & \left(\frac{(f \times f')^n}{\gamma^n} \right)^2 &= \frac{(f \times f')^{2n}}{\gamma^{2n}}, \\ f \times f' \times \frac{(f \times f')^n}{\gamma^n} &= \frac{(f \times f')^n}{\gamma^n} \times f \times f' = \gamma \frac{(f \times f')^{n+1}}{\gamma^{n+1}} \end{aligned}$$

show that, when n becomes infinite (the convergences involved all being uni-

form), $\phi' = \phi$, $\phi \times \phi = \phi$, $f \times f' \times \phi = \phi \times f \times f' = \gamma\phi$. Finally,

$$\frac{\Gamma_n}{\gamma^n} = \frac{(f \times f')^n(1)}{\gamma^n} \rightarrow \phi(1)$$

and therefore $\phi(1) = \kappa$.

THEOREM 15. *Let $f(x)$ be as before. Then there is a (finite or infinite) sequence of real numbers $\gamma_1, \gamma_2, \dots$ and a sequence of a.p. functions $\phi_1(x), \phi_2(x), \dots$, all $\neq 0$, such that $\gamma_1 > \gamma_2 > \dots > 0$, $\phi_n' = \phi_n$, $\phi_n \times \phi_n = \phi_n$, $\phi_n(1) \geq 1$, $\phi_m \times \phi_n = 0$ ($m \neq n$), $f \times f' \times \phi_n = \phi_n \times f \times f' = \gamma_n \phi_n$, and $\gamma_1 \phi_1(x) + \gamma_2 \phi_2(x) + \dots$ converges uniformly to $f \times f'(x)$.*

Apply Theorem 14, and put there $\gamma = \gamma_1$, $\phi(x) = \phi_1(x)$. $\phi_1(1) = \kappa \geq 1$ proves that $\phi_1(x) \neq 0$. Now put $f^* = f - \phi_1 \times f$. Then $f^*(x)$ is a.p. and

$$\begin{aligned} f^* \times f^* &= (f - \phi_1 \times f) \times (f' - f' \times \phi_1) \\ &= f \times f' - \phi_1 \times f \times f' - f \times f' \times \phi_1 + \phi_1 \times f \times f' \times \phi_1 \\ &= f \times f' - \gamma_1 \phi_1 - \gamma_1 \phi_1 + \gamma_1 \phi_1 \times \phi_1 = f \times f' - \gamma_1 \phi_1. \end{aligned}$$

If $f^* \equiv 0$, this shows that $f \times f' = \gamma_1 \phi_1$, that is, the Theorem holds for a sequence consisting of one element. Assume that $f^* \neq 0$.

Then $f \times f' \times \phi_1 = \phi_1 \times f \times f' = \gamma_1 \phi_1$ implies that $(f^* \times f^{*'})^n = (f \times f' - \gamma_1 \phi_1)^n = (f \times f')^n - \gamma^n \phi_1$, and thus

$$\frac{(f^* \times f^{*'})^n(x)}{\gamma^n} = \frac{(f \times f')^n(x)}{\gamma^n} - \phi_1(x) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore if we form $\gamma = \gamma_2$ and $\phi_2(x)$ of Theorem 14, for which we have

$$\frac{(f^* \times f^{*'})^n(x)}{\gamma_2^n} \rightarrow \phi_2(x) \neq 0,$$

then it must be the case that $\gamma_1 > \gamma_2$.

By repeating this process with $f^{**} = f^* - \phi_2 \times f^*$, $f^{***} = f^{**} - \phi_3 \times f^{**}$, \dots we finally find a sequence of real numbers $\gamma_1, \gamma_2, \dots$ and two sequences of a.p. functions $\phi_1(x), \phi_2(x), \dots$ and $f(x), f^*(x), \dots$ with the following properties:

$$\begin{aligned} \gamma_1 &> \gamma_2 > \dots > 0, \quad \phi_n' = \phi_n, \quad \phi_n \times \phi_n = \phi_n, \quad \phi_n(1) \geq 1, \\ f^{(n-1)} \times f^{(n-1)'} \times \phi_n &= \phi_n \times f^{(n-1)} \times f^{(n-1)'} = \gamma_n \phi_n, \quad f^{(n)} = f^{(n-1)} - \phi_n \times f^{(n-1)}, \end{aligned}$$

these sequences ending when an $f^{(n)}$ becomes $\equiv 0$, otherwise never ending.

These rules again imply the relations $f^{(n)} \times f^{(n)'} = f^{(n-1)} \times f^{(n-1)'} - \gamma_n \phi_n$. By adding these relations for all $n=1, \dots, p$, we obtain

$$(*) \quad \gamma_1 \phi_1 + \dots + \gamma_p \phi_p = f \times f' - f^{(p)} \times f^{(p)'}$$

We now wish to prove that $\phi_m \times \phi_n = 0$ for $m \neq n$. Application of (') shows that it is sufficient to consider $m > n$, that is, it is sufficient to prove that $\phi_{n+k+1} \times \phi_n = 0$ for $k=0, 1, 2, \dots$. Consider the equation $f^{(n+k)} \times f^{(n+k)'} \times \phi_n = 0$. For $k=0$, the condition

$$f^{(n)} \times f^{(n)'} \times \phi_n = f^{(n-1)} \times f^{(n-1)'} \times \phi_n - \gamma_n \phi_n \times \phi_n = \gamma_n \phi_n - \gamma_n \phi_n = 0$$

obtains. If it holds for a certain $k=0, 1, 2, \dots$, we have

$$\phi_{n+k+1} \times f^{(n+k)} \times f^{(n+k)'} \times \phi_n = \begin{cases} \phi_{n+k+1} \times (f^{(n+k)} \times f^{(n+k)'} \times \phi_n) = 0, \\ (\phi_{n+k+1} \times f^{(n+k)} \times f^{(n+k)'}) \times \phi_n = \gamma_{n+k+1} \phi_{n+k+1} \times \phi_n, \end{cases}$$

and thus $\phi_{n+k+1} \times \phi_n = 0$. This gives

$$f^{(n+k+1)} \times f^{(n+k+1)'} \times \phi_n = (f^{(n+k)} \times f^{(n+k)'} - \gamma_{n+k+1} \phi_{n+k+1}) \times \phi_n = 0,$$

that is, our equation holds for $k+1$. Therefore it holds for every $k=0, 1, 2, \dots$, and with it, its consequence $\phi_{n+k+1} \times \phi_n = 0$.

Application of $\phi_n \times \dots$ or $\dots \times \phi_n$ to (*) with $p=n-1$ gives

$$\phi_n \times f \times f' = \phi_n \times f^{(n-1)} \times f^{(n-1)'} = \gamma_n \phi_n, \quad f \times f' \times \phi_n = f^{(n-1)} \times f^{(n-1)'} \times \phi_n = \gamma_n \phi_n.$$

The only thing remaining to be proved is the uniform convergence of $\gamma_1 \phi_1(x) + \dots + \gamma_n \phi_n(x)$ to $f \times f'(x)$ as $n \rightarrow \infty$, or, according to (*), the uniform convergence of $f^{(n)} \times f^{(n)'}(x)$ to 0. Now (*) implies, by (3) and (5) of Theorem 12, that $\gamma_1 \phi_1(1) + \dots + \gamma_n \phi_n(1) = (Nf)^2 - (Nf^{(n)})^2 \leq (Nf)^2$, and since $|\phi_n \times \phi_n(x)| \leq \phi_n \times \phi_n(1)$, that is, $|\phi_n(x)| \leq \phi_n(1)$, (*) implies the uniform convergence of $\gamma_1 \phi_1(x) + \gamma_2 \phi_2(x) + \dots$ to $g(x)$ as $n \rightarrow \infty$, where $g(x)$ must be a.p. Hence $f^{(n)} \times f^{(n)'}(x) \rightarrow f \times f'(x) - g(x)$ uniformly. Moreover, the above mentioned convergence implies that $\gamma_n \rightarrow 0$ (because $\phi_n(1) \geq 1$). On the other hand, we have $\Gamma_2/\Gamma_1 < \gamma$, $\Gamma_2 < \gamma \Gamma_1$, $(N[f \times f'])^2 < \gamma(Nf)^2$. If we replace f and γ by $f^{(n)}$ and γ_n we obtain $(N[f^{(n)} \times f^{(n)'}])^2 < \gamma_n(Nf^{(n)})^2 \leq \gamma_n(Nf)^2$. Thus

$$N[f^{(n)} \times f^{(n)'}] \rightarrow 0,$$

implying that $N[f \times f' - g] = 0$ and $g = f \times f'$.

9. Having reached the final result of the E. Schmidt-Weyl theory, we now pass to the approximation theorems. But as we have already mentioned, we are now giving only their proofs and shall discuss their real meaning in the next part.

DEFINITION 8. An a.p. function $\phi(x)$ such that $\phi' = \phi$ and $\phi \times \phi = \phi$ is called a unit. Two units $\phi(x)$ and $\psi(x)$ such that $\phi \times \psi = 0$ are called orthogonal.

THEOREM 16†. For every a.p. function $f(x)$ and every $\epsilon > 0$ there exists a unit $\phi(x)$ such that $N[f - \phi \times f] \leq \epsilon$.

If $f \equiv 0$, then $\phi \equiv 0$. Hence we may assume that $f \not\equiv 0$. Then apply Theorem 13. $\psi_n = \phi_1 + \dots + \phi_n$ is a unit (because $\phi_n' = \phi_n$, $\phi_n \times \phi_n = \phi_n$, $\phi_m \times \phi_n = 0$ for $m \neq n$), and we have

$$\begin{aligned} (N[f - \psi_n \times f])^2 &= \left(f - \sum_{r=1}^n \phi_r \times f\right) \times \left(f' - \sum_{r=1}^n f' \times \phi_r\right) (1) \\ &= \left(f \times f - \sum_{r=1}^n \phi_r \times f \times f - \sum_{r=1}^n f \times f' \times \phi_r + \sum_{\mu, r=1}^n \phi_\mu \times f \times f' \times \phi_r\right) (1) \\ &= \left(f \times f - \sum_{r=1}^n \gamma_r \phi_r - \sum_{r=1}^n \gamma_r \phi_r + \sum_{\mu, r=1}^n \gamma_r \phi_\mu \times \phi_r\right) (1) \\ &= \left(f \times f - \sum_{r=1}^n \gamma_r \phi_r\right) (1) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus $\phi = \psi_n$, for a sufficiently large n , yields the desired result.

THEOREM 17. For every a.p. function $f(x)$ and every $\epsilon > 0$ there exists an a.p. function $g(x)$ such that $|f(x) - g \times f(x)| \leq \epsilon$ for every x .

Following N. Wiener, consider the "translation function" of $f(x)$,

$$e(x) = \text{l.u.b.}_y |f(x^{-1}y) - f(y)|,$$

which was introduced by S. Bochner [2]. As $f(x)$ is a.p., it is easily seen that $e(x)$ is also a.p. Furthermore, $e(x) \geq 0$, $e(1) = 0$. Now define the function

$$F(u) = \begin{cases} 1 - \frac{u}{\epsilon}, & 0 \leq u \leq \epsilon, \\ 0, & u \geq \epsilon. \end{cases}$$

As $F(u)$ is continuous, $\phi(x) = F(e(x))$ is a.p. It is obvious that $\phi(x) \geq 0$, $\phi(1) = 1$; and if $\text{l.u.b.}_y |f(x^{-1}y) - f(y)| > \epsilon$, then $\phi(x) = 0$. Thus $\phi(x) \neq 0$ implies that $|f(x^{-1}y) - f(y)| \leq \epsilon$, and therefore, always $|\phi(x)(f(x^{-1}y) - f(y))| \leq \epsilon \phi(x)$. Consequently, $|M_x[\phi(x)(f(x^{-1}y) - f(y))]| \leq \epsilon M_x \phi(x)$. Now $M_x[\phi(x)(f(x^{-1}y) - f(y))] = \phi \times f(y) - f(y)M_x \phi(x)$, and therefore the function

† Cf. Theorems 28 and 29, where the results of Theorems 16 and 18 will be interpreted and applied.

$$g(y) = \frac{\phi(y)}{M_x \phi(x)}$$

meets the requirements.

THEOREM 18. *For every a.p. function $f(x)$ and every $\epsilon > 0$ there exists an a.p. function $g(x)$ and a unit $\phi(x)$ such that $|f(x) - \phi \times g \times f(x)| \leq \epsilon$ for every x .†*

Choose the function $g(x)$ of Theorem 17 corresponding to $f(x)$ and $\epsilon/2$, and choose the function $\phi(x)$ of Theorem 16 corresponding to $g(x)$ and $\epsilon/(2Nf)$. Then $|f(x) - g \times f(x)| \leq \epsilon/2$, $|g \times f(x) - \phi \times g \times f(x)| \leq N[g - \phi \times g]Nf \leq \epsilon/2$, and therefore $|f(x) - \phi \times g \times f(x)| \leq \epsilon$.

III. THEORY OF LINEAR REPRESENTATIONS OF \mathfrak{G}

10. We define the representations in the usual manner:

DEFINITION 9. *If to every $a \in \mathfrak{G}$ there corresponds a matrix*

$$D(a) = \{D_{\rho\sigma}(a)\} \quad (\rho, \sigma = 1, \dots, s)$$

of degree s such that $D(1) = 1$, (the unit matrix of degree s), $D(ab) = D(a) \cdot D(b)$, then we call $D(a)$ a representation of \mathfrak{G} . (No continuity is assumed.) Two representations $D(a)$ and $D'(a)$ are called equivalent if they are of the same degree s , and if a fixed matrix $U = \{U_{\rho\sigma}\}$, $\rho, \sigma = 1, \dots, s$, exists which transforms one representation into the other: $U^{-1}D(a)U = D'(a)$. A representation $D(a)$ is called reducible (completely reducible) if it is equivalent to a representation $D'(a)$ such that $D'_{\rho\sigma}(a) = 0$ identically (in a) whenever $\rho \leq t$, $\sigma > t$ ($\rho \leq t$, $\sigma > t$ or $\rho > t$, $\sigma \leq t$)‡, for a fixed value of t , $1 \leq t \leq s-1$. Representations without these properties are irreducible (completely irreducible).

For finite groups \mathfrak{G} Frobenius and Schur gave a complete theory of all representations [21, 22]; for continuous groups \mathfrak{G} close analogues of their results were established by Schur for the rotation group in three dimensions, and in much broader generality by Weyl for all compact Lie-groups [30]. These results were extended to all compact groups \mathfrak{G} by Haar [11, pp. 166-169] with the help of his notion of "right-invariant" Lebesgue measure in groups. We shall push the extension further to all groups \mathfrak{G} , but in order to do this it is natural and necessary to restrict the domain of representations of \mathfrak{G} by means of

† Cf. footnote to Theorem 16.

‡ These are the fundamental notions of the Frobenius-Schur theory of group representations [21, 22, 30].

THEOREM 19. *The following conditions on a representation $D(a)$ of \mathfrak{G} are equivalent to each other:*

- A. $D(a)$ is equivalent to a unitary[†] representation.
- B. All elements $D_{\rho\sigma}(a)$ of $D(a)$ are bounded.
- C. All elements $D_{\rho\sigma}(a)$ of $D(a)$ are a.p.

If $D'(a)$ is unitary, then, by the footnote just cited in the case where $\rho = \sigma$,

$$\sum_{\tau=1}^s |D_{\rho\tau}'(a)|^2 = 1, \quad |D_{\rho\tau}'(a)| \leq 1,$$

and all $D_{\rho\tau}'(a)$ are bounded. Therefore the elements $D_{\rho\tau}(a)$ of any $D(a)$ equivalent to $D'(a)$ must also be bounded. Thus A implies B.

If all $D_{\rho\sigma}(a)$ are bounded, every sequence $D_{\rho\sigma}(a_n)$, $n=1, 2, \dots$, contains a subsequence which converges for all $\rho, \sigma=1, \dots, s$. And then, since $D(xa_n) = D(x)D(a_n)$ and $D(a_nx) = D(a_n)D(x)$, the representations $D(xa_n)$ and $D(a_nx)$, and hence all $D_{\rho\sigma}(xa_n)$ and $D_{\rho\sigma}(a_nx)$, converge uniformly. Thus all $D_{\rho\sigma}(a)$ are a.p., that is, B implies C.

If all $D_{\rho\sigma}(a)$ are a.p., so are the expressions $\sum_{\tau=1}^s D_{\rho\tau}(a)\overline{D_{\sigma\tau}(a)}$, and we can form

$$A_{\rho\sigma} = M_s \left[\sum_{\tau=1}^s D_{\rho\tau}(x)\overline{D_{\sigma\tau}(x)} \right].$$

Now

$$\sum_{\tau=1}^s D_{\rho\tau}(a)\overline{D_{\sigma\tau}(a)} = \overline{\sum_{\tau=1}^s D_{\sigma\tau}(a)\overline{D_{\rho\tau}(a)}}$$

and for every system ξ_1, \dots, ξ_s which is not identically zero,

$$\sum_{\rho, \sigma=1}^s \left[\sum_{\tau=1}^s D_{\rho\tau}(a)\overline{D_{\sigma\tau}(a)} \right] \xi_\rho \bar{\xi}_\sigma = \sum_{\tau=1}^s \left| \sum_{\rho=1}^s D_{\rho\tau}(a)\xi_\rho \right|^2 > 0,$$

so that

$$A_{\rho\sigma} = \overline{A_{\sigma\rho}} \quad \text{and} \quad \sum_{\rho, \sigma=1}^s A_{\rho\sigma} \xi_\rho \bar{\xi}_\sigma > 0.$$

Therefore the matrix $A = \{A_{\rho\sigma}\}$ is Hermitian and positive definite. Hence

[†] That is, to a representation $D'(a)$ in which all matrices $D'(a) = \{D'_{\rho\sigma}(a)\}$ ($\rho, \sigma=1, \dots, s$) are unitary. A matrix $U = \{U_{\rho\sigma}\}$ is called unitary if its adjoint $U^* = \{\bar{U}_{\sigma\rho}\}$ is reciprocal to it, that is, if $UU^* = U^*U = 1$, or more explicitly,

$$\sum_{\tau=1}^s U_{\rho\tau} \bar{U}_{\sigma\tau} = \sum_{\tau=1}^s U_{\tau\rho} \bar{U}_{\tau\sigma} = \delta_{\rho\sigma} = \begin{cases} 1, & \rho = \sigma, \\ 0, & \rho \neq \sigma. \end{cases}$$

there exists a matrix $X = \{X_{\rho\sigma}\}$ such that $A = XX^*$. On the other hand,

$$\begin{aligned} A_{\rho\sigma} &= M_x \left[\sum_{\tau=1}^s D_{\rho\tau}(x) \overline{D_{\sigma\tau}(x)} \right] = M_x \left[\sum_{\tau=1}^s D_{\rho\tau}(ax) \overline{D_{\sigma\tau}(ax)} \right] \\ &= M_x \left[\sum_{\rho', \sigma'=1}^s \sum_{\tau=1}^s D_{\rho\rho'}(a) D_{\rho'\tau}(x) \overline{D_{\sigma\sigma'}(a)} \overline{D_{\sigma'\tau}(x)} \right] \\ &= \sum_{\rho', \sigma'=1}^s D_{\rho\rho'}(a) \overline{D_{\sigma\sigma'}(a)} M_x \left[\sum_{\tau=1}^s D_{\rho'\tau}(x) \overline{D_{\sigma'\tau}(x)} \right] \\ &= \sum_{\rho', \sigma'=1}^s D_{\rho\rho'}(a) \overline{D_{\sigma\sigma'}(a)} A_{\rho'\sigma'}, \end{aligned}$$

that is, $A = D(a)AD(a)^*$, or $XX^* = D(a)XX^*D(a)^*$, $X^{-1}D(a)XX^*D(a)^*X^{*-1} = 1$, $(X^{-1}D(a)X)(X^{-1}D(a)X)^* = 1$. In other words, the equivalent representation $X^{-1}D(a)X = D'(a)$ is unitary. Thus C implies A.

Our three statements together prove the equivalence of A, B, and C.

DEFINITION 10. We call normal the representations satisfying one of the equivalent conditions of Theorem 19.

11. The fundamental theorems of the theory of orthogonal representations may now be proved in the classical way [21, 22, 30, 32].

THEOREM 20. Let $D(a)$ and $E(a)$ be completely irreducible normal representations of degrees s and t respectively, and let A be a rectangular matrix with s rows and t columns. If $D(a)A \equiv AE(a)$ for every a , then either $A = 0$ or $s = t$ and $\det A \neq 0$, the latter alternative of course implying the equivalence of $D(a)$ and $E(a)$. If $D(a) = E(a)$, then $A = \alpha 1$ (α being a complex number).

In all these statements (except the last) $D(a)$ and $E(a)$ may be replaced by two equivalent representations. Therefore we may assume them to be unitary. Even then further transformations by unitary matrices X and Y are possible. They carry A into $A' = X^{-1}AY$. Now by such transformations we can obtain $A' = \{A'_{\rho\sigma}\}$, $\rho = 1, \dots, s$; $\sigma = 1, \dots, t$, such that

$$A'_{\rho\sigma} = \begin{cases} c_\rho, & \text{for } \rho = \sigma = 1, \dots, r, \text{ all } c_\rho > 0, \\ 0, & \text{for all other } \rho \text{ and } \sigma, \end{cases}$$

where r is the rank of A' and $r \leq s, r \leq t$. Therefore we may assume that A itself has this form.

Under these conditions the relation $D(a)A = AE(a)$ implies that $D_{\rho\sigma}(a) = 0$ for $\rho > r$ and $\sigma \leq r$, and that $E_{\rho\sigma}(a) = 0$ for $\rho \leq r$ and $\sigma > r$. Since A^* also has the form we assumed for A , $D(a)^* = D(a)^{-1} = D(a^{-1})$, $E(a)^* = E(a)^{-1} = E(a^{-1})$,

we get, by applying $*$ and replacing a by a^{-1} , $A^*D(a)=E(a)A^*$, so that $D_{\rho\sigma}(a)=0$ for $\rho \leq r$ and $\sigma > r$, and $E_{\rho\sigma}(a)=0$ for $\rho > r$ and $\sigma \leq r$. Thus the complete irreducibility of $D(a)$ requires that r be 0 or s , and the complete irreducibility of $E(a)$ requires that r be 0 or t . Hence either $r=0$, in which case $A=0$, or $r=s=t$, in which case $\det A \neq 0$.

If $D(a)=E(a)$, every $A-\alpha 1$ has the same property as A . If α is a root of the characteristic equation of A , we have $\det [A-\alpha 1]=0$, so that our alternative requires that $A-\alpha 1=0$, and $A=\alpha 1$.

THEOREM 21. *Let $D(a)$ and $E(a)$ be completely irreducible normal representations of degrees s and t respectively. If they are inequivalent, we have*

$$D_{\rho\sigma} \times E_{\tau\nu}(x) \equiv 0.$$

Considering $D(a)$ alone, we have

$$D_{\rho\sigma} \times D_{\tau\nu}(x) \equiv \begin{cases} \frac{1}{s} D_{\rho\nu}(x) & \text{for } \sigma = \tau, \\ 0 & \text{for } \sigma \neq \tau. \end{cases}$$

Form the (rectangular) matrix

$$A = \{A_{\tau\sigma}\}, \quad A_{\tau\sigma} = D_{\rho\sigma} \times E_{\tau\nu}(x)$$

for a given choice of ρ , ν , and x . Then

$$\begin{aligned} \sum_{\tau'=1}^t E_{\tau\tau'}(a)A_{\tau'\sigma} &= M_y \left[\sum_{\tau'=1}^t D_{\rho\sigma}(xy^{-1})E_{\tau\tau'}(a)E_{\tau'\nu}(y) \right] \\ &= M_y [D_{\rho\sigma}(xy^{-1})E_{\tau\nu}(ay)] = M_z [D_{\rho\sigma}(z)E_{\tau\nu}(az^{-1}x)], \\ \sum_{\sigma'=1}^s A_{\tau\sigma'}D_{\sigma'\sigma}(a) &= M_y \left[\sum_{\sigma'=1}^s D_{\rho\sigma'}(y)D_{\sigma'\sigma}(a)E_{\tau\nu}(y^{-1}x) \right] \\ &= M_y [D_{\rho\sigma}(ya)E_{\tau\nu}(y^{-1}x)] = M_z [D_{\rho\sigma}(z)E_{\tau\nu}(az^{-1}x)] \end{aligned}$$

(the variable y being replaced by $z=xy^{-1}$ and $z=ya$ respectively), and therefore

$$E(a)A = AD(a).$$

Thus we can apply Theorem 20. If $D(a)$ and $E(a)$ are inequivalent, it results that $A=0$, and if $D(a)=E(a)$, $A=\alpha_{\rho\nu}(x)1_s$ ($\alpha_{\rho\nu}(x)$ being a complex number). This implies (1) that $A_{\tau\sigma}=0$, that is, $D_{\rho\sigma} \times E_{\tau\nu}(x)=0$ if $D(a)$ and $E(a)$ are inequivalent or if $D(a)=E(a)$ and $\sigma \neq \tau$; and (2) that $A_{\tau\sigma}=\alpha_{\rho\nu}(x)$, which is independent of σ , if $D(a)=E(a)$ and $\sigma=\tau$. Hence all the statements of our Theorem are proved if we show that $\alpha_{\rho\nu}(x)=(1/s)D_{\rho\nu}(x)$. This follows immediately from

$$\begin{aligned} s\alpha_{\rho\nu}(x) &= \sum_{\sigma=1}^s A_{\sigma\sigma} = M_y \left[\sum_{\sigma=1}^s D_{\rho\sigma}(xy^{-1})D_{\sigma\nu}(y) \right] \\ &= M_y D_{\rho\nu}(xy^{-1}y) = M_y D_{\rho\nu}(x) = D_{\rho\nu}(x). \end{aligned}$$

THEOREM 22. *For normal representations reducibility and complete reducibility are equivalent, so that irreducibility and complete irreducibility are also equivalent.*

That complete reducibility implies reducibility is obvious. Assume now that $D(a)$ is reducible without being completely reducible. As we can replace $D(a)$ by any equivalent representation, we may assume $D(a)$ to be in the form described in Definition 9. Then there would be a pair of indices ρ and σ such that $D_{\rho\sigma}(x) \equiv 0$. By Theorem 21, this relation implies that $D_{\rho\sigma} \times D_{\sigma\rho} \equiv D_{\rho\rho} \equiv 0$, in spite of the fact that $D_{\rho\rho}(1) = 1$. Thus $D(a)$ must be completely reducible.

12. Our next task is to formulate the connection between the units of Definition 8 and representations. This is accomplished by means of

THEOREM 23. *For every unit $\phi(x)$ there exist a number of inequivalent irreducible unitary representations $D^{(1)}(a), \dots, D^{(u)}(a)$ of degrees s_1, \dots, s_u respectively such that*

$$\phi(x) = \sum_{\omega=1}^u s_{\omega} \sum_{\rho, \sigma=1}^{s_{\omega}} \alpha_{\rho\sigma}^{(\omega)} D_{\rho\sigma}^{(\omega)}(x).$$

Here every matrix $\alpha^{(\omega)} = \{\alpha_{\rho\sigma}^{(\omega)}\}$ is idempotent, that is, $\alpha^{(\omega)*} = \alpha^{(\omega)}$ (cf. footnote on page 465), $(\alpha^{(\omega)})^2 = \alpha^{(\omega)}$.† Conversely, every $\phi(x)$ which is formed in this way (where $D^{(\omega)}(a)$ and $\alpha^{(\omega)}$ satisfy our conditions) is a unit.

By a suitable choice of $D^{(\omega)}(a)$ we can give the matrices $\alpha^{(\omega)}$ the form

$$\alpha_{\rho\sigma}^{(\omega)} = \begin{cases} 1 & \text{for } \rho = \sigma \leq s_{\omega}', \text{ with some } s_{\omega}' = 1, \dots, s_{\omega}, \\ 0 & \text{for all other } \rho \text{ and } \sigma. \end{cases}$$

Consider the (a.p.) solutions $f(x)$ of the equation $\phi \times f = f$. Assume that it is possible to find s solutions g_1, \dots, g_s among them which satisfy the conditions

$$g_{\mu} \times g_{\nu}'(1) = M_x [g_{\mu}(x) \overline{g_{\nu}(x)}] = \begin{cases} 1 & \text{for } \mu = \nu, \\ 0 & \text{for } \mu \neq \nu. \end{cases}$$

Put

† That is,

$$\alpha_{\rho\sigma}^{(\omega)} = \overline{\alpha_{\sigma\rho}^{(\omega)}}, \quad \sum_{\sigma=1}^{s_{\omega}} \alpha_{\rho\sigma}^{(\omega)} \alpha_{\sigma\tau}^{(\omega)} = \alpha_{\rho\tau}^{(\omega)}.$$

$$\psi(x, y) = \phi(xy^{-1}) - \sum_{\mu=1}^s g_{\mu}(x) \overline{g_{\mu}(y)}.$$

Then

$$\begin{aligned} M_{x,y} [|\psi(x, y)|^2] &= M_{x,y} [|\phi(xy^{-1})|^2] - \sum_{\mu=1}^s M_{x,y} [\phi(xy^{-1}) \overline{g_{\mu}(x)} g_{\mu}(y)] \\ &\quad - \sum_{\mu=1}^s M_{x,y} [\overline{\phi(xy^{-1})} g_{\mu}(x) \overline{g_{\mu}(y)}] + \sum_{\mu, \nu=1}^s M_{x,y} [g_{\mu}(x) \overline{g_{\mu}(y)} \overline{g_{\nu}(x)} g_{\nu}(y)]. \end{aligned}$$

By Theorems 9 and 10, the orthogonality properties of g_{μ} , and the relations $\phi \times g_{\mu} = g_{\mu}$, $g'_{\mu} \times \phi = g'_{\mu}$, this turns out to be equal to

$$M_x [|\phi(x)|^2] - s - s + s = (N\phi)^2 - s.$$

Therefore we have $(N\phi)^2 - s \geq 0$, so that $s \leq (N\phi)^2$. Thus the possible numbers s are bounded, and they have a maximal value. Assume that s is this maximal value and choose g_1, \dots, g_s accordingly. If a solution $f(x)$ of $\phi \times f = f$ is such that $f \times g'_{\mu} = 0$ for all $\mu = 1, \dots, s$, then necessarily $f \equiv 0$, for otherwise we could put $g_{s+1} = f/(Nf)$, implying that $Ng_{s+1} = 1$, that is, $g_{s+1} \times g'_{s+1}(1) = 1$, and $g_{s+1} \times g'_{\mu}(1) = 0$ as well as $g_{\mu} \times g'_{s+1}(1) = 0$ for $\mu = 1, \dots, s$, so that

$$g_{\mu} \times g'_{\nu}(1) = \begin{cases} 1 & \text{for } \mu = \nu, \\ 0 & \text{for } \mu \neq \nu, \end{cases} \quad \mu, \nu = 1, \dots, s+1,$$

contradicting our assumption that s is maximal.

If we define $f(x)$ to be $\overline{\psi(x, a)}$, we find by a simple computation that $\phi \times f = f$ and $f \times g'_{\mu}(1) = 0$. Therefore $f(x) \equiv 0$, $\psi(x, a) \equiv 0$, and, as a was arbitrary, $\psi(x, y) \equiv 0$, that is,

$$(\dagger) \quad \phi(xy^{-1}) \equiv \sum_{\mu=1}^s g_{\mu}(x) \overline{g_{\mu}(y)}.$$

As (\dagger) implies that

$$\phi \times f(x) = M_y [\phi(xy^{-1})f(y)] = \sum_{\mu=1}^s M_y [\overline{g_{\mu}(y)}f(y)]g_{\mu}(x),$$

every solution of $\phi \times f = f$ has the form $\sum_{\mu=1}^s \alpha_{\mu} g_{\mu}$ (α_{μ} being complex numbers). It is obvious that $f(xa)$ is a solution along with $f(x)$, so that $g_{\mu}(xa)$ is a solution, and we can write

$$g_{\sigma}(xa) = \sum_{\rho=1}^s D_{\rho\sigma}(a)g_{\rho}(x).$$

The orthogonality properties of g_μ determine the coefficients $D_{\rho\sigma}(a)$ uniquely:

$$D_{\rho\sigma}(a) = M_x[g_\sigma(xa)\overline{g_\rho(x)}] = g'_\rho \times g_\sigma(a).$$

Hence $D(1) = 1_s$, and if we put $D(a) = \{D_{\rho\sigma}(a)\}$, $\rho, \sigma = 1, \dots, s$, we obviously have $D(ab) = D(a)D(b)$, that is, $D(a)$ is a representation. As (§) holds if we replace x and y by xa and ya , that is, if we replace $g_\mu(x)$ and $g_\mu(y)$ by $g_\mu(xa)$ and $g_\mu(ya)$, the transformations $D(a)$ must be unitary.

All this implies that

$$\phi(x) = \sum_{\sigma=1}^s g_\sigma(x)\overline{g_\sigma(1)} = \sum_{\rho, \sigma=1}^s D_{\rho\sigma}(x)g_\rho(1)\overline{g_\sigma(1)},$$

so that $\phi(x)$ is a linear aggregate of all $D_{\rho\sigma}(x)$. Now (cf. Theorem 22) $D(a)$ can be transformed into an equivalent $D'(a)$ which consists of a certain number of irreducible representations $D^{(1)}(a), \dots, D^{(v)}(a)$ (of degrees s_1, \dots, s_v respectively, where $s_1 + \dots + s_v = s$) which succeed each other along the main diagonal, and zeros in all other places. Therefore $\phi(x)$ is also a linear aggregate of the elements $D_{\rho\sigma}'(x)$, that is, of the elements $D_{\rho\sigma}^{(\omega)}(x)$. Now if some representation $D^{(\omega)}(x)$ is equivalent to another representation $D^{(\nu)}(x)$, the elements $D_{\rho\sigma}^{(\omega)}(x)$ are linear aggregates of elements of $D^{(\nu)}(x)$. Therefore, in expressing $\phi(x)$ as a linear aggregate of all elements $D_{\rho\sigma}^{(\omega)}(x)$, it is sufficient to keep only one member of each class of equivalent representations $D^{(\omega)}(x)$. Those representations thus kept may be labeled $D^{(1)}(x), \dots, D^{(u)}(x)$, $u \leq v$. So we finally have the result that $D^{(1)}(a), \dots, D^{(u)}(a)$ (of degrees s_1, \dots, s_u respectively) is a set of inequivalent irreducible unitary representations, and $\phi(x)$ is a linear aggregate of the elements $D_{\rho\sigma}^{(\omega)}(x)$, that is, we can write $\phi(x)$ in the form

$$\phi(x) = \sum_{\omega=1}^u s_\omega \sum_{\rho, \sigma=1}^{s_\omega} \alpha_{\rho\sigma}^{(\omega)} D_{\rho\sigma}^{(\omega)}(x).$$

If by means of this equation we now determine the meaning of $\phi' = \phi$ and $\phi \times \phi = \phi$, remembering that

$$\begin{aligned} (D^{(\omega)}(x))^* &= D^{(\omega)}(x)^{-1} = D^{(\omega)}(x^{-1}), \\ \overline{D_{\sigma\rho}^{(\omega)}(x)} &= D_{\rho\sigma}^{(\omega)}(x^{-1}), \quad \overline{D_{\rho\sigma}^{(\omega)}(x^{-1})} = D_{\sigma\rho}^{(\omega)}(x), \end{aligned}$$

that is, $D_{\sigma\rho}^{(\omega)'}(x) = D_{\rho\sigma}^{(\omega)}(x)$, and

$$D_{\rho\sigma}^{(\omega)} \times D_{\tau\nu}^{(\chi)} = \begin{cases} \frac{1}{s_\omega} D_{\rho\nu}^{(\omega)} & \text{if } \omega = \chi \text{ and } \sigma = \tau, \\ 0 & \text{for all other } \omega, \chi, \rho, \sigma, \tau, \nu, \end{cases}$$

we obtain exactly the conditions in our Theorem. Furthermore it is clear that every matrix $\alpha^{(\omega)} = \{\alpha_{\rho\sigma}^{(\omega)}\}$, being idempotent, can be transformed into the form given at the end of our Theorem. And the inverse transformations of the representations $D^{(\omega)}(x)$, which carry them into equivalent representations, bring about just these $\alpha^{(\omega)}$ -transformations.

13. We choose a system of "representants" for the inequivalent irreducible (normal or orthogonal) representations of \mathfrak{G} :

DEFINITION 11. Let I be the set of all irreducible normal representations of \mathfrak{G} . Call each subset \mathfrak{C} of I which consists of all the elements of I equivalent to one of its elements a class. It is obvious that every element of I belongs to exactly one class. Call the set of all classes C . Each element \mathfrak{C} of C contains unitary representations (since every normal representation is equivalent to a unitary representation). Select one unitary representation from each \mathfrak{C} of C , call it the representant of \mathfrak{C} , denote it by $D(a; \mathfrak{C})^\dagger$, and denote its degree by $s(\mathfrak{C})$.

THEOREM 24. The (a.p.) functions $D_{\rho\sigma}(x; \mathfrak{C})$, \mathfrak{C} in C , ρ and $\sigma = 1, \dots, s(\mathfrak{C})$, have the property that

$$D_{\rho\sigma}(\mathfrak{C}) \times D_{\tau\nu}(\mathfrak{D}) = \begin{cases} \frac{1}{s(\mathfrak{C})} D_{\rho\nu}(\mathfrak{C}) & \text{for } \mathfrak{C} = \mathfrak{D} \text{ and } \sigma = \tau, \\ 0 & \text{for all other } \mathfrak{C}, \mathfrak{D}, \rho, \sigma, \tau, \nu. \end{cases}$$

This implies that

$$M_z[D_{\rho\sigma}(x; \mathfrak{C}) \overline{D_{\tau\nu}(x; \mathfrak{D})}] = \begin{cases} \frac{1}{s(\mathfrak{C})} & \text{for } \mathfrak{C} = \mathfrak{D}, \rho = \nu, \sigma = \tau, \\ 0 & \text{for all other } \mathfrak{C}, \mathfrak{D}, \rho, \sigma, \tau, \nu. \end{cases}$$

The first formula has been proved in Theorem 21. The second formula follows from it if we put the variable equal to 1 and remember that $D_{\tau\nu}(\mathfrak{D})' = \overline{D_{\nu\tau}(\mathfrak{D})}$, $D(1, \mathfrak{D}) = 1$.

Thus the functions $s(\mathfrak{C})^{1/2} D_{\rho\sigma}(x; \mathfrak{C})$ form a "normalized orthogonal" system. This is the basis for the formulation of the usual expansion theorems. The key theorems of the theory will be proved as Theorems 28 and 29.

[†] This does not imply an essential use of the "axiom of choice" because it would in most cases be possible to characterize a $D(a; \mathfrak{C})$ in \mathfrak{C} in a unique way. To abbreviate we shall also use the notation $D(\mathfrak{C})$ omitting the argument a .

DEFINITION 12. If $f(x)$ is an a.p. function, the complex numbers

$$\tilde{\alpha}_{\rho\sigma}(\mathbb{E}) = f \times D_{\rho\sigma}(1; \mathbb{E})' = M_z[f(x)\overline{D_{\rho\sigma}(x; \mathbb{E})}]$$

are called its expansion coefficients. The matrices $\tilde{\alpha}(\mathbb{E}) = \{\tilde{\alpha}_{\rho\sigma}(\mathbb{E})\}$ are called the expansion matrices.

THEOREM 25. If f and g have the expansion matrices $\tilde{\alpha}(\mathbb{E})$ and $\tilde{\beta}(\mathbb{E})$ (\mathbb{E} running over C), $f \pm g$, θf (θ any complex number), f' and $f \times g$ have the expansion matrices $\tilde{\alpha}(\mathbb{E}) \pm \tilde{\beta}(\mathbb{E})$, $\theta \tilde{\alpha}(\mathbb{E})$, $\tilde{\alpha}(\mathbb{E})^*$, and $\tilde{\alpha}(\mathbb{E})\tilde{\beta}(\mathbb{E})$ respectively. A unit ϕ is characterized by the fact that all its expansion matrices are idempotent, so that only a finite number of them can be $\neq 0$.

The statements concerning $f \pm g$ and θf are obvious; as to f' it is sufficient to remark that

$$\begin{aligned} M_z[f'(x)\overline{D_{\rho\sigma}(x; \mathbb{E})}] &= M_z[\overline{f(x^{-1})}D_{\rho\sigma}(x^{-1}; \mathbb{E})'] \\ &= M_z[\overline{f(x)}\overline{D_{\rho\sigma}(x; \mathbb{E})}'] = \overline{M_z[f(x)\overline{D_{\rho\sigma}(x; \mathbb{E})}]} = \overline{\tilde{\alpha}_{\rho\sigma}(\mathbb{E})}. \end{aligned}$$

The following computation proves our statement with regard to $f \times g$ (cf. Theorems 7 and 8):

$$\begin{aligned} M_z[f \times g(x)\overline{D_{\rho\sigma}(x; \mathbb{E})}] &= M_{z,y}[f(y)g(y^{-1}x)\overline{D_{\rho\sigma}(x; \mathbb{E})}] \\ &= M_{y,z}[f(y)g(z)\overline{D_{\rho\sigma}(yz; \mathbb{E})}] \\ &= \sum_{\tau=1}^{s(\mathbb{E})} M_{y,z}[f(y)g(z)\overline{D_{\rho\tau}(y; \mathbb{E})}\overline{D_{\tau\sigma}(z; \mathbb{E})}] \\ &= \sum_{\tau=1}^{s(\mathbb{E})} M_y[f(y)\overline{D_{\rho\tau}(y; \mathbb{E})}]M_z[g(z)\overline{D_{\tau\sigma}(z; \mathbb{E})}] \\ &= \sum_{\tau=1}^{s(\mathbb{E})} \tilde{\alpha}_{\rho\tau}(\mathbb{E})\tilde{\beta}_{\tau\sigma}(\mathbb{E}). \end{aligned}$$

This discussion shows that the idempotence of all expansion matrices $\tilde{\alpha}(\mathbb{E})$ is characteristic of units ϕ , but it follows from Theorem 21 that all those matrices which have a $D(a; \mathbb{E})$ not identical to a $D^{(\omega)}(a)$ for some $\omega = 1, \dots, u$ must vanish, so that only a finite number are different from zero. (We here use the fact that if two functions have the same expansion matrices, they coincide; that is, if all expansion matrices of a function f vanish, then $f(x) \equiv 0$. This follows from Theorem 28 (the proof of which does not depend upon Theorem 25) by putting $f(x) \equiv g(x)$.)

THEOREM 26. If f has the expansion matrices $\tilde{\alpha}(\mathbb{E}) = \{\tilde{\alpha}_{\rho\sigma}(\mathbb{E})\}$ and if $\mathbb{E}_1, \dots, \mathbb{E}_n$ are elements of C , then of all linear aggregates g of the elements $D_{\rho\sigma}(\mathbb{E}_\omega)$, $\omega = 1, \dots, n$; $\rho, \sigma = 1, \dots, s(\mathbb{E}_\omega)$, which can be written in the form

$$g(x) = \sum_{\omega=1}^n s(\mathbb{G}_{\omega}) \sum_{\rho, \sigma=1}^{s(\mathbb{G}_{\omega})} \alpha_{\rho\sigma}(\mathbb{G}_{\omega}) D_{\rho\sigma}(x; \mathbb{G}_{\omega})$$

that one which minimizes the expression $(N[f-g])^2$ is characterized by the property that $\alpha_{\rho\sigma}(\mathbb{G}_{\omega}) = \tilde{\alpha}_{\rho\sigma}(\mathbb{G}_{\omega})$. The minimum value in question is

$$(Nf)^2 - \sum_{\omega=1}^n s(\mathbb{G}_{\omega}) \sum_{\rho, \sigma=1}^{s(\mathbb{G}_{\omega})} |\tilde{\alpha}_{\rho\sigma}(\mathbb{G}_{\omega})|^2.$$

The proof is contained in the well known computation

$$\begin{aligned} (N[f-g])^2 &= M_z[|f(x) - g(x)|^2] \\ &= M_z[|f(x)|^2] - M_z[f(x)\overline{g(x)}] - M_z[\overline{f(x)}g(x)] + M_z[|g(x)|^2] \\ &= (Nf)^2 - \sum_{\omega=1}^n s(\mathbb{G}_{\omega}) \sum_{\rho, \sigma=1}^{s(\mathbb{G}_{\omega})} \overline{\alpha_{\rho\sigma}(\mathbb{G}_{\omega})} M_z[f(x)\overline{D_{\rho\sigma}(x; \mathbb{G}_{\omega})}] \\ &\quad - \sum_{\omega=1}^n s(\mathbb{G}_{\omega}) \sum_{\rho, \sigma=1}^{s(\mathbb{G}_{\omega})} \alpha_{\rho\sigma}(\mathbb{G}_{\omega}) M_z[\overline{f(x)}D_{\rho\sigma}(x; \mathbb{G}_{\omega})] \\ &\quad + \sum_{\omega, \chi=1}^n s(\mathbb{G}_{\omega})s(\mathbb{G}_{\chi}) \sum_{\rho, \sigma=1}^{s(\mathbb{G}_{\omega})} \sum_{\tau, \nu=1}^{s(\mathbb{G}_{\chi})} \overline{\alpha_{\rho\sigma}(\mathbb{G}_{\omega})} \alpha_{\tau\nu}(\mathbb{G}_{\chi}) M_z[D_{\rho\sigma}(x; \mathbb{G}_{\omega})\overline{D_{\tau\nu}(x; \mathbb{G}_{\chi})}] \\ &= (Nf)^2 - \sum_{\omega=1}^n s(\mathbb{G}_{\omega}) \sum_{\rho, \sigma=1}^{s(\mathbb{G}_{\omega})} \overline{\tilde{\alpha}_{\rho\sigma}(\mathbb{G}_{\omega})} \alpha_{\rho\sigma}(\mathbb{G}_{\omega}) \\ &\quad - \sum_{\omega=1}^n s(\mathbb{G}_{\omega}) \sum_{\rho, \sigma=1}^{s(\mathbb{G}_{\omega})} \overline{\tilde{\alpha}_{\rho\sigma}(\mathbb{G}_{\omega})} \alpha_{\rho\sigma}(\mathbb{G}_{\omega}) + \sum_{\omega=1}^n s(\mathbb{G}_{\omega}) \sum_{\rho, \sigma=1}^{s(\mathbb{G}_{\omega})} |\alpha_{\rho\sigma}(\mathbb{G}_{\omega})|^2 \\ &= \left\{ (Nf)^2 - \sum_{\omega=1}^n s(\mathbb{G}_{\omega}) \sum_{\rho, \sigma=1}^{s(\mathbb{G}_{\omega})} |\tilde{\alpha}_{\rho\sigma}(\mathbb{G}_{\omega})|^2 \right\} \\ &\quad + \sum_{\omega=1}^n s(\mathbb{G}_{\omega}) \sum_{\rho, \sigma=1}^{s(\mathbb{G}_{\omega})} |\alpha_{\rho\sigma}(\mathbb{G}_{\omega}) - \tilde{\alpha}_{\rho\sigma}(\mathbb{G}_{\omega})|^2. \end{aligned}$$

THEOREM 27. (Bessel's inequality.) If f has the expansion matrices $(\tilde{\alpha}\mathbb{G}) = \{\tilde{\alpha}_{\rho\sigma}(\mathbb{G})\}$, the number of those \mathbb{G} for which $\tilde{\alpha}(\mathbb{G}) \neq 0$ is at most countably infinite, that is, it is possible to arrange them in a finite or infinite sequence $\mathbb{G}_1, \mathbb{G}_2, \dots$. Then we have

$$(Nf)^2 \geq \sum_{\omega} s(\mathbb{G}_{\omega}) \sum_{\rho, \sigma=1}^{s(\mathbb{G}_{\omega})} |\tilde{\alpha}_{\rho\sigma}(\mathbb{G}_{\omega})|^2.$$

Since, for all other \mathbb{G} 's, $\tilde{\alpha}_{\rho\sigma}(\mathbb{G}) = 0$, we can instead write

$$(Nf)^2 \geq \sum_{\mathbb{G}} s(\mathbb{G}) \sum_{\rho, \sigma=1}^{s(\mathbb{G})} |\tilde{\alpha}_{\rho\sigma}(\mathbb{G})|^2.$$

The number of \mathfrak{C} 's for which

$$s(\mathfrak{C}) \sum_{\rho, \sigma=1}^{s(\mathfrak{C})} |\tilde{\alpha}_{\rho\sigma}(\mathfrak{C})|^2 \geq \epsilon, \quad \epsilon > 0,$$

is, because of the last statement of Theorem 26, certainly finite and $\leq (Nf)^2/\epsilon$. Putting $\epsilon=1, \frac{1}{2}, \frac{1}{3}, \dots$ successively, we see that, with at most countably infinitely many exceptional \mathfrak{C} 's, it is always the case that

$$s(\mathfrak{C}) \sum_{\rho, \sigma=1}^{s(\mathfrak{C})} |\tilde{\alpha}_{\rho\sigma}(\mathfrak{C})|^2 = 0,$$

that is, $\tilde{\alpha}_{\rho\sigma}(\mathfrak{C})=0$. This proves the first statement of our Theorem.

For every n the last statement of Theorem 26 shows that

$$\sum_{\omega=1}^n s(\mathfrak{C}_\omega) \sum_{\rho, \sigma=1}^{s(\mathfrak{C}_\omega)} |\tilde{\alpha}_{\rho\sigma}(\mathfrak{C}_\omega)|^2 \leq (Nf)^2.$$

Hence the sum

$$\sum_{\omega} s(\mathfrak{C}_\omega) \sum_{\rho, \sigma=1}^{s(\mathfrak{C}_\omega)} |\tilde{\alpha}_{\rho\sigma}(\mathfrak{C}_\omega)|^2$$

is convergent (if the number of terms is infinite) and $\leq (Nf)^2$.

THEOREM 28. (*Parseval's equation.*) If f and g have the expansion matrices $\tilde{\alpha}(\mathfrak{C}) = \{\tilde{\alpha}_{\rho\sigma}(\mathfrak{C})\}$ and $\tilde{\beta}(\mathfrak{C}) = \{\tilde{\beta}_{\rho\sigma}(\mathfrak{C})\}$, then

$$M_z[f(x)\overline{g(x)}] = \sum_{\mathfrak{C}} s(\mathfrak{C}) \sum_{\rho, \sigma=1}^{s(\mathfrak{C})} \tilde{\alpha}_{\rho\sigma}(\mathfrak{C}) \overline{\tilde{\beta}_{\rho\sigma}(\mathfrak{C})},$$

where the series $\sum_{\mathfrak{C}}$ contains at most countably infinitely many terms $\neq 0$, and is absolutely convergent (if infinite at all).

If this is proved for $f=g$, we obtain the real part of the statement by replacing our f by $(f+g)/2$ and $(f-g)/2$ and subtracting. Replacing f and g by if and g gives the imaginary part and completes the proof. Hence we may assume that $f=g$, that is, we must show that

$$(Nf)^2 - \sum_{\mathfrak{C}} s(\mathfrak{C}) \sum_{\rho, \sigma=1}^{s(\mathfrak{C})} |\tilde{\alpha}_{\rho\sigma}(\mathfrak{C})|^2 = 0.$$

If we take all finite subsets of C for the $\mathfrak{C}_1, \dots, \mathfrak{C}_n$ in Theorem 26, we see that the left side of the above equation is the greatest lower bound of all $(N[f-g])^2$ if g is any linear aggregate of any finite number of elements $D_{\rho\sigma}(\mathfrak{C})$. Hence we need to show merely that it can be made $\leq \epsilon^2$ for any $\epsilon > 0$. This is accomplished by choosing the unit $\phi(x)$ according to Theorem 16, be-

cause $\phi \times f$ is a linear aggregate of a finite number of elements $D_{\rho\sigma}(\mathfrak{G})$. By Theorem 23 we need to prove this only for the elements $D_{rv}(\mathfrak{D}) \times f$ and it follows that

$$\begin{aligned} D_{rv}(\mathfrak{D}) \times f(x) &= M_v[D_{rv}(xy^{-1}; \mathfrak{D})f(y)] \\ &= \sum_{\lambda=1}^{s(\mathfrak{D})} M_v[D_{r\lambda}(x; \mathfrak{D})D_{\lambda v}(y^{-1}; \mathfrak{D})f(y)] \\ &= \sum_{\lambda=1}^{s(\mathfrak{D})} M_v[D_{\lambda v}(y^{-1}; \mathfrak{D})f(y)]D_{r\lambda}(x; \mathfrak{D}). \end{aligned}$$

THEOREM 29. (*Approximation Theorem.*) For every a.p. function $f(x)$ and every $\epsilon > 0$ there exists a linear aggregate $h(x)$ of a finite number of elements $D_{\rho\sigma}(x; \mathfrak{G})$ (which can be limited to such elements for which the expansion matrix $\bar{\alpha}(\mathfrak{G})$ of f is $\neq 0$) such that $|f(x) - h(x)| \leq \epsilon$ for every x .

By Theorem 18 we may put $h = \phi \times f \times g$, so that we need to prove merely that $\phi \times f \times g$ is a linear aggregate of the desired kind. By Theorem 23 we may consider $D_{rv}(\mathfrak{D}) \times f \times g$. The last formula of the preceding proof shows that this is a linear aggregate of a finite number of elements $D_{r\lambda}(\mathfrak{D})$. This formula gives for the coefficient of $D_{r\lambda}(\mathfrak{D})$ in $D_{\rho\sigma}(\mathfrak{D}) \times f \times g$ (replace its f by $f \times g$)

$$\begin{aligned} M_v[D_{\lambda v}(y^{-1}; \mathfrak{D})f \times g(y)] &= M_v[\overline{D_{\lambda v}(y; \mathfrak{D})}f \times g(y)] \\ &= M_v[f \times g(y)\overline{D_{\lambda v}(y; \mathfrak{D})}]. \end{aligned}$$

Thus it is the expansion coefficient of $D_{\lambda\lambda}(\mathfrak{D})$ in $f \times g$, and this is equal to zero if the expansion matrix of $D_{\lambda\lambda}(\mathfrak{D})$ in f is zero (cf. the statement of Theorem 25 concerning $f \times g$).

THEOREM 30. Each a.p. function is the limit of a uniformly convergent sequence of functions each of which is a linear aggregate of a finite number of elements $D_{\rho\sigma}(\mathfrak{G})$, and conversely.

The statement follows from Theorem 29 by putting $\epsilon = 1, \frac{1}{2}, \frac{1}{3}, \dots$ in succession. The converse statement is a consequence of the a.p. character of all elements $D_{\rho\sigma}(\mathfrak{G})$.

IV. ALMOST PERIODICITY AND CLOSED FAMILIES OF FUNCTIONS

14. Parts I-III give a fairly complete theory of a.p. functions in an arbitrary group \mathfrak{G} , absolutely free from the customary restriction of continuity. We now introduce restrictions of this type, but in a more general manner, by considering certain families of functions.

DEFINITION 13. A set S of functions $f(x)$ (defined in \mathfrak{G} with complex numbers as values) is called a closed family (cl.f.) if it has the following properties:

- (1) If $f(x)$ is in S , every $f(xa)$ is in S .
- (2) If $f(x)$ is in S , every $f(ax)$ is in S .
- (3) If $f(x)$ is in S , every $\alpha f(x)$ is in S .
- (4) If $f(x)$ and $g(x)$ are in S , $f(x) \pm g(x)$ is in S .
- (5) If $f_1(x), f_2(x), \dots$ are in S and if $f_n(x)$ converges uniformly to $f(x)$ as $n \rightarrow \infty$, then $f(x)$ is in S .

THEOREM 31. If S is a cl.f. and contains either f or g , then it contains $f \times g$; if $D(a) = \{D_{\rho\sigma}(a)\}$ is an irreducible normal representation, S contains every $D_{\rho\sigma}$ if it contains one $D_{\rho\sigma}$; if the system of representative irreducible normal representations $D(a; \mathfrak{G})$ is given (cf. Definition 11), and if S contains f , then S contains all elements $D_{\rho\sigma}(\mathfrak{G})$ of every $D(\mathfrak{G})$ whose expansion matrix $\tilde{\alpha}(\mathfrak{G})$ (cf. Definition 12) is $\neq 0$.

Therefore Theorems 28 (Parseval's equation), 29 (Approximation Theorem), and 30 remain true if we restrict ourselves throughout to functions in S .

If f belongs to S , $f \times g$ belongs to S by the Remark following Definition 6. The case where g belongs to S can be reduced to the case where f belongs to S by replacing ab by ba in \mathfrak{G} . If a $D_{\rho\sigma}$ belongs to S , every $D_{\rho'\sigma'}$ belongs to it, since, by Theorem 21, $s^2 D_{\rho'\rho} \times D_{\rho\sigma} \times D_{\sigma\sigma'} = D_{\rho'\sigma'}$. Finally,

$$\begin{aligned} M_y[f(y)D_{\sigma'\sigma}(y^{-1}x; \mathfrak{G})] &= M_y\left[f(y) \sum_{\tau=1}^s D_{\sigma'\tau}(y^{-1}; \mathfrak{G}) D_{\tau\sigma}(x; \mathfrak{G})\right] \\ &= \sum_{\tau=1}^s M_y[f(y)D_{\sigma'\tau}(y^{-1}; \mathfrak{G})] D_{\tau\sigma}(x; \mathfrak{G}) = \sum_{\tau=1}^s M_y[f(y)\overline{D_{\tau\sigma'}(y; \mathfrak{G})}] D_{\tau\sigma}(x; \mathfrak{G}), \end{aligned}$$

that is,

$$\begin{aligned} f \times D_{\sigma'\sigma}(\mathfrak{G}) &= \sum_{\tau=1}^s \tilde{\alpha}_{\tau\sigma'}(\mathfrak{G}) D_{\tau\sigma}(\mathfrak{G}), \\ D_{\rho\rho'}(\mathfrak{G}) \times f \times D_{\sigma'\sigma}(\mathfrak{G}) &= \frac{1}{s} \tilde{\alpha}_{\rho'\sigma'}(\mathfrak{G}) D_{\rho\sigma}(\mathfrak{G}). \end{aligned}$$

Hence if $\tilde{\alpha}(\mathfrak{G}) \neq 0$ for a given \mathfrak{G} , that is, if any $\tilde{\alpha}_{\rho'\sigma'}(\mathfrak{G}) \neq 0$, and if f belongs to S , then $f \times D_{\sigma'\sigma}(\mathfrak{G})$, $D_{\rho\rho'}(\mathfrak{G}) \times f \times D_{\sigma'\sigma}(\mathfrak{G})$, and $D_{\rho\sigma}(\mathfrak{G})$ in turn belong to S .

If we keep these facts in mind we see that the proofs of Theorems 28, 29, and 30 still hold in S .

DEFINITION 14. If a topology T is given in \mathfrak{G}^\dagger we denote the set of all T -continuous functions by $[T]$.

THEOREM 32. If a topology T is given in \mathfrak{G}^\dagger in which ab is continuous in a for a fixed b , and in b for a fixed a , then $[T]$ is a cl.f.

The statement is obvious.

The a.p. functions of a cl.f. S are determined (in the manner described in Theorem 30; cf. the last statement of Theorem 31) by the elements $D_{\rho\sigma}(\mathfrak{G})$ belonging to it. This greatly facilitates the determination of all a.p. functions of a given cl.f. S .

15. We shall discuss some examples in detail.

EXAMPLE 1. Let $\mathfrak{G} = \mathfrak{G}_{\text{rat}}$ be the set of all rational numbers with addition as the rule of composition. As the group is Abelian, all irreducible representations are of degree 1, $D(a; \mathfrak{G}) = \{D_{\rho\sigma}(a; \mathfrak{G})\}$, $\rho, \sigma = 1$, so that we have a single element $D_{11}(a; \mathfrak{G}) = \phi(a; \mathfrak{G})$. The fact that this is a unitary representation is expressed by the relation

$$(*) \quad \phi(a)\phi(b) = \phi(a+b), \quad |\phi(a)| = 1.$$

Every rational number can be written in the form $a = m/n!$ ($m=0, \pm 1, \pm 2, \dots$; $n=1, 2, \dots$). Now put

$$\phi\left(\frac{1}{n!}\right) = e^{2\pi\lambda_n i}, \quad 0 \leq \lambda_n < 1.$$

Then

$$(**) \quad \phi(a) = \phi\left(\frac{m}{n!}\right) = e^{2\pi\lambda_n m i}.$$

Since

$$\frac{n+1}{(n+1)!} = \frac{1}{n!},$$

it follows that

$$(**) \quad (n+1)\lambda_{n+1} \equiv \lambda_n \pmod{1} \quad (n=1, 2, \dots)$$

and, on the other hand, it is clear that $(*)$ makes $(**)$ a definition prescribing a unique value for $\phi(a)$ which satisfies $(*)$. So the general solution is $(**)$, with the further condition $(**)$. An alternative way of writing $(**)$ is

$$(***) \quad \lambda_{n+1} \equiv \frac{\lambda_n + p_n}{n+1}, \quad p_n = 0, 1, \dots, n; \quad n=1, 2, \dots$$

[†] See first footnote on page 447.

EXAMPLE 2. Take the same $\mathfrak{G} = \mathfrak{G}_{\text{rat}}$, but take its normal topology $T = T_0$ (distance $|a-b|$) and consider $S = [T_0]$. The question then is, for which λ_n 's does the $\phi(a)$ of (**) belong to $[T_0]$? That is, when is it T_0 -continuous? It is obvious that this means that $n!\lambda_n$ is bounded, and as (**) implies that $n!\lambda_n = \lambda_1 + 1!p_1 + \dots + (n-1)!p_{n-1}$, it means that only a finite number of the p_m 's are $\neq 0$. Thus $n!\lambda_n$ is ultimately constant, say λ , and we have

$$(\S) \quad \phi(a) = e^{2\pi\lambda a i} \quad (\lambda \text{ real}). \dagger$$

EXAMPLE 3. Take the same $\mathfrak{G} = \mathfrak{G}_{\text{rat}}$, but take its p -adic topology $T = T_p$ ($p=2, 3, 5, \dots$ a prime number; distance is then 2^{N_0} , where N_0 is the minimal exponent $N=0, \pm 1, \pm 2, \dots$ for which the least denominator of $p^N(a-b)$ is not divisible by p) and consider $S = [T_p]$. The question is, for which λ_n 's is the $\phi(a)$ of (**) T_p -continuous? In T_p , $p^\nu/n! \rightarrow 0$ as $\nu \rightarrow \infty$ ($n=1, 2, \dots$, but fixed), so that $\exp(2\pi\lambda_n p^\nu i) \rightarrow 1$, $\lambda_n p^\nu \rightarrow 0 \pmod{1}$, which implies, of course, that there is a $\nu = \nu_n$ for which $\lambda_n p^\nu$ is an integer. This can be expressed in the following manner: there is a ν for which $\lambda_1 p^\nu$ is an integer, and p_n in (**) is divisible by the greatest divisor of $n+1$ which is prime to p . On the other hand, it is not difficult to see that this condition is sufficient.

EXAMPLE 4. Let $\mathfrak{G} = \mathfrak{G}_{\text{real}}$ be the set of all real numbers with addition as the rule of composition. Again we first determine all a.p. functions, that is, all irreducible unitary representations. This again means solving (*), but now with a and b running over all real numbers. Equation (*) can be solved by the following procedure:

Choose a rational linear basis of the set of real numbers, that is, a set B such that for every real number a the equation $a = \alpha_1 \xi_1 + \dots + \alpha_n \xi_n$ ($n=1, 2, \dots$; $\alpha_1, \dots, \alpha_n$ rational numbers, all $\neq 0$; ξ_1, \dots, ξ_n different elements of B) has exactly one solution [12, pp. 459-462]. For every ξ of B we can define the quantity

$$\Gamma^{(a)}(\xi) = \begin{cases} \alpha_m & \text{if } \xi = \text{some } \xi_m, \\ 0 & \text{if } \xi \neq \text{each } \xi_m, \end{cases} \quad m = 1, \dots, n;$$

then $\Gamma^{(a)}(\xi)$ is always rational, we have

$$a = \sum_{\xi \in B} \Gamma^{(a)}(\xi) \xi,$$

where only a finite number of terms are $\neq 0$, and thus $\Gamma^{(a+b)}(\xi) = \Gamma^{(a)}(\xi) + \Gamma^{(b)}(\xi)$. From this it follows at once that every solution $\phi(a)$ of (*) for real a 's is of the form

\dagger Thus there exist discontinuous a.p. functions of a rational variable. This fact was proved by Ursell [28, Second Note].

$$(\dagger) \quad \phi(a) = \prod_{\xi \in B} \phi_{\xi}(\Gamma^{(a)}(\xi)),$$

where each $\phi_{\xi}(c)$ is a solution of (*) for rational c 's, and thus only a finite number of factors are $\neq 1$. Conversely, it is obvious that every $\phi(a)$ in (\dagger) is a solution. Therefore the general solution is given by (\dagger) if, for every ξ of B , we choose a $\phi_{\xi}(c)$ from (**) and (***) with $\lambda_n = \lambda_{\xi, n}$ and $p_n = p_{\xi, n}$ dependent on ξ .

EXAMPLE 5. Take the same $\mathfrak{G} = \mathfrak{G}_{\text{real}}$, but consider the set of all Lebesgue-measurable functions, $S = S_m$, which is obviously a c.f. The question is, which functions $\phi(a)$ of (\dagger) are Lebesgue-measurable? As they are solutions of $\phi(a)\phi(b) = \phi(a+b)$ and $|\phi(a)| = 1$, we can infer from their measurability that they must be of the form

$$(\S) \quad \phi(a) = e^{2\pi\lambda a i} \quad (\lambda \text{ real}). \dagger$$

EXAMPLE 6. Take the same $\mathfrak{G} = \mathfrak{G}_{\text{real}}$, but take its normal topology $T = T_0$ (distance $|a-b|$) and consider $S = [T_0]$. The question is, which functions $\phi(a)$ of (\dagger) are T_0 -continuous? As every T_0 -continuous function is measurable, all such functions must be of the form (\S); and as all functions (\S) are continuous, this again gives the general solution.||

EXAMPLE 7. Take the same $\mathfrak{G} = \mathfrak{G}_{\text{real}}$, but in it take a new topology $T = T(\lambda_1, \dots, \lambda_k)$, where the only relation $n_1\lambda_1 + \dots + n_k\lambda_k = 0$, with $n_1, \dots, n_k = 0, \pm 1, \pm 2, \dots$, shall be $n_1 = \dots = n_k = 0$; distance is defined by¶

$$\begin{aligned} & \left[|e^{2\pi\lambda_1 a i} - e^{2\pi\lambda_1 b i}|^2 + \dots + |e^{2\pi\lambda_k a i} - e^{2\pi\lambda_k b i}|^2 \right]^{1/2} \\ & = 2[\sin^2 \pi\lambda_1(a-b) + \dots + \sin^2 \pi\lambda_k(a-b)]^{1/2}. \end{aligned}$$

Hence the condition $a_n \rightarrow a$ in $T(\lambda_1, \dots, \lambda_k)$ as $n \rightarrow \infty$ means that $a_n \rightarrow a$ with

† This is analogous to a result of Fréchet [9] who discussed $f(a)+f(b)=f(a+b)$. Cf. also Sierpinski [23] and Banach [1]. The simplest way to prove our statement is this:

Put $\psi_{\epsilon}(a) = \int_a^{a+\epsilon} \phi(x) dx$. Then $\psi_{\epsilon}(a)$ is continuous in a and satisfies $\psi_{\epsilon}(a)\phi(b) = \psi_{\epsilon}(a+b)$. If we had $\psi_{\epsilon}(a) = 0$ for every ϵ , then, as $(\partial/\partial \epsilon)\psi_{\epsilon}(a)$ is equal to $\phi(a+\epsilon)$ except over a set of measure zero, it would lead to a function $\phi(a+\epsilon) = 0$, except over a set of measure zero, which contradicts the condition $|\phi(a+\epsilon)| = 1$. Thus we can find ϵ_0 and a_0 such that $\psi_{\epsilon_0}(a_0) \neq 0$, and then our equation shows that $\phi(b) = \psi_{\epsilon_0}(a_0+b)/\psi_{\epsilon_0}(a_0)$, that is, $\phi(b)$ is continuous. Then $\psi_{\epsilon_0}(a)$ is differentiable, so that (by our last equation) $\phi(b)$ is also. Now we differentiate $\phi(a)\phi(b) = \phi(a+b)$ and get $\phi'(a)\phi(b) = \phi'(a+b)$, that is, $\phi'(b) = \beta\phi(b)$ when $a=0$. This means that $\phi(a) = \alpha e^{\beta a}$, and our original conditions make $\alpha=1$, $\beta = 2\pi\lambda i$, λ real.

|| Thus there exist discontinuous a.p. functions of a real variable, but they are all non-measurable. These facts have also been proved by Ursell [28, First Note].

¶ For $k=1$ this is not only a new topology in $\mathfrak{G}_{\text{real}}$, but this also implies an identification of elements congruent mod $1/\lambda_1$. After this identification it is the normal topology. For $k>1$ it implies no identifications, but it is a new topology.

respect to mod $1/\lambda_1, \dots$, and mod $1/\lambda_k$ simultaneously. Therefore $\mathcal{G}_{\text{real}}$ is compact when metrically completed in this topology and every uniformly $T(\lambda_1, \dots, \lambda_k)$ -continuous function is a.p. (cf. Theorem 36). The question is, which functions $\phi(a)$ of (†) are uniformly $T(\lambda_1, \dots, \lambda_k)$ -continuous? As $T(\lambda_1, \dots, \lambda_k)$ -continuity implies T_0 -continuity, they must have the form (§), that is, $\phi(a) = e^{2\pi\lambda a i}$. Now the condition $a_n \rightarrow a$ with respect to mod $1/\lambda_1, \dots$, and mod $1/\lambda_k$ should imply that $\phi(a_n) \rightarrow \phi(a)$ so that $e^{2\pi\lambda a_n i} \rightarrow e^{2\pi\lambda a i}$. This is the case if and only if

$$\lambda = n_1\lambda_1 + \dots + n_k\lambda_k, \quad n_1, \dots, n_k = 0, \pm 1, \pm 2, \dots.*$$

EXAMPLE 8. Let \mathcal{G} be a semi-simple Lie group.† The determination of all a.p. functions again means the determination of all irreducible unitary representations (which now of course need not be of degree 1). But such a representation is always continuous in the normal topology T_0 of \mathcal{G} .‡ Therefore all a.p. functions in this \mathcal{G} are automatically T_0 -continuous, in contrast with Examples 1, 2, and 4, 6 (cf. footnotes † and ‡ on pages 478 and 479 respectively). Thus there is no need to discuss $S = [T_0]$ separately.

16. Examples 1–8 sufficiently illustrate the various possibilities of combining a.p. functions with topology to make further comment unnecessary. We shall now investigate another phenomenon.

THEOREM 33. Let \mathcal{G} be a group and S a cl.f. of functions in it. The following conditions on two elements a and b of \mathcal{G} are equivalent:

- A. $D(a; \mathcal{G}) = D(b; \mathcal{G})$ (that is, $D_{\rho\sigma}(a; \mathcal{G}) = D_{\rho\sigma}(b; \mathcal{G})$) for every \mathcal{G} of C for which the elements $D_{\rho\sigma}(\mathcal{G})$ belong to S .
- B. $D(a) = D(b)$ (that is, $D_{\rho\sigma}(a) = D_{\rho\sigma}(b)$) for every normal representation for which the elements $D_{\rho\sigma}$ belong to S .
- C. $f(a) = f(b)$ for every a.p. function in S .

A is a special case of B, so that B implies A.

As every $D_{\rho\sigma}(x)$ is a.p. (Theorem 19 and Definition 10), B is a special case of C so that C implies B.

Finally, Theorems 30 and 31 show that A implies C.

Our three statements together prove the equivalence of A, B, and C.

DEFINITION 15. We call two elements a and b of \mathcal{G} which satisfy one of the equivalent conditions of Theorem 33 *S-coherent* (if S is the set of all functions, we abbreviate this to *coherent*). We denote the set of those elements which are *S-coherent* (coherent) with 1 by \mathcal{G}^S (\mathcal{G}_0).

* That is, we are led to the Bohl-Esclangon [3] quasi-periodic functions with the basis $\lambda_1, \dots, \lambda_n$. Cf. also H. Bohr [4, II, pp. 111–117].

† For a detailed discussion of this notion cf. E. Cartan [7].

‡ This is a most remarkable difference between the behavior of Abelian Lie groups (cf. Example 4) and semi-simple Lie groups. It was discovered by B. L. van der Waerden [29, p. 785].

THEOREM 34. \mathfrak{G}^S is an invariant subgroup of \mathfrak{G} , and if $S = [T]$ for a topology T of \mathfrak{G} , then \mathfrak{G}^S is T -closed. Those elements of \mathfrak{G} which are coherent with a given a form the coset of \mathfrak{G}_0^S in \mathfrak{G} belonging to a .

Consider the condition B in Theorem 33 (either A or C could also be used). If a and b belong to \mathfrak{G}_0^S we have $D(ab) = D(a)D(b) = 1$, $D(a^{-1}) = D(a)^{-1} = 1$, that is, a^{-1} and ab belong to it; if only a belongs to \mathfrak{G}_0^S , we have $D(b^{-1}ab) = D(b)^{-1}D(a)D(b) = D(b)^{-1}D(b) = 1$, that is, $b^{-1}ab$ also belongs to it. If $S = [T]$, every $D(a)$ is T -continuous, each set $D(a) = 1$ is T -closed, and so their common part \mathfrak{G}_0^S is also. That a and b are coherent means that we always have $D(a) = D(b)$, $D(a^{-1}b) = D(a)^{-1}D(b) = 1$, that is, that $a^{-1}b$ belongs to \mathfrak{G}_0^S . Hence the elements b form exactly the coset of \mathfrak{G}_0^S in \mathfrak{G} belonging to a .

DEFINITION 16. If $\mathfrak{G}_0^S = 1$ ($\mathfrak{G}_0 = 1$) we call \mathfrak{G} and S (\mathfrak{G}) maximally a.p.; if $\mathfrak{G}_0^S = \mathfrak{G}$ ($\mathfrak{G}_0 = \mathfrak{G}$) we call \mathfrak{G} and S (\mathfrak{G}) minimally a.p.

These two cases are indeed the two extremes which can occur. If \mathfrak{G} and S are minimally a.p., then for every a.p. $f(x)$ of S we always have $f(a) = f(1)$, that is, the constants are the only a.p. functions in S . And for every normal representation $D(a)$ with the elements $D_{\rho\rho}(a)$ in S , it must be $D(a) = D(1) = 1$, so that if $D(a)$ is irreducible its degree must be 1. If, on the other hand, \mathfrak{G} and S are maximally a.p., then there exists, for every pair a and b in \mathfrak{G} , $a \neq b$, a \mathfrak{C} from C such that all $D_{\rho\rho}(\mathfrak{C})$ are in S with $D(a; \mathfrak{C}) \neq D(b; \mathfrak{C})$, and an a.p. function $f(x)$ in S such that $f(a) \neq f(b)$. Even more is true:

THEOREM 35. If $f(x)g(x)$ is in S whenever $f(x)$ and $g(x)$ are in S , and if \mathfrak{G} and S are maximally a.p., then, for any finite set a_1, \dots, a_n of distinct elements of \mathfrak{G} and any set of complex numbers $\alpha_1, \dots, \alpha_n$, an a.p. function $f(x)$ exists in S with the prescribed values $f(a_1) = \alpha_1, \dots, f(a_n) = \alpha_n$.

If $a \neq b$, there is an a.p. function $g(x)$ in S such that $g(a) \neq g(b)$, so that

$$h(x) = \frac{g(x) - g(b)}{g(a) - g(b)}$$

is an a.p. function in S with $h(a) = 1$ and $h(b) = 0$. For every pair a and b ($a \neq b$), choose such a function $h(x)$ and denote it by $h(a, b; x)$. Then

$$f(x) = \sum_{\nu=1}^n \alpha_{\nu} \prod_{\mu=1, \mu \neq \nu}^n h(a_{\mu}, b_{\mu}; x)$$

has all the properties required.

There are also some other ways to characterize \mathfrak{G}_0^S , but we shall not discuss them here.

17. If \mathfrak{G} and S are maximally a.p., we can introduce a topology by means of their a.p. functions. In this connection the following notions are of importance:

DEFINITION 17. *If \mathfrak{G} and S are maximally a.p. we define a topology FS in \mathfrak{G} by considering the following "neighborhoods" $\mathfrak{N}(a)$ of an element a of \mathfrak{G} : Choose a finite number of a.p. functions f_1, \dots, f_n and an $\epsilon > 0$; then $\mathfrak{N}(a) = \mathfrak{N}(a; f_1, \dots, f_n, \epsilon)$ is the set of all b 's such that $|f_1(a) - f_1(b)| < \epsilon, \dots, |f_n(a) - f_n(b)| < \epsilon$. (If S is the set of all functions we abbreviate this to F .)*

One sees at once that FS satisfies Hausdorff's Axioms (cf. first footnote on page 447).

DEFINITION 18. *If two topologies T_1 and T_2 for a set \mathfrak{S} are given, T_1 is called weaker than T_2 if every T_1 -neighborhood of an element a of \mathfrak{S} contains a T_2 -neighborhood of a .*

Obviously, every set which is closed or open in the T_1 -sense, and every function which is continuous in the T_1 -sense, has the same property in the T_2 -sense. Thus for a group \mathfrak{G} , $[T_1]$ is a subset of $[T_2]$ (cf. Definition 14). On the other hand, it is obvious that if S_1 and S_2 are c.f. and S_2 is a subset of S_1 , then FS_1 is weaker than FS_2 .

We intend to go more deeply into the theory of $[T]$ and FS on another occasion. At present let us merely remark that for every \mathfrak{G} (even for a non-topologically given \mathfrak{G}) F is a topology determined by \mathfrak{G} alone (if \mathfrak{G} is maximally a.p.). Discussion of Examples 1 and 4 shows without much difficulty that $\mathfrak{G}_{\text{rat}}$ and $\mathfrak{G}_{\text{real}}$ are maximally a.p. (even with their c.f. $[T_0]$ or $[T_p]$ and $[T_0]$ or $[T(\lambda_1, \dots, \lambda_k)]$ respectively (for $k > 1$, cf. footnote* on page 480)) and that their F 's are very "strong"; the condition $a_n \rightarrow a$ in F as $n \rightarrow \infty$ means that all a_n 's, with a finite number of exceptions, are equal to a . On the other hand, Example 8 shows that, for a semi-simple Lie group \mathfrak{G} (if it is maximally a.p.), $F = F[T_0]$. Theorem 36 shows that if \mathfrak{G} is compact in T_0 , \mathfrak{G} and $[T_0]$ are maximally a.p. and $F[T_0] = T_0$. Thus, if \mathfrak{G} is a semi-simple and compact Lie group, it is maximally a.p. and $F = T_0$.

18. The case where a group \mathfrak{G} and a topology T have the properties that \mathfrak{G} and $[T]$ are maximally a.p. and $F[T] = T$ is of particular importance.

THEOREM 36. *\mathfrak{G} and $[T]$ are maximally a.p. and $F[T] = T$ in each of the two following cases (in case B, $F[T] = T$ should be understood to mean only that the condition $a_n \rightarrow a$ as $n \rightarrow \infty$ is equivalent to it in both senses):*

A. \mathfrak{G} is compact in T .

† See first footnote on page 447.

B. \mathfrak{G} is locally compact[†] and separable[†] in T , \mathfrak{G} is an Abelian group, and ab and a^{-1} are T -continuous in a and b , and a respectively.[‡]

If \mathfrak{G} is compact in T , every continuous $f(x)$ is a.p.: for \mathfrak{G} being compact, $f(x)$ is uniformly continuous; if any sequence a_1, a_2, \dots is given, we can extract from it a subsequence a_{n_1}, a_{n_2}, \dots which converges to a limit a , and then we have the result that $f(xa_{n_i}) \rightarrow f(xa)$ and $f(a_{n_i}x) \rightarrow f(ax)$ uniformly as $i \rightarrow \infty$. Thus $[T]$ consists only of a.p. functions.

Now it is possible to define a distance $D(a, b)$ in \mathfrak{G} which is equivalent to the topology T [27, 25]. $f(x) = D(a, x)$ belongs to $[T]$ and we have the result that $f(a) = 0 \neq f(b)$, proving that \mathfrak{G} and $[T]$ are maximally a.p.; and the neighborhood $\mathfrak{N}(a; f, \epsilon)$ (cf. Definition 17) is the sphere with the center a and the radius ϵ , proving that T is weaker than $F[T]$. But $F[T]$ is obviously weaker than T , and therefore $F[T] = T$. Thus A is proved.

The proof of B will be given at the end of Part V.

Minimally a.p. groups likewise exist, for example, the group $\mathfrak{g}_{(n)}$ of all linear transformations of determinant 1 in the real euclidean space of n dimensions, $n=2, 3, \dots$. As it is a semi-simple Lie group, indeed even simple [6], all its bounded representations are continuous [29]; as it is a linear group, their continuity implies their differentiability [14, p. 37]. Hence we need only determine those irreducible representations of $\mathfrak{g}_{(n)}$ which arise from "infinitesimal representations," and see if there exist any bounded ones among them. Now these representations and their traces (characteristics) are known [70; 30, pp. 287, 300], and only the identity, $D(a) \equiv 1$, has a bounded trace, so no other representation can be bounded. Application of Theorem 33, criterion A, shows that $\mathfrak{g}_{(n)}$ is minimally a.p.

The group \mathfrak{g}' of all transformations $y = ax + b$ (a and b real, $a > 0$) is neither minimally nor maximally a.p., as a simple discussion shows.

V. ABELIAN GROUPS

19. We assume throughout Part V that the assumptions of Theorem 36, case B (which we shall finally prove), hold; thus we assume that a group \mathfrak{G} and a topology T are given, that \mathfrak{G} is locally compact and separable in T and Abelian, and that ab, a^{-1} are T -continuous.

Under the above topological assumptions, A. Haar has shown the existence of a right-invariant Lebesgue integral [11, pp. 166-167]. Thus it is possible to define for complex-valued functions $f(x)$ defined in \mathfrak{G} (i) a notion of measura-

[†] "Locally compact" means that each element a has a conditionally compact neighborhood [13, p. 107]; as a group \mathfrak{G} is homogeneous it is sufficient to postulate this for the element 1. "Separable" means that there exists a countably infinite "equivalent system of neighborhoods"; if the topology is originated by a distance notion, one may postulate the existence of a countably infinite everywhere dense subset [13, p. 125, and p. 229, Axiom 10].

[‡] See first footnote on page 447.

bility, (ii) a notion of summability, (iii) an integral $\int_{\mathfrak{G}} f(x) dx$. On the basis of (i) and (ii), moreover, it is possible to do this in such a manner that (i)–(iii) have all the formal properties of these notions as in the usual Lebesgue theory, and besides are invariant under the substitution of $f(xa)$ for $f(x)$.

We now consider all measurable functions $f(x)$ in \mathfrak{G} for which $|f(x)|^2$ is summable, that is, $\int_{\mathfrak{G}} |f(x)|^2 dx$ is finite. These functions form a Hilbert space $\mathfrak{H}_{\mathfrak{G}}$ if we define the inner product (f, g) to be $\int_{\mathfrak{G}} f(x) \overline{g(x)} dx$ †, provided that \mathfrak{G} is infinite, which we will assume to be the case. (If it is finite, it is compact and falls under case A.) In $\mathfrak{H}_{\mathfrak{G}}$,

$$(\#) \quad O_a f(x) = f(xa)$$

defines a linear and unitary operation (that is, an operation which leaves (f, g) invariant), and it follows that

$$(\# \#) \quad O_a O_b = O_{ab}.$$

Now we use the Abelian character of \mathfrak{G} , by virtue of which $(\# \#)$ implies that O_a and O_b commute. As O_a is unitary, its adjoint† is $O_a^* = O_a^{-1}$ and, by $(\# \#)$, $O_a^* = O_{a^{-1}}$. Thus every O_b commutes with every O_a and O_a^* , and the set of all operators O_a has been called Abelian [16, p. 389]. Therefore a theorem proved by the author applies to this set: there exists a bounded Hermitian operator R such that every O_a is a function of R ,

$$(\# \#) \quad O_a = \phi_a(R),$$

where $\phi_a(\lambda)$ is a complex-valued function of the variable λ .§ That the functions $\phi_a(\lambda)$ can be used for the discussion of the group \mathfrak{G} has been noted by Haar and successfully applied to countably infinite Abelian groups [10, p. 131]; cf. also Wiener and Paley [33]. Theorem 37 will be an application of this idea in the full generality allowed by Haar's right-invariant Lebesgue integral. It must be remarked, however, that Haar's method of discussing countably infinite Abelian groups has been considerably simplified by Wiener and Paley [33], but that their simplification seems not to apply to our general case, and that we have to use Haar's original method.

THEOREM 37. *If \mathfrak{G} and T fulfill the assumptions formulated at the beginning of this part (that is, if \mathfrak{G} is locally compact, separable, and Abelian), there exists a function in two variables $\phi(a, \lambda)$ (a in \mathfrak{G} , λ real) with the following properties:*

† For the modern theory of Hilbert space cf. J. v. Neumann [15, pp. 63–70, 108–111]. Cf. further M. H. Stone [24, pp. 1–32].

§ The notion of a function of an operator is due originally to F. Riesz. More general forms have been given to it by J. v. Neumann [17, pp. 202–213] and M. H. Stone [24, pp. 221–241]. The theorem in question has been proved by J. v. Neumann [17, p. 214].

$\phi(a, \lambda)$ is a Baire function in (a, λ) ,* and there exists a "resolution of the identity" $E(\lambda)$ such that

$$(\dagger) \quad (O_a f, g) = \int_{-\infty}^{\infty} \phi(a, \lambda) d(E(\lambda)f, g) \dagger$$

identically in a and $f(x)$ and $g(x)$.

For the function $\phi(a, \lambda) = \phi_a(\lambda)$ in (\dagger) , the theorem mentioned in the footnote§ on page 484 leads to all our statements except for the Baire character of $\phi(a, \lambda)$ in (a, λ) (it would show the Baire character in λ , but we need it also in a).

We know that finite linear aggregates of functions $f_{\mathfrak{D}}$ where \mathfrak{D} is a conditionally compact open set, therefore having finite measure, and

$$f_{\mathfrak{D}}(a) = \begin{cases} 1 & \text{for } a \text{ in } \mathfrak{D}, \\ 0 & \text{elsewhere,} \end{cases}$$

are everywhere dense in our functional space [15, p. 110]. From now on "everywhere dense" will be interpreted in the sense of the distance

$$D(f, g) = \|f - g\| = [(f - g, f - g)]^{1/2} = \left[\int_{\mathfrak{G}} |f(x) - g(x)|^2 dx \right]^{1/2}$$

but not in the sense of the distance l.u.b. $_x |f(x) - g(x)|$. Now, if \mathfrak{D}_1 is an open set the closure of which is part of \mathfrak{D} , we can find a continuous function§

$$f_{\mathfrak{D}, \mathfrak{D}_1}(a) \begin{cases} = 1 & \text{for } a \text{ in } \mathfrak{D}, \\ = 0 & \text{for } a \text{ not in } \mathfrak{D}, \\ \geq 0 \text{ and } \leq 1 & \text{elsewhere.} \end{cases}$$

If we let \mathfrak{D}_1 converge to \mathfrak{D} , then $f_{\mathfrak{D}, \mathfrak{D}_1}(a)$ converges everywhere to, and is majorized by, $f_{\mathfrak{D}}(a)$, so that $f_{\mathfrak{D}}(a)$ is its limit in the sense of the distance $\|f - g\|$. Therefore continuous functions f which are $\neq 0$ only in conditionally compact sets are everywhere dense in our functional space. Since $a_n \rightarrow a$ implies $xa_n \rightarrow xa$, we have $f(xa_n) \rightarrow f(xa)$ for these functions, and, by the second property,

$$\|O_{a_n} f - O_a f\| = \left[\int_{\mathfrak{G}} |f(xa_n) - f(xa)|^2 dx \right]^{1/2} \rightarrow 0.$$

Hence $a_n \rightarrow a$ implies $O_{a_n} f \rightarrow O_a f$ for an everywhere dense set of f 's, but as all

* That is, it can be obtained from continuous functions in (a, λ) by successive limiting processes wherein the limit is always taken of everywhere convergent sequences.

† $\int_{-\infty}^{\infty}$ is a Lebesgue-Stieltjes integral over λ . For an explanation of the terminology used, see [15, p. 92] or [24, p. 174].

§ This is a problem of Fréchet, first solved by Hahn. Cf. [26, Anhang III, p. 290].

O_b 's are unitary operators, and therefore uniformly continuous in f , the implication holds for every f . Consequently O_af is a continuous function in a for every fixed f .

A simple computation shows, after substituting $E(\mu)g$ in (†), at the end of Theorem 37 in place of g [cf. 17, p. 206],

$$(O_af, E(\mu)g) = \int_{-\infty}^{\mu} \phi(a, \lambda) d(E(\lambda)f, g).$$

Now choose a complete normalized orthogonal system f_1, f_2, \dots , put $f = g = f_n$, $n = 1, 2, \dots$, multiply by 2^{-n} , and add. The infinite series thus obtained in the left- and right-hand members converge uniformly since $(O_af, E(\mu)g)$ and $(E(\lambda)f, g)$ are both $\leq \|f\| \|g\|$ in absolute value. The result is

$$\sum_{n=1}^{\infty} 2^{-n} (O_af_n, E(\mu)f_n) = \int_{-\infty}^{\mu} \phi(a, \lambda) d \left[\sum_{n=1}^{\infty} 2^{-n} (E(\lambda)f_n, f_n) \right].$$

$(O_af_n, E(\mu)f_n)$ is continuous in a and continuous on the right in μ , $(E(\lambda)f_n, f_n)$ is continuous on the right in λ and monotonically increasing, and the same properties hold for the uniformly convergent sums

$$F(a, \mu) = \sum_{n=1}^{\infty} 2^{-n} (O_af_n, E(\mu)f_n), \quad G(\lambda) = \sum_{n=1}^{\infty} 2^{-n} (E(\lambda)f_n, f_n).$$

Thus $F(a, \mu)$ and $G(\lambda)$ are Baire functions, the latter is monotonically increasing, and

$$F(a, \mu) = \int_{-\infty}^{\mu} \phi(a, \lambda) dG(\lambda).$$

If we consider $G(\lambda)$ as the variable (instead of λ), then the well known theorem on the differentiability of integrals shows that†

$$\lim_{\delta, \epsilon \rightarrow 0^+} \frac{F(a, \mu + \delta) - F(a, \mu - \epsilon)}{G(\mu + \delta) - G(\mu - \epsilon)}$$

exists and equals $\phi(a, \mu)$ except, however, for a set of μ 's dependent on a whose $\xi = G(\mu)$ -image§ is a set (of real numbers) of Lebesgue measure zero. Now the function

$$\phi_1(a, \lambda) = \begin{cases} \lim_{\delta, \epsilon \rightarrow 0^+} \frac{F(a, \mu + \delta) - F(a, \mu - \epsilon)}{G(\mu + \delta) - G(\mu - \epsilon)} & \text{when this limit exists,} \\ 0 & \text{otherwise} \end{cases}$$

† Cf. [5, pp. 544–545]. Analogous results concerning “central derivatives” of F with respect to G are due to Daniell [8].

§ If $G(\mu)$ is discontinuous at $\mu = \mu_0$, the image of $\mu = \mu_0$ is supposed to be the whole jump-interval $G(\mu_0 - 0) \leq \xi \leq G(\mu_0 + 0)$.

is obviously a Baire function, and the set Σ of λ 's for which $\phi_1(a, \lambda) \neq \phi(a, \lambda)$ has a $\xi = G(\mu)$ -image of Lebesgue measure zero. Since $2^*G(\mu) - (E(\mu)f_n, f_n)$ is monotonically increasing, the $\xi = (E(\mu)f_n, f_n)$ -image of Σ is also of Lebesgue measure† zero, and therefore every $\xi = (E(\mu)f, f)$ -image is of Lebesgue measure zero [17, p. 213, the last remark of Part II].

Hence we may replace $\phi(a, \lambda)$ in (‡) by $\phi_1(a, \lambda)$, and this will not affect the validity of (‡) for $f = g$; now if we replace our f by $(f+g)/2$ and $(f-g)/2$ and subtract, we get the real part of the general (‡); if we replace f and g by if and g , we get its imaginary part, and prove it altogether. Thus $\phi_1(a, \lambda)$ meets all our requirements.

THEOREM 38. *Under the assumptions of Theorem 37, $\phi(a, \lambda)$ can even be chosen as a continuous function in a satisfying the equations*

$$\phi(ab, \lambda) = \phi(a, \lambda) \phi(b, \lambda), \quad |\phi(a, \lambda)| = 1.$$

A simple computation shows [17, p. 206] that

$$\begin{aligned} (O_a O_b f, g) &= \int_{-\infty}^{+\infty} \phi(a, \lambda) \phi(b, \lambda) d(E(\lambda)f, g), \\ (O_a^* O_b f, g) &= \int_{-\infty}^{+\infty} |\phi(a, \lambda)|^2 d(E(\lambda)f, g); \end{aligned}$$

on the other hand,

$$(O_a b f, g) = \int_{-\infty}^{+\infty} \phi(ab, \lambda) d(E(\lambda)f, g), \quad (f, g) = \int_{-\infty}^{+\infty} d(E(\lambda)f, g).$$

Now $O_a O_b = O_{ab}$, $O_a^* O_a = 1$; therefore the right sides of our equations are equal. An analogous computation shows that if we substitute $E(\mu)g$ for g [17, p. 206] and subtract, we get

$$\begin{aligned} \int_{-\infty}^{\mu} (\phi(ab, \lambda) - \phi(a, \lambda) \phi(b, \lambda)) d(E(\lambda)f, g) &= 0, \\ \int_{-\infty}^{\mu} (|\phi(a, \lambda)|^2 - 1) d(E(\lambda)f, g) &= 0. \end{aligned}$$

Putting $f = g$ shows that the equations of our Theorem hold except for a set of λ 's (depending on the pair a and b and on a respectively), the $\xi = (E(\lambda)f, f)$ -image of which has Lebesgue measure zero† (this condition holds for all f 's simultaneously). Returning to the complete normalized orthogonal

† The Lebesgue measure of the $\xi = H(\mu)$ -image of a set \mathfrak{S} is $\int_{\mathfrak{S}} dH(\mu)$, and therefore, if it is 0 for $H(\mu)$, it will be 0 for every other function $K(\mu)$ for which $H(\mu) - K(\mu)$ is monotonically increasing [cf. 17, p. 198, rule d, and p. 199].

system f_1, f_2, \dots in the proof of Theorem 37, and to the corresponding

$$G(\lambda) = \sum_{n=1}^{\infty} 2^{-n} (E(\lambda) f_n, f_n),$$

we see that also the $\xi = G(\lambda)$ -images have Lebesgue measure zero (this follows for each $\sum_{n=1}^N 2^{-n} (E(\lambda) f_n, f_n)$ from the integral formula in the footnote on page 487, and for

$$G(\lambda) = \sum_{n=1}^{\infty} 2^{-n} (E(\lambda) f_n, f_n)$$

from the fact that the difference is monotonically increasing, ≥ 0 , and $\leq \sum_{n=N+1}^{\infty} 2^{-n} = 1/2^N$.

For a -sets and b -sets (that is, subsets of \mathfrak{G}) we have a measure, namely, the Haar-Lebesgue measure. For λ -sets (that is, sets of real numbers) we shall consider the Lebesgue measure of the $\xi = G(\lambda)$ -image (cf. the footnote§ on page 486), and call it the λ -measure. All these measures have the formal properties of the Lebesgue measure.* We can, by the analogue of the process which leads from linear to plane measure [cf. 18, p. 588] use these measures to define measures with similar properties for (a, b) -sets, (a, λ) -sets, and (a, b, λ) -sets. If we use these defining processes, the theorem of Fubini holds for all combinations of the variables a, b, λ because its proof [5, pp. 622–628] applies unchanged.

As we are dealing with Baire functions, the (a, b, λ) - and (a, λ) -sets for which $\phi(ab, \lambda) \neq \phi(a, \lambda)\phi(b, \lambda)$ and $|\phi(a, \lambda)| \neq 1$ are Borel sets and therefore measurable. Hence Fubini's theorem can be applied to them; as for fixed a, b and λ respectively they give λ -sets of zero λ -measure, they are sets of zero (a, b, λ) -measure and (a, λ) -measure themselves. This again implies that if λ does not belong to a certain (fixed) set \mathfrak{S}_1 of zero λ -measure, and if a does not belong to a certain set $\mathfrak{S}_2^{(\lambda)}$ (depending on λ) of zero (Haar) measure, then we have the result that, if b does not belong to a certain set $\mathfrak{S}_3^{(\lambda, a)}$ (depending on λ and a) of zero (Haar) measure, then $\phi(ab, \lambda) = \phi(a, \lambda)\phi(b, \lambda)$, and, at any rate, $|\phi(a, \lambda)| = 1$.

Now choose a conditionally compact open set \mathfrak{D} . If we had, for a certain λ ,

$$\int_{\mathfrak{D}} \phi(x, \lambda) dx = 0$$

for every \mathfrak{D} , this would imply that

$$\int_{\mathfrak{M}} \phi(x, \lambda) dx = 0$$

* By this we mean that they satisfy Carathéodory's postulates I–V [5, pp. 238–239, 258].

for every measurable set \mathfrak{M} , and thus $\phi(x, \lambda) = 0$ except for an x -set of λ -measure zero. This contradicts the fact that we have $|\phi(x, \lambda)| = 1$ except for an x -set of λ -measure zero. Therefore choose \mathfrak{D} such that

$$\int_{\mathfrak{D}} \phi(x, \lambda) dx \neq 0,$$

and denote the set of all (ax) 's (x in \mathfrak{D}) by $a\mathfrak{D}$.

Assume λ in \mathfrak{S}_1 and a in $\mathfrak{S}_2^{(\lambda)}$. Then we have $\phi(ax, \lambda) = \phi(a, \lambda)\phi(x, \lambda)$ if x is not in $\mathfrak{S}_3^{(\lambda, a)}$ and this implies that

$$\begin{aligned} \phi(a, \lambda) \int_{\mathfrak{D}} \phi(x, \lambda) dx &= \int_{\mathfrak{D}} \phi(ax, \lambda) dx = \int_{a\mathfrak{D}} \phi(y, \lambda) dy, \\ \phi(a, \lambda) &= \frac{\int_{a\mathfrak{D}} \phi(x, \lambda) dx}{\int_{\mathfrak{D}} \phi(x, \lambda) dx}. \end{aligned}$$

Now, by well known theorems on Lebesgue integrals, the numerator is continuous in a , the denominator is constant and $\neq 0$, and for this argument we need not restrict a to $\mathfrak{S}_2^{(\lambda)}$ (λ , of course, is in \mathfrak{S}_1). Hence we may define a continuous function $\phi_2(a, \lambda)$ by putting it equal to

$$\frac{\int_{a\mathfrak{D}} \phi(x, \lambda) dx}{\int_{\mathfrak{D}} \phi(x, \lambda) dx}$$

(λ in \mathfrak{S}_1); and we have $\phi_2(a, \lambda) = \phi(a, \lambda)$ if a is not in $\mathfrak{S}_2^{(\lambda)}$.

In this case, if b is not in $\mathfrak{S}_2^{(\lambda)}$ and ab is not in $\mathfrak{S}_2^{(\lambda)}$, we have $\phi_2(ab, \lambda) = \phi_2(a, \lambda)\phi_2(b, \lambda)$. But as we except only a b -set of zero (Haar) measure, this holds in an everywhere dense b -set, and thus, for reasons of continuity, for every b . So the above formula is true for every b , and $|\phi_2(a, \lambda)| = 1$ is true if a is not in $\mathfrak{S}_2^{(\lambda)}$. For the same continuity reasons, therefore, there are no a -exceptions at all. Thus (if λ is not in \mathfrak{S}_1) $\phi_2(a, \lambda)$ meets the requirements of our theorem if it can replace $\phi(a, \lambda)$.

By definition, $\phi_2(a, \lambda)$ is a Baire function in (a, λ) . Hence the (a, λ) -set for which $\phi(a, \lambda) \neq \phi_2(a, \lambda)$ is a Borel set and therefore measurable. Hence Fubini's theorem applies to it, and since for a fixed λ , except in a set of zero λ -measure (\mathfrak{S}_1), it gives an a -set of zero (Haar) measure ($\mathfrak{S}_2^{(\lambda)}$), it is a set of zero (a, λ) -measure itself. This again implies that if a does not belong to a certain (fixed) set \mathfrak{S}'_1 of zero (Haar) measure, then $\phi(a, \lambda) = \phi_2(a, \lambda)$ provided λ does not belong to a certain set $\mathfrak{S}'^{(a)}$ (depending on a) of zero λ -measure. If we change $\phi_2(a, \lambda)$ for the λ 's of \mathfrak{S}_1 into 1, we obtain its continuity in a and the equations of our Theorem for all λ 's without exceptions, and the state-

ment just now proved still holds if we replace $\mathfrak{S}_2'(a)$ by $\mathfrak{S}_2'(a) + \mathfrak{S}_1$ which is also a set of zero λ -measure.

If a does not belong to \mathfrak{S}_1' , we have $\phi(a, \lambda) = \phi_2(a, \lambda)$ except for a λ -set with zero λ -measure, that is, with a $\xi = G(\lambda)$ -image (cf. the footnote§ on page 486) of zero Lebesgue measure. This proves, as at the end of the proof of Theorem 37, that

$$(O_a f, g) = \int_{-\infty}^{+\infty} \phi_2(a, \lambda) d(E(\lambda)f, g)$$

identically in $f(x)$ and $g(x)$ if a does not belong to \mathfrak{S}_1' . Since \mathfrak{S}_1' has zero (Haar) measure, the domain of validity in a is everywhere dense. But both sides are continuous functions of a : this was shown for the left side at the beginning of the proof of Theorem 37, and follows for the right side from the continuity of $\phi_2(a, \lambda)$ in a for all λ 's. Hence our equation holds for all a 's.

Thus $\phi_2(a, \lambda)$ meets all our requirements.

THEOREM 39. *If the assumptions of Theorems 37 and 38 are satisfied, and if ab and a^{-1} are continuous in (a, b) and in a respectively, then the condition $\phi(a, \lambda) = \phi(b, \lambda)$ for all λ 's is equivalent to the condition $a = b$, and the condition $\phi(a_n, \lambda) \rightarrow \phi(a, \lambda)$ as $n \rightarrow \infty$ for all λ 's is equivalent to the condition $a_n \rightarrow a$ as $n \rightarrow \infty$.*

The first statement follows from the second by putting $a_1 = a_2 = \dots = b$. The necessity of the criterion in the second statement is obvious, as all the functions $\phi(a, \lambda)$ are continuous in a . So the only thing we need to prove is its sufficiency.

Therefore suppose that $\phi(a_n, \lambda) \rightarrow \phi(a, \lambda)$ as $n \rightarrow \infty$ for all λ 's. Then (§) in Theorem 37 shows that $(O_{a_n} f, g) \rightarrow (O_a f, g)$ as $n \rightarrow \infty$ for any $f(x)$ and $g(x)$ of our functional space. Now let \mathfrak{D} be a conditionally compact open set, and define

$$f_{\mathfrak{D}}(x) = \begin{cases} 1 & \text{for } x \text{ in } \mathfrak{D}, \\ 0 & \text{elsewhere.} \end{cases}$$

Put $f(x) = f_{\mathfrak{D}}(x)$ and $g(x) = f_{\mathfrak{D}}(ax)$. Then

$$(O_{a_n} f, g) = \int_{\mathfrak{D}} f(a_n x) \overline{f(ax)} dx,$$

$$(O_a f, g) = \int_{\mathfrak{D}} |f(ax)|^2 dx > 0,$$

and $(O_{a_n} f, g) \rightarrow (O_a f, g)$ implies that if n is sufficiently large, $(O_{a_n} f, g) \neq 0$ and therefore $f(a_n x) \overline{f(ax)} \neq 0$. Hence there is an x for which $a_n x$ and ax both belong

to \mathfrak{D} , and as $a_n a^{-1} = (a_n x)(ax)^{-1}$, $a a_n^{-1}$ can be written in the form uv^{-1} , where u and v both belong to \mathfrak{D} . As every open set has conditionally compact subsets, this holds for every open \mathfrak{D} .

Now let \mathfrak{N} be a neighborhood of a . Then we can find an open set \mathfrak{D} for which every uv^{-1} , u and v in \mathfrak{D} , belongs to \mathfrak{N} . (Here is where the extra continuity assumptions are used.) Then our result shows that if n is sufficiently large, a_n belongs to \mathfrak{N} . This means that $a_n \rightarrow a$ as $n \rightarrow \infty$.

Theorems 37 and 38, combined with Theorem 19, show that each $\phi(a, \lambda)$, when considered as an a -function, is a.p. and belongs to $[T]$. Therefore Theorem 39 proves exactly the statements of Theorem 36, case B.

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WARING'S PROBLEM FOR CUBIC FUNCTIONS*

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1. Introduction. L. E. Dickson† has proved that all integers sufficiently large are sums of nine values of $f(x) = x + \epsilon(x^3 - x)/6$, where ϵ is prime to 3. In §6 of this paper the author considers the above function $f(x)$ with $\epsilon = 3a$. For $a \equiv 0$ or $1 \pmod{3}$ he obtains the same result as Dickson obtained for ϵ prime to 3. However, for $a \equiv 2 \pmod{3}$ it is proved that every integer sufficiently large is expressible as a sum of *ten* values of $f(x)$.

Certain classes of cubic functions with the square term present are treated in §§2-5, inclusive, the results being stated in Theorems 1, 2, and 3. These results are analogous to those stated for polynomials without square term.

In the same paper Dickson showed that *all* positive integers are sums of nine values of $f(x) = (x^3 + 2x)/3$ and stated the possibility of such a theorem for $f(x) = (x^3 + 5x)/6$. Miss Frances Baker‡ proved a universal theorem for representation of weight nine by $f(x) = (x^3 + x)/2$. The only cubic functions of the form $f(x) = x + \epsilon(x^3 - x)/6$ for which it is possible to obtain a universal theorem giving representation of weight nine are those for which ϵ takes one of the values 1, \dots , 6. The author proves in §7 that every integer may be represented as a sum of fifteen values of $f(x) = x^3 + 3(x^2 - x)$ for values ≥ 0 of x . Since 41 requires fifteen values this is the best theorem obtainable.

2. Determination of all functions (1) having certain properties. We consider cubic functions of the form

$$(1) \quad f(x) = \frac{ax^3 + b_0x^2 + cx}{d}, \quad a > 0, \quad b_0 \neq 0,$$

where a , b_0 , c and d are integers having no common divisor greater than 1. Further, in order that a true Waring's Problem be considered, it is stipulated that the coefficients of $f(x)$ must satisfy the following conditions:

(a) that the values of $f(x)$ be positive integers for all integral values ≥ 0 of x ,

(b) that the function have the value 1 for some integral value $\xi \geq 0$ of x . The quantities $f(1)$, $f(2)$ and $f(3)$ will be integral if d divides each of

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† *Waring's problem for cubic functions*, these Transactions, vol. 36 (1934), pp. 1-12.

‡ *A Contribution to the Waring Problem for Cubic Functions*, Doctoral Dissertation, University of Chicago, 1934.

$a+b_0+c$, $8a+4b_0+2c$, and $27a+9b_0+3c$. Eliminate c and b_0 from these three expressions; the result of this process is that d divides $6a$. Consequently d divides each of $2b_0$ and $6c$. If d has a prime factor $p > 3$, p divides each of a , b_0 , and c contrary to hypothesis. Thus the only prime factors of d are 2 and 3. Since $f(\xi) = 1$, d must satisfy

$$(2) \quad d = a\xi^3 + b_0\xi^2 + c\xi.$$

Case I, $d = 6\omega$. Since ω divides each of a , b_0 and c , then $\omega = 1$. From (2), ξ is a positive divisor of 6. Since $d = 6$ divides $2b_0$, let $b_0 = 3b$.

I₁, $\xi = 1$. Thus, from (2), $c = 6 - a - 3b$, and

$$f(x) = \frac{a}{6}(x^3 - x) + \frac{b}{2}(x^2 - x) + x.$$

If $b < 0$, write $b = -b_1$. Then necessary and sufficient conditions that $f(x)$ satisfy property (a) above are $b > 0$, or if $b < 0$ then $0 < b_1 \leq a+2$ if $a \geq 3$, and $0 < b_1 \leq (4a+4)/3$ if $a = 1$ or 2.

I₂, $\xi = 2$. Thus $c = 3 - 4a - 6b$ and

$$f(x) = \frac{a}{6}(x^3 - x) + \frac{b}{2}(x^2 - x) + \frac{1}{2}(1 - a - b)x.$$

In order that $f(x)$ be integral a and b must have different parity. From $f(1) \geq 0$, $b \leq 1 - a$ requires $b < 0$. Let $b = -b_1$; thus $b_1 \geq a - 1$. Also, $f(3) \geq 0$, $f(4) \geq 0$ and $f(5) \geq 0$ require $b_1 \leq \eta$, where $\eta = (4a+1)/2$ if $a = 1$ and $\eta = (5a+3)/3$ if $a > 1$. Accordingly, the conditions $a - 1 \leq b_1 \leq \eta$ and $a - b_1 \equiv 1 \pmod{2}$ are necessary and sufficient that

$$f(x) = \frac{a}{6}(x^3 - x) - \frac{b_1}{2}(x^2 - x) + \frac{1}{2}(1 - a + b_1)x$$

satisfy conditions (a) and (b).

I₃, $\xi = 3$. With $c = 2 - 9a - 9b$ the conditions $f(1) \geq 0$, $f(2) \geq 0$, and $f(4) \geq 0$ require that $b = -b_1$, $b_1 > 0$ and $(5a-2)/3 \leq b_1 \leq (7a+2)/3$. The function $f(x)$ becomes

$$f(x) = \frac{a}{6}(x^3 - x) - \frac{b_1}{2}(x^2 - x) + \frac{1}{3}(1 - 4a + 3b_1)x.$$

Further, $a \equiv 1 \pmod{3}$.

I₄, $\xi = 6$. Conditions (a) and (b) require

$$f(x) = \frac{a}{6}(x^3 - x) - \frac{b_1}{2}(x^2 - x) + \frac{1}{6}(1 - 35a + 15b_1)x,$$

where a and b_1 are such that $a \equiv 3b_1 - 1 \pmod{6}$ and $(20a-1)/6 \leq b_1 \leq (13a+1)/3$.

Case II, $d=3\omega$, ω odd. As in I, $\omega=1$ and $b_0=3b$. From (2), $\xi=1$ or 3.

II₁, $\xi=1$. With $c=3-a-3b$,

$$f(x) = \frac{a}{3}(x^3 - x) + b(x^2 - x) + x.$$

Condition (a) requires that $b \geq (-3-8a)/6$.

II₂, $\xi=3$. Since $c=1-9a-9b$,

$$f(x) = \frac{a}{3}(x^3 - x) - b_1(x^2 - x) + \frac{1}{3}(1 - 8a + 6b_1)x.$$

Condition (a) requires that $a \equiv 2 \pmod{3}$ and that $(5a-1)/3 \leq b_1 \leq (7a+1)/3$.

Case III, $d=2\omega$, $(\omega, 3)=1$. According to hypothesis $\omega=1$; from (2) $\xi=1$ or 2.

III₁, $\xi=1$. Then

$$f(x) = \frac{a}{2}(x^3 - x) + \frac{b_0}{2}(x^2 - x) + x,$$

where $b_0 \geq -2-3a$.

III₂, $\xi=2$. Then

$$f(x) = \frac{a}{2}(x^3 - x) + \frac{b_0}{2}(x^2 - x) + \frac{1}{2}(1 - 3a - b_0)x,$$

where $a+b_0 \equiv 1 \pmod{2}$, $b_0 < 0$ and $3a-1 \leq -b_0 \leq 5a+1$.

Case IV, $d=1$. From (2), $\xi=1$. This requires

$$f(x) = a(x^3 - x) + b_0(x^2 - x) + x,$$

where $b_0 \geq -1-3a$.

Consider

$$(3) \quad f(x) = \frac{p}{6}(x^3 - x) + \frac{q}{2}(x^2 - x) + ux, \quad p > 0, \quad q \neq 0,$$

where p , q and u are integers satisfying necessary and sufficient conditions, as stated in Cases I-IV, that $f(x)$ be integral and ≥ 0 for all integral values ≥ 0 of x . The substitution $x=X+t$, with $q=-tp$, transforms (3) into

$$(4) \quad f(x) = F(X) + \alpha,$$

where

$$\begin{aligned} F(X) &= \frac{p}{6}(X^3 - X) + gX, \\ \alpha &= ut + \frac{1}{6}(3pt^2 - 2pt^3 - pt) \quad (\text{an integer}), \\ g &= u - \frac{1}{2}pt(t-1). \end{aligned}$$

In §§3, 4, and 5 the only values of p and q considered are those for which the above transformation is possible, i.e., values such that $q = -tp$, t an integer.

3. The functions (5) for $(p, 3) = 1$. The functions to be studied in this section are

$$(5) \quad F(X) = \frac{p}{6}(X^3 - X) + gX \quad (p > 0, (p, 3) = 1, p \text{ and } q \text{ integers}).$$

The investigation is entirely analogous to that of L. E. Dickson in his Transactions paper mentioned heretofore except that the inequalities require $X \geq |t|$. This ensures that $x \geq 0$.

We prove

THEOREM 1. *Let the triple of integers p, q, u be given satisfying the conditions stated at the end of §2, $(p, 3) = 1$, and let α be defined as under (4). Then there exist integers C and v such that every integer $\geq C \cdot 3^v + 9\alpha$ is a sum of nine values of (3) for integral values ≥ 0 of x .*

Let $|t| \leq 3^\delta$, δ being an integer ≥ 0 .

The following three lemmas are necessary.

LEMMA 1. *Let the integers t and δ be defined as above. Corresponding to any positive integer s there exists an integer m' such that s is congruent to $F(3m')$ modulo 3^n , where $|t| \leq 3m' < 3^{n+1} + 3^\delta$.*

Define Δ by

$$\Delta = F(z + 3r) - F(z) = \frac{p}{2}(3rz^2 + 9r^2z + 9r^3 - r) + 3gr.$$

It may be proved by induction that $\Delta \not\equiv 0 \pmod{3^n}$ if and only if $r \not\equiv 0 \pmod{3^n}$. Let m'' be an arbitrary integer such that $0 \leq m'' < 3^n$ and let k' be an integer such that $0 \leq k' < 3^n$, $k' < m''$. Then, if m' and k are defined by $m' = m'' + 3^{n-1}$ and $k = k' + 3^{n-1}$, we obtain $m' - k = m'' - k' \not\equiv 0 \pmod{3^n}$. Use this value $m' - k$ as an r in Δ . Then

$$F(3m') - F(3k) = F\{3k + 3(m' - k)\} - F(3k) \not\equiv 0 \pmod{3^n}.$$

Since m'' ranges over a complete residue system modulo 3^n , m' does likewise, hence the same is true of $F(3m')$. From $3m' = 3m'' + 3^s$ it follows that $3^s \leq 3m' < 3^{n+1} + 3^s$. This proves the lemma.

Lemma 2 is taken directly from the Dickson Transactions paper.

LEMMA 2. *If η is an odd constant integer, $v(\eta-v)$ is even and can be made congruent to any assigned even integer modulo 2^k by choice of an integer v .*

LEMMA 3. *If $n \geq \max(3, \delta)$, $g \leq 13p+1$, and $3m < 3^{n+1}p + 3^s$, then $F(3m) < 3^{3n}\gamma$, where*

$$(6) \quad \gamma = \frac{p}{2}(9p^3 + 9p^2 + 4p + 1).$$

Since $X^3 - X$ is monotone increasing,

$$\begin{aligned} F(3m) &= \frac{p}{6}(27m^3 - 3m) + 3gm \leq \frac{p}{2}(9m^3 - m) + 3(13p+1)m \\ &< \frac{p}{2}(9 \cdot 3^{3n}p^3 + 9 \cdot 3^{2n+s}p^2 + 9 \cdot 3^{n+2s-1}p + 9 \cdot 3^{s-3} - 3^n p - 3^{s-1}) \\ &\quad + (13p+1)(3^{n+1}p + 3^s) \\ &< \frac{3^{3n}p}{2}(9p^3 + 9p^2 + 4p + 1) = 3^{3n}\gamma. \end{aligned}$$

According to Lemma 1 every integer s may be written as $s = F(3m') + 3^n M'$, where M' is an integer. Substituting $z = 3m'$ and $r = 3^n y$ in Δ , and writing $\Delta = 3^n E$, we obtain

$$(7) \quad E = (p/2)(3yz^2 + 9 \cdot 3^n y^2 z + 9 \cdot 3^{2n} y^3 - y) + 3gy;$$

also, with $m = m' + 3^n y$,

$$F(3m) - F(3m') = F(z + 3r) - F(z) = \Delta = 3^n E.$$

Thus

$$(8) \quad s = F(3m) + 3^n M,$$

where $M = M' - E$ is an integer. Later we will choose y such that $0 \leq y < p$. With these values of y and m' the upper bound used in Lemma 3 is obtained, $|t| \leq 3m < 3^{n+1}p + 3^s$.

Consideration of the values of p , q , and u of the different sub-cases in I and II of §2 shows that $13p+1$ is the upper bound* of g ; this was used in Lemma 3. For example, in Case I₄,

* If universal theorems are desired it is advantageous to lower this upper bound, this being possible when a particular function is being considered. The upper bound of $F(3m_i)$ may usually be lowered by consideration apart from the general theory.

$$g = \frac{1}{6}(1 - 35p - 18q) - \frac{q^2}{2p} \leq -3q \leq 13p + 1,$$

since $1 - 35p < 0$ and $-3q \leq 13p + 1$.

The integer s lies in some interval

$$C \cdot 3^{3n} \leq s < C \cdot 3^{3n+3}$$

and thus in one of the sub-intervals

$$3^{i-1}C \cdot 3^{3n} \leq s_i < 3^i C \cdot 3^{3n} \quad (i = 1, 2, 3).$$

Since $f(x) \geq 0$, $F(X) \geq -\alpha$, and thus $-\alpha \leq F(3m_i) < 3^{3n}\gamma$. From $3^{i-1}C \cdot 3^{3n} \leq F(3m_i) + 3^n M_i < 3^i C \cdot 3^{3n}$ and the last inequality we obtain

$$(9) \quad (3^{i-1}C - \gamma)3^{2n} < M_i \leq 3^i C \cdot 3^{2n} + \frac{\alpha}{3^n}.$$

Six functional values to be used in the representation of s have the sum

$$(10) \quad T_i = \sum_{j=1}^3 \{F(3^n - X_j) + F(3^n + X_j)\} = p(3^{3n} + 3^n Q_i - 3^n) + 6g \cdot 3^n,$$

where $Q_i = \sum_{j=1}^3 X_j^2$. The two remaining values to be used are given by

$$(11) \quad \phi_i = F(v_i) + F(w_i) = (v_i + w_i) \left[\frac{p}{6}(v_i^2 - v_i w_i + w_i^2 - 1) + g \right].$$

Let $v_i + w_i = 3b_i 3^n$, where b_i is an odd positive integer. Then $\phi_i = 3^n B_i$, where

$$(12) \quad B_i = 3b_i \left[\frac{p}{6} \{9 \cdot 3^{2n} b_i^2 - 3v_i(3b_i \cdot 3^n - v_i) - 1\} + g \right].$$

A necessary and sufficient condition that there exist values of X of the forms $3m_i$, $3^n - X_j$, $3^n + X_j$, v_i and w_i ($j=1, 2, 3$) for which s_i is expressible as the sum of nine values of (5) is that

$$s_i = F(3m_i) + 3^n M_i = F(3m_i) + \phi_i + T_i,$$

or

$$(13) \quad \begin{aligned} 3^n M_i &= \phi_i + T_i = 3^n B_i + p(3^{3n} + 3^n Q_i - 3^n) + 6g \cdot 3^n, \\ pQ_i &= M_i - B_i - p(3^{2n} - 1) - 6g. \end{aligned}$$

The value of Q_i as defined in (13) will be shown later to be integral.

We proceed to introduce inequalities which will enable us to choose the desired constants b_i and which will ensure that the arguments of $F(X)$ are $\geq |t|$.

Choose v_i and Q_i such that

$$(14) \quad 3^{\delta} \leq v_i \leq 3b_i \cdot 3^n - 3^{\delta}, \quad 0 \leq Q_i \leq 3^{2n-2}.$$

The first inequality of (14) implies that $3^{\delta} \leq w_i \leq 3b_i \cdot 3^n - 3^{\delta}$ and thus that $v_i \geq |t|$, $w_i \geq |t|$. The second inequality requires that $X_i \leq 3^{n-1}$; thus $3^n - X_i \geq 3^n - 3^{n-1} = 2 \cdot 3^{n-1} \geq 3^{\delta} \geq |t|$ if $n \geq \delta + 1$.

Let $V_i = v_i - 3b_i \cdot 3^n / 2$. From (13), $Q_i \geq 0$ if $V_i^2 \leq A_i$, where

$$(15) \quad A_i = \left\{ \frac{M_i - p(3^{2n} - 1) - 6g}{3b_i} - g \right\} \frac{2}{p} - \frac{3}{4} b_i^2 \cdot 3^{2n} + \frac{1}{3};$$

also $Q_i \leq 3^{2n-2}$ if $V_i^2 \geq G_i$, where $G_i = A_i - 2 \cdot 3^{2n-1} / b_i$.

The inequalities (14) will be satisfied if the following are satisfied:

$$(16) \quad \begin{aligned} A_i &\geq 0, \quad A_i^{1/2} \geq V_i, \quad V_i \geq 0, \\ G_i &\geq 0, \quad G_i^{1/2} \leq V_i, \quad A_i \leq \frac{3}{2} b_i \cdot 3^n - 3^{\delta}. \end{aligned}$$

This gives the range on v_i , viz.,

$$(17) \quad G_i^{1/2} + \frac{3}{2} b_i \cdot 3^n \leq v_i \leq A_i^{1/2} + \frac{3}{2} b_i \cdot 3^n.$$

The two inequalities $G_i \geq 0$ and $A_i^{1/2} \leq (3/2)b_i \cdot 3^n - 3^{\delta}$ together with (17) are sufficient that (16) be satisfied. Accordingly $G_i \geq 0$ if

$$M_i \geq \left\{ \left(\frac{3}{4} b_i^2 3^{2n} - \frac{1}{3} \right) \frac{p}{2} + g \right\} 3b_i + 6g + p(3^{2n} + 3^{2n-2} - 1) = l_i,$$

and $A_i \leq (3b_i \cdot 3^n / 2 - 3^{\delta})^2$ if

$$M_i \leq \left\{ \left(3b_i^2 3^{2n} - 3b_i 3^{n+\delta} + 3^{2\delta} - \frac{1}{3} \right) \frac{p}{2} + g \right\} 3b_i + 6g + p(3^{2n} - 1) = L_i.$$

The relation $l_i \leq M_i \leq L_i$ will be satisfied if $l_i \leq$ the lower bound in (9) and $L_i \geq$ the upper bound in (9). From these last inequalities we obtain

$$(18) \quad \frac{l_i}{3^{2n}} + \gamma \leq 3^{i-1}C \leq \frac{L_i}{3^{2n+1}} - \frac{\alpha}{3^{3n+1}} \quad (i = 1, 2, 3).$$

When n is sufficiently large, i.e., $n - \delta$ is sufficiently large, certain terms of (18) having a power of 3 in the denominator are negligibly small. The constants b_1 , b_2 , b_3 and C are determined so as to satisfy (18) with these terms omitted; that is,

$$(19) \quad \frac{9}{8}pb_i^2 + \frac{10}{9}p + \gamma \leq 3^{i-1}C \leq \frac{3}{2}pb_i^2 + \frac{p}{3}.$$

Then for $n \geq n_1$, say, these same constants will satisfy (18). Write (19) in the form $I_i \leq 3^{i-1}C \leq S_i$.

The method used here for the choice of b_1, b_2, b_3 and C differs somewhat from that used by Dickson. For $p=1$ and $p=2$ the following choice of these constants satisfies (19):

| p | b_1 | b_2 | b_3 | C |
|-----|-------|-------|-------|------|
| 1 | 5 | 7 | 11 | 168 |
| 2 | 9 | 13 | 19 | 1760 |

Case $p \equiv 1 \pmod{3}$, $p=3e+1$, e an integer ≥ 0 . Take b_1, b_2 and b_3 as linear combinations of e . For the coefficient of e in b_1 choose the least even integer for which $I_1 \leq S_1$ as far as the coefficient of e^4 is concerned* and for the constant term the value of b_1 displayed in the above table. The coefficient of e in b_2 is taken as the least even integer for which the coefficient of e^4 in $S_2/3$ is \geq that in I_1 , the constant term being chosen as before. Similarly, choose b_3 such that the coefficient of e^4 in $S_3/9$ is \geq the maximum of the coefficients of e^4 in I_1 and $I_2/3$. Take C to be the quartic polynomial in e whose coefficients are integers not less than the corresponding coefficients in $I_3/9$ and differing from them by at most a quantity less than unity. The following constants satisfy (19):

$$\begin{aligned} b_1 &= 8e + 5, \quad b_2 = 12e + 7, \quad b_3 = 18e + 11, \\ C &= 2228e^4 + 4806e^3 + 3830e^2 + 1329e + 168. \end{aligned}$$

Case $p \equiv 2 \pmod{3}$, $p=3e+2$, e an integer ≥ 0 . Apply the method outlined above subject to the explanation given in the footnote. Choose

$$\begin{aligned} b_1 &= 10e + 9, \quad b_2 = 14e + 13, \quad b_3 = 20e + 19, \\ C &= 3740e^4 + 12,456e^3 + 15,510e^2 + 8551e + 1760. \end{aligned}$$

The coefficients in C were chosen as above from the corresponding coefficients in I_1 . This choice satisfies (19).

We prove that Q_i is an integer. From (13), $M_i \equiv M_i' - E \equiv B_i + 6g \pmod{p}$, and from (7), $E \equiv 3gy \pmod{p}$. The definition of g , the respective values of u in Cases I and II of §2, and $q \equiv 0 \pmod{p}$ give $g \equiv u \pmod{p}$, $(u, p) = 1$, and thus $(g, p) = 1$. Accordingly, y may be chosen such that E is congruent

* In some cases, with this choice of the coefficient of e in b_1 it is not possible to choose the coefficient of e^4 in C to satisfy $I_1 \leq C \leq S_1$ and $I_2 \leq 3C \leq S_2$. In these cases take the next even integer as this coefficient.

to any assigned integer modulo p , and thus such that $E \equiv M_i' - B_i - 6g \pmod{p}$. Hence Q_i is an integer.

The range D_i of values of v_i , from (17), is

$$D_i = A_i^{1/2} - G_i^{1/2} = A_i^{1/2} \{1 - (1 - \mu_i)^{1/2}\} = \frac{A_i^{1/2} \mu_i}{1 + (1 - \mu_i)^{1/2}}$$

$$= A_i^{1/2} \cdot \frac{3^{2n-1}}{b_i A_i} \cdot \frac{2}{1 + (1 - \mu_i)^{1/2}} > \frac{3^{2n-1}}{b_i A_i^{1/2}},$$

where

$$\mu_i = \frac{2 \cdot 3^{2n-1}}{b_i A_i}.$$

From (15) and (9)

$$A_i < 3^{2n} \left[\frac{2 \cdot 3^i \cdot C}{3b_i p} - \frac{2}{3b_i} - \frac{3}{4} b_i^2 + 1 \right].$$

Therefore

$$D_i > \frac{3^{n-1}}{b_i \left(\frac{2 \cdot 3^i \cdot C}{3b_i p} - \frac{2}{3b_i} - \frac{3}{4} b_i^2 + 1 \right)^{1/2}},$$

which for $n \geq n_2$, say, exceeds 8.

The quantity Q_i is representable as the sum of three integral squares. For, from (13), $2pQ_i \equiv 2M_i - 12g - 2B_i \pmod{8p}$. Take $3b_i \cdot 3^n = \eta$ in Lemma 2 and choose v_i modulo 8 such that $v_i(3b_i \cdot 3^n - v_i) \equiv 2\zeta_i \pmod{8}$, where ζ_i is an arbitrary integer. Thus, from (12),

$$2B_i \equiv -3pb_i \cdot v_i(3b_i \cdot 3^n - v_i) + 6gb_i \equiv 6gb_i - 6pb_i \zeta_i \pmod{8p}.$$

By choice of y we made $M_i \equiv 6g + 3gb_i \pmod{p}$, from which $M_i = 6g + 3gb_i + k_i p$, where k_i is an integer. Substitute these relations for M_i and $2B_i$ into the above congruence for $2pQ_i$. Thus

$$2pQ_i \equiv 2k_i p + 6pb_i \zeta_i \pmod{8p},$$

$$Q_i \equiv k_i + 3b_i \zeta_i \pmod{4}.$$

Since $(3b_i, 4) = 1$, ζ_i can be chosen such that $Q_i \equiv 1 \pmod{4}$.*

It has been shown that every integer $s \geq C \cdot 3^{2\nu}$, where $\nu = \max(3, \delta + 1, n_1, n_2)$, is a sum of nine values of (5) for values of $X \geq |t|$, the arguments of $F(X)$ being $3m_i$, $3^n - X_j$, $3^n + X_j$, v_i and w_i ($j = 1, 2, 3$). Thus every integer

* A sufficient condition that Q_i be representable as the sum of three integral squares is that Q_i be not of the form $4^a(8b+7)$, where $a \geq 0$, $b \geq 0$, a and b being integers. See Landau, *Vorlesungen über Zahlentheorie*, p. 123, Theorem 187.

$s \geq C \cdot 3^{3v} + 9\alpha$ is a sum of nine values of $f(x) = F(X) + \alpha$ for $X \geq |t|$ or $x \geq 0$. Theorem 1 is immediate.

4. Functions (5) with $p = 3p_1$, $p_1 \not\equiv 2g \pmod{3}$. The following theorem will be proved:

THEOREM 2. *Let the integers p , q and u satisfy the conditions $p = 3p_1$, $p_1 \not\equiv 2g \pmod{3}$, $q = -3p_1t$, t an integer, $u = 1$ or $(1 - 3p_1 - q)/3$, and let α be defined by (4). Then for each such triple p , q and u there exist constants C and v such that every integer $s \geq C \cdot 3^{3v} + 9\alpha$ is a sum of nine integral values ≥ 0 of (3) for integral values ≥ 0 of x .*

For this section the function defined in (5) becomes

$$(20) \quad G(X) = \frac{p_1}{2}(X^3 - X) + gX, \quad p_1 > 0, \quad g = u - \frac{3}{2}p_1t(t-1).$$

Let $|t| \leq 3^\delta$, δ being an integer ≥ 0 .

LEMMA 4. *For each integer s there exists an integer m' , $|t| \leq 3^\delta \leq m' < 3^n + 3^\delta$, such that $s \equiv G(m') \pmod{3^n}$.*

When we note that $g \equiv u \equiv 1$ or $2 \pmod{3}$ according as $G(X)$ comes under Cases III₁, IV, or III₂ of §2, and denote $m'' + 3^\delta$ by m' , $k' + 3^\delta$ by k , the proof of Lemma 4 is analogous to that of Lemma 1.

Consideration of the possible values of p , q , and u as noted in III₁, III₂, and IV gives $g \leq 1$.

LEMMA 5. *If $n \geq \delta + 1$, $g \leq 1$ and $3^\delta \leq m < 3^n(3p_1 + 1) + 3^\delta$, then $G(m) < 3^{3n}\gamma$, where*

$$\gamma = \frac{p_1}{2}[(3p_1 + 1)^3 + (3p_1 + 1)^2 + (3p_1 + 1) + 3].$$

For,

$$\begin{aligned} G(m) &< \frac{p_1}{2}[\{3^n(3p_1 + 1) + 3^\delta\}^3 - \{3^n(3p_1 + 1) + 3^\delta\}] \\ &\quad + 3^n(3p_1 + 1) + 3^\delta \\ &< \frac{3^{3n}p_1}{2}[(3p_1 + 1)^3 + (3p_1 + 1)^2 + (3p_1 + 1) + 1] + 3^n(3p_1 + 2) \\ &< 3^{3n}\gamma. \end{aligned}$$

The integer s may be written, by Lemma 4 and the method used to obtain (7) and (8), in the form

$$(21) \quad s = G(m) + 3^n M \quad (M = M' - E \text{ an integer}),$$

where

$$(22) \quad E = \frac{p_1}{2}(3yz^2 + 3 \cdot 3^n y^2 z + 3^{2n} y^3 - y) + gy.$$

The inequalities and equalities (9)–(19) inclusive along with the arguments relative to them are applicable to this section when F is replaced by G , and p by $3p_1$. The constants b_1, b_2, b_3 and C must satisfy

$$(23) \quad \frac{27}{8} p_1 b_1^3 + \frac{10}{3} p_1 + \gamma \leq 3^{i-1} C \leq \frac{9}{2} p_1 b_1^3 + p_1.$$

For $p_1 = 1, 2$, and 3 the following values of these constants satisfy (23):

| p_1 | b_1 | b_2 | b_3 | C |
|-------|-------|-------|-------|------|
| 1 | 5 | 7 | 11 | 505 |
| 2 | 9 | 13 | 19 | 5330 |
| 3 | 9 | 13 | 19 | 9061 |

Case $p_1 \equiv 0 \pmod{3}$, $p_1 = 3e$, e an integer ≥ 1 . Choose

$$b_1 = 8e + 1, \quad b_2 = 12e + 1, \quad b_3 = 18e + 1,$$

$$C = 6683e^4 + 2100e^3 + 324e^2 + 30e.$$

This set of values satisfies (23). This C is obtained from the polynomials in e which represent I_1 and $I_3/9$ when the above values of b_1, b_2 and b_3 are substituted in them.

Case $p_1 \equiv 1 \pmod{3}$, $p_1 = 3e + 1$, e an integer ≥ 0 . The values

$$b_1 = 8e + 5, \quad b_2 = 12e + 7, \quad b_3 = 18e + 11,$$

$$C = 6683e^4 + 14,432e^3 + 11,505e^2 + 3992e + 505$$

satisfy (23).

Case $p_1 \equiv 2 \pmod{3}$, $p_1 = 3e + 2$, e an integer ≥ 0 . The values

$$b_1 = 8e + 9, \quad b_2 = 12e + 13, \quad b_3 = 18e + 19,$$

$$C = 6683e^4 + 25,529e^3 + 36,223e^2 + 22,575e + 5330$$

satisfy (23).

The quantity Q , as defined by (13) with $p = 3p_1$ is an integer. For, from (22),

$$2E \equiv (2g - p_1)y \pmod{3p_1}.$$

Case p_1 is odd. From the definition of g , $2g \equiv 2u \equiv 1$ or $2 \pmod{p_1}$, and so $(2g - p_1, p_1) = 1$. This, together with $(2g - p_1, 3) = 1$, gives $(2g - p_1, 3p_1) = 1$. Accordingly, by choice of y modulo $3p_1$, $2E$ may be made congruent to any assigned integer modulo $3p_1$, from which it follows that the same is true of E .

Case p_1 is even, $p_1 = 2p_2$. Since $g - p_2 \equiv 1 \pmod{p_2}$, $(g - p_2, p_2) = 1$, and so $(g - p_2, 3p_2) = 1$. Therefore, from this and $E \equiv (g - p_2)y \pmod{3p_2}$, it follows that E may be made congruent to any assigned integer modulo $3p_2$ by choice of y modulo $3p_2$. Write $E = k + \rho \cdot 3p_2$, where k and ρ are integers, k being arbitrary and $0 \leq k < 3p_2$. Let E' be the expression E with y replaced by $y + 3p_2$. Hence

$$E' - E = 3p_2^2 [3z^2 + 3^{n+1}z(2y + 3p_2) + 3^{2n}(3y^2 + 9yp_2 + 9p_2^2) - 1] + 3gp_2.$$

When p_1 is even, g is odd, and hence $E' - E$ is an odd multiple of $3p_2$. Accordingly, if we choose y modulo $3p_1$ we obtain for each value of k two values of ρ , one even and one odd. Thus there are $3p_1$ values of $E \equiv k + \rho \cdot 3p_2 \pmod{3p_1}$, where $0 \leq k < 3p_2$, $\rho = 0, 1$, and these values are incongruent modulo $3p_1$, each to each. Hence, by choice of y modulo $3p_1$, E may be made congruent to any assigned integer modulo $3p_1$.

Choose y such that $E \equiv M_i' - B_i - 6g \pmod{3p_1}$. This choice, according to (13), makes Q_i an integer.

As in §3, for n sufficiently large, $n \geq n_2$ say, $D_i > 8$. From (13) $6p_1Q_i \equiv 2M_i - 12g - 2B_i \pmod{8p_1}$. Using Lemma 2 we may make $v_i(3b_i \cdot 3^n - v_i) \equiv 2\zeta_i \pmod{8}$, where ζ_i is arbitrary. Accordingly, $2B_i \equiv 6gb_i - 18p_1b_i\zeta_i \pmod{8p_1}$. By the above choice of y , $M_i \equiv B_i + 6g \equiv 3gb_i + 6g \pmod{3p_1}$, and hence $M_i \equiv 6g + 3gb_i + h_i \cdot 3p_1$, where h_i is an integer. Substituting these expressions for $2B_i$ and M_i in the above congruence involving $6p_1Q_i$, we obtain

$$6p_1Q_i \equiv 6h_i p_1 + 18p_1b_i\zeta_i \pmod{8p_1},$$

$$Q_i \equiv h_i + 3b_i\zeta_i \pmod{4}.$$

Choose v_i such that the corresponding value of ζ_i makes $Q_i \equiv 1 \pmod{4}$. Accordingly, Q_i is representable as the sum of three integral squares.

This completes the proof that every integer $s \geq C \cdot 3^{3\nu}$, where $\nu = \max(\delta + 1, n_1, n_2)$, and C has been determined, is a sum of nine values of $G(X)$ for $X \geq |t|$, the arguments of $G(X)$ being m_i , $3^n - X_i$, $3^n + X_i$, v_i and w_i ($j = 1, 2, 3$). Hence every $s \geq C \cdot 3^{3\nu} + 9\alpha$ is a sum of nine positive integral values of $f(x)$ given by (3) with $p = 3p_1$, the arguments of the functions $f(x)$ being derived from those above by means of $x = X + t$. Theorem 2 is immediate.

5. Functions (5) for $p = 3p_1$, $p_1 \equiv 2g \pmod{3}$. This section deals with functions of the form (20) where the restrictions on p_1 are not as strong as

those stated in Theorem 2. The results of this section include those of the last* but since the weight of the representation of integers sufficiently large has to be increased to ten, §4 gives better results for the special functions considered there.

We prove

THEOREM 3. *Let integers p_1, q and u be given satisfying the conditions $p_1 \equiv 2g \pmod{3}$, $q = -3p_1t$, t an integer, u as in Theorem 2, and let α be defined by (4). Then there exist integers C and v such that every integer $\geq C \cdot 5^{3v} + 10\alpha$ is a sum of ten values of the function (3) with this triple p_1, q and u as its coefficients, for positive integral values of x .*

The theory in this section differs from that in the preceding sections in three main particulars:

- (1) two values of the function $G(X)$, instead of one, are subtracted initially from the integer (see Lemma 7),
- (2) the prime 5 is used instead of 3,
- (3) the interval in which s lies is divided into five sub-intervals instead of three.

The fact that a lemma analogous to Lemma 1 cannot be obtained for the functions considered here, even with the modulus changed to any prime up to 23 inclusive, necessitates the first change noted above.

LEMMA 6. *The positive integers s and n being given, there exist integers k_1, k_2 and τ such that $s \equiv G(k_1 + 2\tau \cdot 5^n) + G(k_2 + 2\tau \cdot 5^n) \pmod{5^n}$, where $0 \leq k_1 < 5^n$, $0 \leq k_2 < 5^n$, $0 \leq \tau < 5$.*

This lemma is proved by induction on n . For $n=1$ we considered all possible combinations of values of p_1 and q modulo 5 and showed in each case that it is possible to choose integers k_1 and k_2 for which $s \equiv G(k_1) + G(k_2) \pmod{5}$,

$$p_1(3k_1^2 + 3k_2^2 - 2) + 4g \not\equiv 0 \pmod{5}, \quad 0 \leq k_1 < 5 \quad \text{and} \quad 0 \leq k_2 < 5.$$

For example, if $p_1 \equiv 1, g \equiv 1 \pmod{5}$, then

$$\begin{aligned} 0 &\equiv G(0) + G(0), & 1 &\equiv G(1) + G(2), & 2 &\equiv G(1) + G(1), \\ 3 &\equiv G(4) + G(4), & 4 &\equiv G(4) + G(2) \pmod{5}; \end{aligned}$$

these values of k_1 and k_2 satisfy the conditions stated. Now, as the induction hypothesis, let the integers k_1 and k_2 exist such that $s \equiv G(k_1) + G(k_2) + k \cdot 5^n$,

* The constants b_i ($i=1, \dots, 5$) and C are not calculated here for $p_1 \equiv 0 \pmod{3}$. This would have to be done before the results of Theorem 3 could be applied to all functions considered in §4.

k being an integer, where $0 \leq k_1 < 5^n$, $0 \leq k_2 < 5^n$, and $p_1(3k_1^2 + 3k_2^2 - 2) + 4g \not\equiv 0 \pmod{5}$. Then, if τ is an integer,

$$(24) \quad \begin{aligned} G(k_1 + 2\tau \cdot 5^n) + G(k_2 + 2\tau \cdot 5^n) \\ \equiv s + 5^n[\{p_1(3k_1^2 + 3k_2^2 - 2) + 4g\}\tau - k] \pmod{5^{n+1}}. \end{aligned}$$

Since, by the hypothesis for the induction, the coefficient of τ in the square bracket of (24) is prime to 5, we may choose τ modulo 5 such that the coefficient of 5^n is congruent to zero modulo 5. Thus

$$s \equiv G(k_1 + 2\tau \cdot 5^n) + G(k_2 + 2\tau \cdot 5^n) \pmod{5^{n+1}}.$$

The induction is complete.

Substitute $h_1 = k_1 + 2\tau \cdot 5^n$ and $h_2 = k_2 + 2\tau \cdot 5^n$ in Lemma 6. We obtain

LEMMA 7. For any given integers s and n there exist integers h_1 and h_2 such that $s \equiv G(h_1) + G(h_2) \pmod{5^n}$, where $0 \leq h_1 < 9 \cdot 5^n$ and $0 \leq h_2 < 9 \cdot 5^n$.

Choose δ and n such that $|t| \leq 5^\delta$ and $n \geq \delta$. Let $m_1 = h_1 + 5^n$ and $m_2 = h_2 + y \cdot 5^n$, where $1 \leq y \leq 3p_1$. Substitution of m_1 and m_2 into $G(X)$ gives $G(m_1) \equiv G(h_1) \pmod{5^n}$ and $G(m_2) = G(h_2) + 5^n E$, where

$$(25) \quad E = \frac{p_1}{2}(3h_2^2 y + 3 \cdot 5^n h_2 y^2 + 5^{2n} y^3 - y) + gy.$$

Combining these results with Lemma 7 we obtain

$$(26) \quad \begin{aligned} s &\equiv G(m_1) + G(h_2) \pmod{5^n}, \\ s &= G(m_1) + G(h_2) + 5^n M' = G(m_1) + G(m_2) + 5^n M, \end{aligned}$$

where M' is an integer, $M = M' - E$, and $|t| \leq 5^n \leq m_1 < 10 \cdot 5^n$, $|t| \leq 5^n \leq m_2 < (3p_1 + 9)5^n$.

LEMMA 8. If $n \geq 1$, $g \leq 1$, $5^n \leq m_1 < 10 \cdot 5^n$, $5^n \leq m_2 < (3p_1 + 9)5^n$, then $-2\alpha \leq G(m_1) + G(m_2) < 5^{3n}\gamma$, where

$$(27) \quad \gamma = \frac{p_1}{2}[(3p_1 + 9)^3 + 1002].$$

The statement $g \leq 1$ in §4 holds here. Substitution of the upper bounds for m_1 and m_2 into $G(m_1)$ and $G(m_2)$ gives

$$G(m_1) < \frac{1001}{2} p_1 \cdot 5^{3n}, \quad G(m_2) < \frac{p_1}{2}[(3p_1 + 9)^3 + 1]5^{3n}.$$

Since $f(x) = G(X) + \alpha \geq 0$, then $G(X) \geq -\alpha$. The statement of the lemma follows.

As stated formerly, the interval $C \cdot 5^{3n} \leq s < C \cdot 5^{3n+3}$ is subdivided such that

$$(28) \quad 3^{i-1}C \cdot 5^{3n} \leq s_i < 3^i C \cdot 5^{3n} \quad (i = 1, 2, 3, 4, 5).$$

The following replacements will transform the relations used in §3 into those used here. Replace 3^n by 5^n and p by $3p_1$ throughout; in (9) replace α by 2α ; replace $v_i + w_i = 3b_i \cdot 3^n$ by $v_i + w_i = 5b_i \cdot 5^n$; in (14) replace 3 by 5; let $V_i = v_i - 5b_i \cdot 5^n/2$. As a result we obtain inequalities corresponding to (19),

$$(29) \quad \frac{125}{8}p_1b_i^3 + \frac{78}{25}p_1 + \gamma \leq 3^{i-1}C \leq \frac{125}{6}p_1b_i^3 + p_1 \quad (i = 1, \dots, 5).$$

The constants b_i ($i = 1, \dots, 5$) and C are chosen in accordance with (29) as follows:

$$p_1 \equiv 1 \pmod{3},$$

$$b_1 = 8e + 9, \quad b_2 = 12e + 13, \quad b_3 = 18e + 19, \quad b_4 = 24e + 27, \quad b_5 = 36e + 39,$$

$$C = 30,497e^4 + 106,839e^3 + 134,404e^2 + 70,596e + 12,759;$$

$$p_1 \equiv 2 \pmod{3}, \text{ the same } b_i \text{ as for } p_1 \equiv 1 \pmod{3}, \text{ and}$$

$$C = 30,497e^4 + 117,126e^3 + 167,074e^2 + 107,572e + 27,165.$$

The quantity Q_i is an integer. For, from (25), $E \equiv gy \pmod{3p_1}$. Also $g \equiv 1 \pmod{3p_1}$ or $2g \equiv 1 \pmod{3p_1}$ according as the cases being considered are III₁ and IV or III₂; thus $(g, 3p_1) = 1$. Accordingly we may choose y such that E is congruent to any assigned integer modulo $3p_1$; choose y such that $E \equiv M_i' - B_i - 6g \pmod{3p_1}$. From (13) with the above replacements we see that Q_i is an integer.

To prove that Q_i is representable as the sum of three integral squares, we proceed as follows. Equation (13) with $p = 3p_1$ and 3 replaced by 5 gives $6p_1Q_i \equiv 2M_i - 12g - 2B_i \pmod{8p_1}$. The replacements described above give

$$B_i = 5b_i \left[\frac{p_1}{2} \{ 25b_i^2 \cdot 5^{2n} - 3v_i(5b_i \cdot 5^n - v_i) - 1 \} + g \right],$$

and thus

$$2B_i \equiv 10gb_i - 30p_1b_i\zeta_i \pmod{8p_1},$$

where ζ_i arises from the use of Lemma 2, as described in §3, and is arbitrary. Also,

$$B_i \equiv \frac{5b_i p_1}{2} (b_i^2 - 1) + 5b_i g \pmod{3p_1}.$$

Since, according to the last paragraph, $M_i \equiv B_i + 6g \pmod{3p_1}$, then

$$M_i = 6g + 5b_i g + \frac{5b_i p_1}{2}(b_i^2 - 1) + 3k_i p_1 \quad (k_i \text{ integral}).$$

This gives

$$6p_1 Q_i \equiv 30p_1 b_i \zeta_i + 6k_i p_1 \pmod{8p_1},$$

$$Q_i \equiv b_i \zeta_i + k_i \pmod{4}.$$

Choose ζ_i such that $Q_i \equiv 1 \pmod{4}$.

It has been shown that every integer $\geq C \cdot 5^{3\nu}$, where C and ν have been determined, is a sum of ten positive integral values of the function (20), p_1 and g being given, for integral values of $X \geq |t|$. The statement of the theorem follows.

6. Cubic functions without square term. L. E. Dickson, in his Transactions paper, did not consider the Waring problem for functions $f(x) = x + \epsilon(x^3 - x)/6$ where ϵ is a multiple of 3. Frances Baker* considered the problem for functions of the above form where $\epsilon = 3a$, a odd and $a \equiv 1 \pmod{3}$.

The work contained in Chapter I of Miss Baker's thesis, with two or three minor changes, holds equally well when the only restriction on a is $a \not\equiv 2 \pmod{3}$. For $a = 3e$ the following constants b_1 , b_2 , b_3 and C satisfy the inequalities in her paper corresponding to (19) of this paper:

$$b_1 = 14e + 9, \quad b_2 = 20e + 13, \quad b_3 = 30e + 19,$$

$$C = 30,497e^4 + 57,713e^3 + 36,552e^2 + 7719e + 14.$$

The proofs which it is necessary to change are contained on pages 12 and 13 of her paper, these changes being in accordance with similar work contained in the previous part of this paper. This results in the following theorem, Miss Baker's results being included:

THEOREM 4. *To each positive integer a , $a \not\equiv 2 \pmod{3}$, there correspond positive integers C and ν such that every integer $\geq C \cdot 3^{3\nu}$ is a sum of nine values of*

$$(30) \quad f(x) = x + \frac{a}{2}(x^3 - x)$$

for integral values ≥ 0 of x .

Consider the problem for the functions (30) with $a \equiv 2 \pmod{3}$. The results are stated in

THEOREM 5. *To each positive integer a , $a \equiv 2 \pmod{3}$, there correspond positive integers C and ν such that every integer $\geq C \cdot 5^{3\nu}$ is a sum of ten values of (30) for integral values ≥ 0 of x .*

* A Contribution to the Waring Problem for Cubic Functions, Doctoral Dissertation, University of Chicago, 1934.

The proof of Theorem 5 parallels that of Theorem 3, the major changes being necessitated by the requirement $x \geq 0$ instead of $X \geq |t|$ as heretofore required in this paper. This is due to the fact that it is not necessary to transform linearly our original function into another without square term before the theory is applied. By choosing $m_1 = h_1$ instead of $m_1 = h_1 + 5^n$ we lower the upper bound for $G(m_1)$ as contained in Lemma 7. The inequalities corresponding to (14) would be

$$0 \leq v_i \leq 5b_i \cdot 5^n \quad \text{and} \quad 0 \leq Q_i \leq 5^{2n}.$$

Choose the constants b_i ($i = 1, \dots, 5$) and C so that

$$\frac{125}{8}ab_i^3 + \gamma + 6a \leq 3^{i-1}C \leq \frac{125}{6}ab_i^3 + a \quad (i = 1, \dots, 5)$$

are satisfied, the choice being

$$b_1 = 8e + 5, \quad b_2 = 12e + 7, \quad b_3 = 16e + 11, \quad b_4 = 24e + 15,$$

$$b_5 = 34e + 21,$$

$$C = 27,365e^4 + 66,465e^3 + 62,661e^2 + 27,136e + 4681.$$

The remainder of the proof is so similar to that of Theorem 3 that it is needless to repeat it here.

Theorems 4 and 5 complete the general theory for cubic functions without square term.

7. Universal theorems. A universal theorem for weight nine is possible for only two functions of the type considered in Theorems 4 and 5. The function (30) has the values

$$f(0) = 0, \quad f(1) = 1, \quad f(2) = 2 + 3a.$$

If $f(2) \geq 12$, the integer 11 has a representation of weight eleven as a sum of functions (30), i.e., $11 = 11f(1)$. Accordingly, $a = 1, 2$, and 3 are the only values of a for which universal theorems of weight ten are possible. The case $a = 1$ was considered by Miss Baker* and a universal theorem of weight nine was obtained. The case $a = 2$ reduces to the problem of cubes† for which the result is well known. For $a = 3$, $f(2) = 11$, and $21 = f(2) + 10f(1)$, a representation of weight eleven.

We prove

* Loc. cit.

† L. E. Dickson, *Simpler proofs of Waring's theorem on cubes, with various generalizations*, these Transactions, vol. 30 (1928), pp. 1-18.

THEOREM 6. Every integer ≥ 0 is a sum of fifteen values of

$$(31) \quad f(x) = x^3 + 3(x^2 - x)$$

for integral values ≥ 0 of x .

This function (31) is the function (3) for $p=3p_1$, $p_1=2$, $q=6$, $u=1$. These values of p and p_1 satisfy the hypothesis of Theorem 3, and so, for ν sufficiently large, every integer $\geq C \cdot 5^{3\nu} + 50$ is a sum of ten values of (31). The integers C and ν calculated from the theory are $C=27,165$ and $\nu=8$. We have to prove that all integers $< 27,165 \times 5^{24} + 50$ can be represented as a sum of fifteen values of (31).

A table of minimum weights of the representations of integers 1-1000 shows that integers 298-1000, inclusive, have weight 8; 169, 83 and 41 are the largest integers of weights 11, 13 and 15, respectively.

Apply the following theorem* to the data given above:

THEOREM 7. Let a polynomial $f(x)$ take integral values ≥ 0 for all integers $x \geq 0$; let $f(x+1) - f(x)$ increase with x . Suppose that every integer n for which $l < n \leq g + f(0)$ is a sum of $k-1$ values of $f(x)$ for integers $x \geq 0$. Let m be the maximum integer for which $f(m+1) - f(m) < g - l$. Then every integer N for which $l + f(0) < N \leq g + f(m+1)$ is a sum of k values of $f(x)$ for integers $x \geq 0$.

Seven applications of Theorem 7 lead to the result that all integers \leq a constant greater than $27,165 \times 5^{24} + 50$ are sums of fifteen values of (31). The proof of Theorem 6 is complete.

8. Generalization.† In this section we show that the theory contained in §§3, 4 and 5 holds for values of the parameters p , q and u of (3) subject only to the condition $q = -tp$. We shall consider §3. If, in Lemma 3, we use $g \leq |g|$ in place of $g \leq 13p+1$, the only effect is to alter certain terms of (18). When we pass to (19) these terms drop out, so the same set of b_1, b_2, b_3 and C will suffice. In the proof that Q_i is an integer there is the restriction $(g, p) = \theta = 1$. However, if $\theta > 1$, $F(X)$, and likewise any sum of values of $F(X)$, would be a multiple of θ . As stated at the beginning of §2, functions of this type do not enter into the theory. Accordingly, the parameter g is arbitrary and the result stated above follows for §3. Similar results for §§4 and 5 are immediate.

* L. E. Dickson, *Waring's problem for cubic functions*, these Transactions, vol. 36 (1934), pp. 1-12. This is Dickson's Theorem 3.

† Section appended June 30, 1934. This is analogous to theory recently obtained by L. E. Dickson.

THE APPLICATION OF THE THEORY OF ADMISSIBLE NUMBERS TO TIME SERIES WITH CONSTANT PROBABILITY*

BY

FRANCIS REGAN

I. INTRODUCTION

The idea of admissible numbers in probability is a new one. Its development has come about during the past few years through the researches of Copeland.† Admissible numbers furnish a method for testing the consistency of the assumptions of the theory of probability and also serve as a guide for setting up sets of assumptions. The problem of testing consistency is a very extensive one, since in almost every branch of the theory of probability new assumptions are made.

This paper is an extension of the concept of admissibility to time series. A time series is a sequence of occurrences, which are represented by a set of points on a time axis. These points must satisfy a certain law; namely, that there is a definite probability of getting a point in any interval. This probability may vary according to the length of the interval, or according to the length of the interval and the position of the interval.

In order that a probability situation may have any meaning from the statistical point of view, it must be capable of being repeated a large number of times under similar circumstances. A type of time series which is dealt with here has these repetitions given directly by the time series. In order that this may be the case, the probability of obtaining a point in any interval must be a periodic function of the position of the interval; that is, the coordinate of the left hand extremity.

Since this set of points possesses the property necessary to use the statistical point of view of probability without any modifications, then if it is to satisfy the fundamental assumptions of the theory of probability, it is neces-

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† A. H. Copeland, *Admissible numbers in the theory of probability*, American Journal of Mathematics, vol. 50, No. 4, Oct., 1928. *Independent event histories*, loc. cit., vol. 51, No. 4, Oct., 1929. *Admissible numbers in the theory of geometrical probability*, loc. cit., vol. 53, No. 1, Jan., 1931. *The theory of probability from the point of view of admissible numbers*, Annals of Mathematical Statistics, vol. 3, No. 3, Aug., 1932.

sary that sequences of successes and failures be represented by the digits of admissible numbers. The number of conditions imposed upon these points has the power of the continuum, since, for every interval, a different set of conditions is obtained. It is the purpose of this paper to show that these conditions are consistent and can be satisfied.

The time series will be represented by the set of points $\tau_1 < \tau_2 < \dots < \tau_i < \dots$. Let $f(\alpha, \tau, t)$ be the probability of α points of the series lying in an interval of length τ , beginning at time t . It follows from the past discussion, that it is necessary for $f(\alpha, \tau, t)$ to be periodic in t . The k th interval I_k of the series is defined as $t + (k-1)\Lambda < h \leq t + \tau + (k-1)\Lambda$, where Λ is a period or an integral multiple of a period and $t + \tau \leq \Lambda$. Let $x(\alpha, \tau, t, \Lambda)$ be a number such that its k th digit is one if there are exactly α points in I_k and zero otherwise. Since this paper deals with constant probability, it may be seen that the probability of α points of the series lying in τ is independent of t , and that every Λ is a period. In this case, the time series $\tau_1 < \tau_2 < \dots < \tau_i < \dots$ will be constructed so that the number $x(\alpha, \tau, t, \Lambda)$ is an element of the set $A[f(\alpha, \tau, t)]^*$ for every α, τ, t and Λ , where $\tau = m \cdot 2^{-\sigma+1}$, $t = r \cdot 2^{-\sigma+1}$ and $\Lambda = \rho \cdot 2^{-\sigma+1}$, where α, ρ, m, r and σ are positive integers and $r + m \leq \rho$.[†] The construction of this series that will be given is, in a sense, a numerical construction and is probably about the simplest mathematical construction possible.

The case where the probability of α points lying in τ is dependent on t , will be discussed in a subsequent paper.

II. PROBABILITY OF AT LEAST ONE POINT LYING IN AN INTERVAL

1. When the probability does not depend upon the beginning time, it is seen[‡] that the probability of α points lying in an interval of length τ is

$$f(\alpha, \tau, 0) = [e^{-m\tau}(m\tau)^\alpha]/\alpha!.$$

Since m is the ratio constant which determines the unit of time, there will be no loss in generality by choosing m equal to one.

For the present, we shall be concerned with the case in which at least one point lies in τ . A geometrical construction of this time series may be formed

* The set $A[f(\alpha, \tau, t)]$ is the set of all admissible numbers associated with the probability function $f(\alpha, \tau, t)$.

† It should be observed that the function $f(\alpha, \tau, t)$ is independent of t , hence $f(\alpha, \tau, t) = f(\alpha, \tau, 0)$. Also the number $x(\alpha, \tau, t, \Lambda)$ is independent of t , since it is an element of the set $A[f(\alpha, \tau, t)]$; that is, the numbers $x(\alpha, \tau, 0, \Lambda)$ and $x(\alpha, \tau, t, \Lambda)$ are members of the same set $A[f(\alpha, \tau, 0)]$, provided that $\tau = m \cdot 2^{-\sigma+1}$, $\Lambda = \rho \cdot 2^{-\sigma+1}$, $t = 0$ in the first number and $t = r \cdot 2^{-\sigma+1}$ in the second, where $r + m \leq \rho$.

‡ See Fry, *Probability and its Engineering Uses*, pp. 216-27 and pp. 232-35. A more rigorous proof than given by Fry will be published later in a joint paper by the author and Professor A. H. Copeland.

to illustrate the phenomena arising from the probability function $[1-f(0, \tau, 0)]$. Construct a set of segments of length τ on a line. A set of points $\tau_1 < \tau_2 < \dots < \tau_i < \dots$ is distributed along this line in such a manner that the probability of at least one point lying in an interval of length τ is $[1-f(0, \tau, 0)]$. Here the k th interval I_k is $(k-1)\tau < h \leq k\tau$.* In I_k , there may be no points or at least one. Let us define $\sim x(0, \tau, 0, \tau)^\dagger$ such that its k th digit is one if there is at least one point in I_k and zero otherwise. The success ratio is

$$p_N[\sim x_r] = \sum_{k=1}^N \frac{x^{(k)}}{N}, \ddagger$$

and we demand that $p(\sim x_r) = 1-f(0, \tau, 0)$, where

$$p(\sim x_r) = \lim_{N \rightarrow \infty} p_N(\sim x_r).$$

A physical illustration of a time series of this character is the occurrences of quakes in a certain region. Let us take for example, the quakes actually occurring from 1490 A.D. to 1930 A.D. in the region which is now Mexico, assuming that records of such could have been kept. These data would be applicable to the above. Let τ represent the time span of ten years and the set of points properly graphed the quakes. Here $\sim x_r$ will be defined such that its k th digit is one if there is at least one quake in the k th decade and zero otherwise. The success ratio is

$$p_{44}[\sim x_r] = \sum_{k=1}^{44} \frac{x^{(k)}}{44}.$$

If these records of quakes were kept up indefinitely this success ratio would approach $(1-e^{-10m})$.

2. We have constructed a time series which illustrates exactly how a physical event would be dealt with. It becomes our problem to build up an imaginary series which logically follows from the physical, but one that satisfies the laws of admissibility. In testing the consistency of the assumptions made in developing the probability function, the time series must possess certain properties, one of which is an unlimited number of occurrences. These properties will be exhibited in the next paragraph where the series will be developed.

The time series may be represented by a set of points $\tau_1 < \tau_2 < \dots < \tau_i$

* Here $t=0$ and $\tau=\Delta$.

† The symbol \sim means "not." Let $\sim x(0, \tau, 0, \tau)$ be represented by $\sim x_r$.

‡ See Copeland, *Admissible numbers in the theory of probability*, American Journal of Mathematics, vol. 50, p. 536.

$< \dots$ on the positive τ -axis. These points satisfy the following law. Let this axis be divided into periods or intervals and let the k th interval I_k be defined as $(k-1)\rho \cdot 2^{-\sigma+1} < h \leq k \cdot \rho \cdot 2^{-\sigma+1}$, where ρ and σ are positive integers. We are considering the case where $l=0$, $\Lambda=\tau=\rho \cdot 2^{-\sigma+1}$. Let $\sim x_\tau$ be such that the k th digit is one if there is at least one point of the time series in I_k and zero otherwise. The points of the time series must possess the property that $p[\sim x_\tau] = [1 - e^{-\tau}]$ for every ρ and σ .

3. We shall now construct the imaginary series. We shall construct a finite set of points of the series in such a manner that the conditions described above are satisfied to a certain degree of approximation when applied to unit intervals. We will call this set of points the first stage. The set of points for the second stage will be selected in such a manner that the conditions will hold to a certain degree of approximation when applied to unit intervals or intervals of length one half. For the third stage, the conditions can be applied to intervals of length 1, $\frac{1}{2}$, or $\frac{1}{4}$. We let N_1 be the number of intervals employed in the first stage. In the second stage, there will be N_2 unit intervals or $2 \cdot N_2$ half unit intervals, etc. The choice of numbers $N_1, N_2, \dots, N_s, \dots$ will be determined at a later point in the paper. We shall construct the N_1 contiguous unit intervals of the first stage on the τ -axis, beginning at time zero, the origin. The points of the time series will occupy the mid-points of these intervals. Let X_1 be a member of the set $A(1 - e^{-1})$ and let the i th unit interval contain a point of the time series if and only if the i th digit of X_1 equals one. If j_i is the smallest number j such that $j \cdot p_j(X_1) = i$, the points which we have constructed have the coordinates $\tau_i = j_i - \frac{1}{2}$. We have now constructed the first $N_1 \cdot p_{N_1}(X_1)$ points of the time series.

To the N_1 unit intervals we will add $2 \cdot N_2$ intervals of length $\frac{1}{2}$. We will allow the next set of points of the time series to occupy the mid-points of these intervals. Those intervals which will contain points of the time series will be determined by the digits of a number X_2 which is a member of the set $A(1 - e^{-2^{-1}})$. In general the s th set of points will occupy the mid-points of $2^{s-1} \cdot N_s$ intervals of length 2^{-s+1} . Those intervals which will contain points of the series will be determined by the digits of a number X_s which is a member of the set $A(1 - e^{-2^{-s+1}})$.

Let $\nu_s = N_1 + N_2 + \dots + N_{s-1}$ ($\nu_1 = 0$) and

$$\gamma_s = \sum_{k=1}^{s-1} 2^{k-1} \cdot N_k \cdot p_{2^{k-1} \cdot N_k}(X_k) \quad (\gamma_1 = 0).$$

Then after the s th set of points has been chosen, we have determined γ_{s+1} points and these points all lie in the interval from 0 to ν_{s+1} . Let s_i be such that $\gamma_{s_i} < i \leq \gamma_{s_i+1}$ and let j_i be the smallest j such that $j \cdot p_j(X_{s_i}) = i - \gamma_{s_i}$. Then the coordinates of the points of the time series are given by the equations

$$(1) \quad \tau_i = \nu_{s_i} + j_i \cdot 2^{-s_i+1} - 2^{-s_i}.$$

For example suppose that $i = \gamma_2 + 1$; then $s_i = 2$ and the point τ_i lies to the right of the point ν_2 . Let us suppose further that 3 is the smallest integer j such that $j \cdot p_i(X_2) = i - \gamma_{s_i} = 1$. Then τ_i is the mid-point of the third half unit interval to the right of the point ν_2 ; that is, $\tau_i = \nu_2 + 3 \cdot 2^{-1} - 2^{-2}$.

Let us consider a further example. The time series consists of the sequence of points $\tau_1, \tau_2, \dots, \tau_i, \dots$ and the associated number $\sim x(0, 1, 0, 1) = \sim x_1$ has the following sequence of digits:

$$\begin{aligned} & \cdot X_1^{(1)} X_1^{(2)} \cdots X_1^{(N_1)} (X_2^{(1)} \vee X_2^{(2)}) (X_2^{(3)} \vee X_2^{(4)}) \cdots \\ & (X_2^{(2N_2-1)} \vee X_2^{(2N_2)}) (X_3^{(1)} \vee X_3^{(2)} \vee X_3^{(3)} \vee X_3^{(4)}) \cdots, \end{aligned}$$

where $X_2^{(1)} \vee X_2^{(2)} = 1$ whenever one or both of the numbers is equal to one, and $X_2^{(1)} \vee X_2^{(2)} = 0$ otherwise, etc. We have the equation $p(X_1) = 1 - e^{-1}$ and we will prove that

$$\begin{aligned} p[(1/2)X_2 \vee (2/2)X_2] \\ = p[(1/4)X_3 \vee (2/4)X_3 \vee (3/4)X_3 \vee (4/4)X_3] = \cdots = 1 - e^{-1}. \end{aligned}$$

We will show that, for a proper choice of the numbers $N_1, N_2, \dots, N_s, \dots$, $p(\sim x_1) = 1 - e^{-1}$. The numbers $\sim x_{1/2}, \sim x_{1/4}$, etc. can be treated in a similar manner.

4. The time series is defined when the numbers N_s are defined. We shall prove the following theorem.

THEOREM 1. *If the time series consists of the points $\tau_1, \tau_2, \tau_3, \dots$ satisfying the conditions*

$$\tau_i = \nu_{s_i} + j_i \cdot 2^{-s_i+1} - 2^{-s_i}, \quad \nu_s = N_1 + N_2 + \cdots + N_{s-1} \quad (\nu_1 = 0),$$

(H₁) s_i is such that $\gamma_{s_i} < i \leq \gamma_{s_i+1}$,

$$\gamma_s = \sum_{k=1}^{s-1} 2^{k-1} \cdot N_k \cdot p_{2^{k-1} \cdot N_k}(X_k) \quad (\gamma_1 = 0),$$

X_s is a member of the set $A(1 - e^{-2^{-s+1}})$,

j_i is the smallest j such that $j \cdot p_i(X_{s_i}) = i - \gamma_{s_i}$,

then the numbers $N_1, N_2, \dots, N_s, \dots$ can be so chosen that for every τ satisfying the conditions

$$\tau = \rho \cdot 2^{-\sigma+1},$$

(H₂) ρ and σ are positive integers and $0 < \rho \leq 2^{2\sigma-1}$,

the corresponding number $\sim x$, is an element of the set $A(1 - e^{-\tau})$.

† See Lemma 1 in the second paragraph below.

We are considering N_s/τ digits of each of these numbers. It may be seen from these numbers that the first digit of U_s is one if and only if at least one of the digits $X_s^{(1)}, X_s^{(2)}, \dots, X_s^{(\rho \cdot 2^{s-\sigma})}$ is one. By analyzing the digits $X_s^{(1)}$ to $X_s^{(\rho \cdot 2^{s-\sigma})}$ inclusive, it is seen that if at least one of these is one, then there is at least one point of the time series in the first period of the s -stage; that is, of the interval $I_{(\nu_s/\tau)+1}$. Hence the $\{(\nu_s/\tau)+1\}$ th digit of $\sim x_r$ is the same as the first digit of U_s . This process may be continued for the N_s/τ digits of U_s .

Previously, we made use of the following lemma:

LEMMA 1. If X_s is an element of $A(p)$, the number $\{\sum_{q=1}^n (q/n) X_s \cdot \mathbf{v}\}$ is a member of the set $A[1 - (1-p)^n]$.

We know that

$$(a) \quad \sim \sum_{q=1}^n \left(\frac{q}{n}\right) X_s \cdot \mathbf{v} = \prod_{q=1}^n \sim \left(\frac{q}{n}\right) X_s \cdot *$$

Since the numbers $(q/n) X_s$ ($q=1, 2, \dots, n$) are independent, then $\sim (q/n) X_s$ ($q=1, 2, \dots, n$) are independent.† From (a), we get

$$\sum_{q=1}^n \left(\frac{q}{n}\right) X_s \cdot \mathbf{v} = \sim \prod_{q=1}^n \sim \left(\frac{q}{n}\right) X_s \cdot,$$

and hence, we have

$$p \left[\sum_{q=1}^n \left(\frac{q}{n}\right) X_s \cdot \mathbf{v} \right] = 1 - p \left[\prod_{q=1}^n \sim \left(\frac{q}{n}\right) X_s \cdot \right],$$

or

$$p \left[\sum_{q=1}^n \left(\frac{q}{n}\right) X_s \cdot \mathbf{v} \right] = [1 - (1-p)^n].$$

We see that $\sum_{q=1}^n (q/n) X_s \cdot \mathbf{v}$ has the desired probability, but we must now show that

$$p \left[\prod_{i=1}^k \left(\frac{r_i}{m}\right) \left\{ \sum_{q=1}^n \left(\frac{q}{n}\right) X_s \cdot \mathbf{v} \right\} \right] = [1 - (1-p)^n]^k,$$

* Here the symbol $\sim (q/n) X_s \cdot$ represents the number $\{ \sim (1/n) X_s \cdot \sim (2/n) X_s \cdot \dots \sim (n/n) X_s \cdot \}$. Such symbols used in this paper will have similar meanings. For the truth of this equality, see Copeland, *The theory of probability from the point of view of admissible numbers*, Annals of Mathematical Statistics, vol. 3, Aug., 1932, p. 149.

† See Copeland, *Admissible numbers in the theory of probability*, American Journal of Mathematics, Oct., 1928, Theorem 5, p. 542.

for every set of numbers, r_1, r_2, \dots, r_k , such that $0 < r_i \leq m$ and $r_i \neq r_j$ if $i \neq j$.
Since

$$\sum_{q=1}^n \left(\frac{q}{n}\right) X_s \mathbf{v} = \sim \prod_{q=1}^n \sim \left(\frac{q}{n}\right) X_s \cdot,$$

then

$$\left(\frac{r_i}{m}\right) \left[\sum_{q=1}^n \left(\frac{q}{n}\right) X_s \mathbf{v} \right] = \sim \prod_{q=1}^n \sim \left(\frac{q + (r_i - 1)n}{mn}\right) X_s \cdot^*.$$

Then

$$\prod_{i=1}^k \left(\frac{r_i}{m}\right) \left[\sum_{q=1}^n \left(\frac{q}{n}\right) X_s \mathbf{v} \right] = \prod_{i=1}^k \sim \prod_{q=1}^n \sim \left(\frac{q + (r_i - 1)n}{mn}\right) X_s \cdot.$$

The numbers r_i are chosen such that for every set r_1, r_2, \dots, r_k , we have $0 < r_i \leq m$ and $r_i \neq r_j$ if $i \neq j$. Then the numbers $\sim \{ [q + (r_i - 1)n] / (mn) \} X_s$ are independent, and hence the numbers $\prod_{q=1}^n \sim \{ [q + (r_i - 1)n] / (mn) \} X_s \cdot^\dagger$ are independent. We now conclude that

$$p \left[\prod_{i=1}^k \left(\frac{r_i}{m}\right) \left\{ \sum_{q=1}^n \left(\frac{q}{n}\right) X_s \mathbf{v} \right\} \right] = [1 - (1 - p)^n]^k.$$

Therefore the lemma is proved.

The following fundamental lemma will enable us to determine the choice of the numbers N_s .

LEMMA 2. *If there exists a sequence $U_1, U_2, \dots, U_s, \dots$ and a monotonic non-increasing sequence ϵ_s such that $\lim_{s \rightarrow \infty} \epsilon_s = 0$, and if $H_1, H_2, \dots, H_s, \dots$ and $J_1, J_2, \dots, J_s, \dots$ are two sets of integers such that*

$$(a) \quad |p_N(U_{s+1}) - p| < \epsilon_s/3, \text{ if } N \geq H_s,$$

and

$$(b) \quad |p_N(U_s) - p| + [\mu_s + H_s]/J_s < \epsilon_s/3, \text{ if } N \geq J_s,$$

where $\mu_s = J_1 + J_2 + \dots + J_{s-1}$ ($\mu_1 = 0$), and if x is such that its digits $\mu_s + 1$ to μ_{s+1} are the same as the digits 1 to J_s of U_s , where $s_0 \leq s$, then

$$(c) \quad |p_N(x) - p| < \epsilon_s,$$

provided $\mu_{s+1} \leq N \leq \mu_{s+2}$ and $s_0 \leq s$.

* See Copeland, *Admissible numbers in the theory of probability*, American Journal of Mathematics, Oct., 1928, p. 539.

† Since these numbers are independent, their negations are also.

By hypothesis, the digits μ_s+1 to μ_{s+1} of x are the same as the digits 1 to J_s of U_s , where $s_0 \leq s$. Hence, we have

$$x = .x^{(1)}x^{(2)} \dots x^{(k)} \dots x^{(\mu_s)} \dots x^{(\mu_s+1)} \dots x^{(\mu_{s+1})} \dots$$

and

$$U_s = .u_s^{(1)}u_s^{(2)} \dots u_s^{(k)} \dots u_s^{(J_s)} \dots,$$

where the $x^{(k)}$ and $u_s^{(k)}$ are ones or zeros and $x^{(\mu_s+1)} = u_s^{(1)}$, $x^{(\mu_s+2)} = u_s^{(2)}$, \dots , $x^{(\mu_{s+1})} = u_s^{(J_s)}$, where $s_0 \leq s$.*

Since

$$(d) \quad p_N(x) = \sum_{k=1}^N \frac{x^{(k)}}{N} \quad \text{and} \quad p_{J_s}(U_s) = \sum_{k=1}^{J_s} \frac{u_s^{(k)}}{J_s},$$

it follows that

$$|N \cdot p_N(x) - J_s \cdot p_{J_s}(U_s)| \leq N - J_s,$$

when $N \geq \mu_{s+1}$.

Hence

$$(e) \quad \left| p_N(x) - \left[\frac{J_s}{N} \right] p_{J_s}(U_s) \right| \leq \frac{N - J_s}{N} \leq \frac{\mu_s + H_s}{J_s} < \frac{\epsilon_s}{3},$$

if $\mu_{s+1} \leq N \leq \mu_{s+1} + H_s$.

Combining (b) and (e), we obtain

$$(f) \quad \left| p_N(x) - \left[\frac{J_s}{N} \right] p \right| < \frac{\epsilon_s}{3} \left[1 + \left(\frac{J_s}{N} \right) \right] \leq 2 \frac{\epsilon_s}{3}.$$

But

$$(g) \quad \left| p - \left[\frac{J_s}{N} \right] p \right| = \frac{N - J_s}{N} \cdot p < \frac{\mu_s + H_s}{J_s} < \frac{\epsilon_s}{3},$$

if $\mu_{s+1} \leq N \leq \mu_{s+1} + H_s$.

Therefore, adding (f) and (g) we get

$$(h) \quad |p_N(x) - p| < \epsilon_s,$$

if $\mu_{s+1} \leq N \leq \mu_{s+1} + H_s$.

We also know that

$$(i) \quad p_{N-\mu_{s+1}}(U_{s+1}) = \sum_{k=1}^{N-\mu_{s+1}} \frac{u_{s+1}^{(k)}}{N-\mu_{s+1}},$$

provided $N \geq \mu_{s+1}$.

* It may be noted that the first μ_{s_0} digits may be arbitrarily defined for x .

Using (d) and (i), we may form

$$(j) \quad |Np_N(x) - J_s p_{J_s}(U_s) - (N - \mu_{s+1})p_{N-\mu_{s+1}}(U_{s+1})|,$$

and find for what values of N this expression will be less than $N\epsilon_s/3$. We find that

$$(k) \quad |p_N(x) - [J_s/N]p_{J_s}(U_s) - ([N - \mu_{s+1}]/N)p_{N-\mu_{s+1}}(U_{s+1})| \\ \leq [N - J_s - (N - \mu_{s+1})]/N = \mu_s/N \leq \epsilon_s/3,$$

if $\mu_{s+1} + H_s \leq N \leq \mu_{s+2}$.

Combining (a), (b) and (k), we obtain

$$(l) \quad |p_N(x) - [\{J_s + N - \mu_{s+1}\}/N]p| < [2 - \mu_s/N]\epsilon_s/3 < 2\epsilon_s/3.$$

But

$$(m) \quad |p - [\{J_s + N - \mu_{s+1}\}/N]p| = (\mu_s/N)p < \epsilon_s/3,$$

if $\mu_{s+1} + H_s \leq N \leq \mu_{s+2}$.

Adding (l) and (m), we get

$$(n) \quad |p_N(x) - p| < \epsilon_s,$$

if $\mu_{s+1} + H_s \leq N \leq \mu_{s+2}$.

It follows from (h) and (n) that

$$|p_N(x) - p| < \epsilon_s,$$

provided $\mu_{s+1} \leq N \leq \mu_{s+2}$, where $s_0 \leq s$.

In order to prove Theorem 1, we shall put it in the form of the fundamental lemma.

Since U_s is an element of $A(1 - e^{-\tau})$, it follows that

$$p \left[\prod_{i=1}^k \left(\frac{r_i}{n} \right) U_s \right] = [1 - e^{-\tau}]^k.$$

Let $\epsilon_1, \epsilon_2, \dots, \epsilon_s, \dots$ be a decreasing sequence of positive numbers having the limit zero. We can choose two sets of integers $M_1, M_2, \dots, M_s, \dots$ and $N_1, N_2, \dots, N_s, \dots$ such that

$$(1') \quad \left| p_N \left[\prod_{i=1}^k \left(\frac{r_i}{n} \right) U_{s+1} \right] - [1 - e^{-\tau}]^k \right| < \frac{\epsilon_s}{3},$$

whenever $N \geq M_s/(n\tau)$, and

$$(2') \quad \left| p_N \left[\prod_{i=1}^k \left(\frac{r_i}{n} \right) U_s \right] - [1 - e^{-\tau}]^k \right| + \frac{\nu_s + M_s}{N_s} < \frac{\epsilon_s}{3},$$

when $N \geq N_s/(n\tau)$, and where $\nu_s = N_1 + N_2 + \dots + N_{s-1}$. The sequence M_s must be chosen so that $M_s/(n\tau)$ is an integer. It has been stated that $\nu_s/(n\tau)$ and $N_s/(n\tau)$ are integers.

It is understood that conditions (1') and (2') hold for every set of positive integers $n, \sigma, \rho, r_1, r_2, \dots, r_k$, such that $n \leq s, \sigma \leq s, 0 < \rho \leq 2^{2\sigma-1}, r_i \leq n$ and $k \leq n$, where $r_i \neq r_j$ if $i \neq j$.

The first ν_s/τ digits of $\sim x_r$ are found directly from the time series, for instance the first digit of $\sim x_r$ is one if there is at least one point of the time series in I_1 . When $\sigma \leq s$, we have shown that the digits $(\nu_s/\tau) + 1$ to ν_{s+1}/τ of $\sim x_r$ are the same as the digits 1 to N_s/τ of U_s . Then the digits $(\nu_s/(n\tau)) + 1$ to $\nu_{s+1}/(n\tau)$ of $[\Pi_{i=1}^k (r_i/n) \sim x_r]$ are the same digits as 1 to $N_s/(n\tau)$ of $[\Pi_{i=1}^k (r_i/n) U_s]$.

We may now make the comparison of the theorem with the lemma as follows. The numbers

$$\left[\prod_{i=1}^k \left(\frac{r_i}{n} \right) U_s \right], [1 - e^{-\tau}]^k, \frac{M_s}{n\tau}, \frac{N_s}{n\tau}, \frac{\nu_s}{n\tau}, \sigma$$

and

$$\left[\prod_{i=1}^k \left(\frac{r_i}{n} \right) \sim x_r \right]$$

have taken the places of $U_s, p, H_s, J_s, \mu_s, s_0$ and x respectively. The numbers N_s have now been selected and hence the time series is determined so that for every ρ and $\sigma, \sim x_r$ is an element of $A(1 - e^{-\tau})$.

5. Since a time series has been obtained so that the conditions of admissibility are satisfied in the period τ , it is natural to inquire whether similar conditions hold when, within the period τ , sub-intervals of the form $2^{-\sigma+1}$ are omitted from consideration.

We shall consider m intervals of the form $2^{-\sigma+1}$ in the period τ . Let these intervals begin at $\rho_i \cdot 2^{-\sigma+1}, i = 1, 2, \dots, m$, where $\rho_m + 1 \leq \rho$ and $m \leq \rho$. The sub-intervals of I_k will be defined as follows: $(k-1)\tau + \rho_i \cdot 2^{-\sigma+1} < h \leq (k-1)\tau + (\rho_i + 1)2^{-\sigma+1}, i = 1, 2, \dots, m$. Corresponding to $\sim x_r$ of §2, we defined a number $\sim x_r^*$ such that its k th digit is one if there exists at least one point of the time series in the m intervals of I_k and zero otherwise. The points of the time series must possess the property that $p[\sim x_r^*]$ is equal to $(1 - e^{-\tau'})$, for every integral value of m, σ and ρ , where $m \leq \rho$.

Since the same type of time series is used for this case, then this series is defined when the corresponding numbers N_s are defined.†

With these remarks, we come to

* This symbol written in full is $\sim x(0, \tau', t, \Lambda)$, where $\tau' = m \cdot 2^{-\sigma+1}, t = 0$ and $\Lambda = \tau = \rho \cdot 2^{-\sigma+1}$.

† The numbers N_s used here are not necessarily the same numerically as in the preceding discussion.

THEOREM 2. *If the conditions (H_1) of Theorem 1 are satisfied, then the numbers N_1, N_2, \dots, N_s can be so chosen that for every τ and τ' satisfying the conditions*

$$(H_2) \quad \begin{aligned} \tau &= \rho \cdot 2^{-\sigma+1}, \\ \tau' &= m \cdot 2^{-\sigma+1}, \end{aligned}$$

where m, ρ and σ are positive integers, $m \leq \rho$ and $0 < \rho \leq 2^{2\sigma-1}$, the corresponding number $\sim x_{\tau'}$ is an element of the set $A(1 - e^{-\tau'})$.

In proving this theorem, we employ the following scheme. The number which characterizes the m intervals at the σ -stage for the event that succeeds if there is at least one point in the m intervals of I_k , is

$$\{[(\rho_1 + 1)/\rho]X_{\sigma} \vee [(\rho_2 + 1)/\rho]X_{\sigma} \vee \dots \vee [(\rho_m + 1)/\rho]X_{\sigma}\},$$

which is an element of $A(1 - e^{-\tau'})$, which follows from Lemma 1. At the s -stage, the number that characterizes this event for the m intervals of I_k is

$$\left[\sum_{i=1}^m \{[(\rho_i \cdot 2^{s-\sigma} + 1)/(\rho \cdot 2^{s-\sigma})] \vee [(\rho_i \cdot 2^{s-\sigma} + 2)/(\rho \cdot 2^{s-\sigma})] \vee \dots \vee [(\rho_i \cdot 2^{s-\sigma} + 2^{s-\sigma})/(\rho \cdot 2^{s-\sigma})] \vee \} \right] X_s \vee,$$

which is an element of $A(1 - e^{-\tau'})$. Let this number be represented by W_s . When $s \geq \sigma$ and $\nu_s/\tau < k \leq \nu_{s+1}/\tau$, then there exists at least one point in the m intervals of I_k if and only if the $(k - \nu_s/\tau)$ th digit of W_s is one. Hence, when $s \geq \sigma$, the digits $(\nu_s/\tau) + 1$ to ν_{s+1}/τ of $\sim x_{\tau'}$ are the same as the digits 1 to N_s/τ of W_s . The numbers N_s are chosen so that $\nu_s/(n\tau)$ is an integer, if $n \leq s$, $\sigma \leq s$, and $0 < \rho \leq 2^{2\sigma-1}$.

Let ϵ_s be a decreasing sequence of positive numbers having the limit zero. We can select a set of integers $M_1, M_2, \dots, M_s, \dots$, such that

$$(a) \quad \left| p_N \left\{ \prod_{i=1}^k \left(\frac{r_i}{n} \right) W_{s+1} \right\} - \{1 - e^{-\tau'}\}^k \right| < \frac{\epsilon_s}{3},$$

if $N \geq M_s/(n\tau)$, where the sequence M_s has been chosen so that $M_s/(n\tau)$ is an integer. We can choose a second set of integers $N_1, N_2, \dots, N_s, \dots$, such that

$$(b) \quad \left| p_N \left\{ \prod_{i=1}^k \left(\frac{r_i}{n} \right) W_s \right\} - \{1 - e^{-\tau'}\}^k \right| + \frac{\nu_s + M_s}{N_s} < \frac{\epsilon_s}{3},$$

when $N \geq N_s/(n\tau)$ and $\nu_s = N_1 + N_2 + \dots + N_{s-1}$.

Conditions (a) and (b) hold for every $\rho_1, \rho_2, \dots, \rho_m, m, n, \rho$ and σ such that $\rho_m + 1 \leq \rho, m \leq \rho, n \leq s, 0 < \rho \leq 2^{2\sigma-1}, \sigma \leq s$, and for every set of numbers, r_1, r_2, \dots, r_k , such that $r_i \leq n, k \leq n$, where $r_i \neq r_j$ if $i \neq j$.

When $\sigma \leq s$, we see that the digits $(\nu_s/\tau) + 1$ to ν_{s+1}/τ of $\sim x_{\tau'}$ are the same as the digits 1 to N_s/τ of W_s . It will be seen from the definition of $\sim x_{\tau'}$, that the digits 1 to ν_s/τ of $\sim x_{\tau'}$ have been determined from the m intervals of I_1 to $I_{\nu_s/\tau}$. The digits $(\nu_s/(n\tau)) + 1$ to $\nu_{s+1}/(n\tau)$ of $[\prod_{i=1}^k (r_i/n) \sim x_{\tau'}]$ are the same as 1 to $N_s/(n\tau)$ of $[\prod_{i=1}^k (r_i/n) W_s]$.

The fundamental lemma may now be applied. The numbers

$$\left[\prod_{i=1}^k (r_i/n) W_s \right], [1 - e^{-\tau'}]^k, M_s/(n\tau), N_s/(n\tau), \nu_s/(n\tau), \sigma$$

$$\text{and } \left[\prod_{i=1}^k (r_i/n) \sim x_{\tau'} \right]$$

have taken the places of $U_s, p, H_s, J_s, \mu_s, s_0$ and x respectively. This shows that the numbers N_s have been chosen so that the time series is definitely determined in such a way that for every m, ρ , and σ , where $m \leq \rho, \sim x_{\tau'}$ is an element of $A(1 - e^{-\tau'})$.

If $m = \rho$, then $\rho_1 = 0$ and this case reduces to Theorem 1. Furthermore, if $\rho_1 = r$ and $\rho_m = r + m - 1$, then $\sim x_{\tau'}$ becomes $\sim x(0, \tau, t, \Lambda)$ where $\tau = m \cdot 2^{-\sigma+1}$, $t = r \cdot 2^{-\sigma+1}$ and $\Lambda = \rho \cdot 2^{-\sigma+1}$.

III. PROBABILITY OF α POINTS LYING IN AN INTERVAL

6. The same type of time series that was defined in §2 will be used here. As formerly, the intervals are closed on the right and open on the left. The k th interval I_k is $(k-1)\tau < h \leq k\tau$. Let us now define the number $x(\alpha, \tau, 0, \tau)^*$ such that its k th digit is one if exactly α points lie in I_k and zero otherwise. The points of the time series must be distributed on the time axis so that $p[x_r]$ is equal to $[(\tau^\alpha e^{-\tau})/\alpha!]$, for every integral value of ρ, σ and α .

7. As before, the numbers N_s must be chosen so that a consistent time series will be defined. The fundamental theorem here is the following:

THEOREM 3. *If the conditions (H_1) of Theorem 1 are satisfied, then the numbers $N_1, N_2, \dots, N_s, \dots$ can be chosen for every τ satisfying the conditions*

$$(H_2) \quad \tau = \rho \cdot 2^{-\sigma+1};$$

ρ, α and σ are positive integers, where $0 < \rho \leq 2^{2\sigma-1}$, so that the corresponding number x_r is a member of the set $A[(\tau^\alpha e^{-\tau})/\alpha!]$.

Let us define ρ, τ, β_s and α , such that $0 < \rho \leq 2^{2\sigma-1}, \tau = \rho \cdot 2^{-\sigma+1}, \beta_s = \rho \cdot 2^{s-\sigma}$

* Let $x(\alpha, \tau, 0, \tau)$ be denoted by x_r .

and $\alpha \leq \beta_s$. At the s -stage there are β_s divisions of τ . Let us consider an event which succeeds if there is a point in each of α such intervals and no points in the remaining $(\beta_s - \alpha)$ intervals. The number corresponding to this situation is

$$U_s = \sum^{\beta_s C_\alpha} \left\{ \prod_{i=1}^{\alpha} \left(\frac{q_i}{\beta_s} \right) X_s \cdot \prod_{i=\alpha+1}^{\beta_s} \left(\frac{q_i}{\beta_s} \right) X_s \right\} \vee, *$$

where $q_i \neq q_{i'}$ if $i \neq i'$ in every term of the symbolic sum (\vee). This number U_s is admissible,[†] and is a member of the set $A[\beta_s C_\alpha (1 - e^{-\tau/\beta_s})^\alpha (e^{-\tau/\beta_s})^{\beta_s - \alpha}]$. Since

$$\lim_{s \rightarrow \infty} \beta_s C_\alpha (1 - e^{-\tau/\beta_s})^\alpha (e^{-\tau/\beta_s})^{\beta_s - \alpha} = [\tau^\alpha e^{-\tau}]/\alpha!, \ddagger$$

we can choose s and N in such a manner that the difference

$$p_N \left\{ \prod_{i=1}^k \left(\frac{r_i}{n} \right) U_s \right\} - \{[\tau^\alpha e^{-\tau}]/\alpha!\} *$$

is arbitrarily small. We have to satisfy conditions analogous to (1') and (2') of §4 but since

$$\beta_s C_\alpha (1 - e^{-\tau/\beta_s})^\alpha (e^{-\tau/\beta_s})^{\beta_s - \alpha} \neq (\tau^\alpha e^{-\tau})/\alpha!$$

we have an additional problem. The sequence ϵ_s can no longer be arbitrary and the conditions analogous to (1') and (2') cannot hold for every s which is greater than or equal to σ . We will show that there exists a monotonic non-increasing sequence ϵ_s such that $\lim_{s \rightarrow \infty} \epsilon_s = 0$ and two sequences of integers $M_1, M_2, \dots, M_s, \dots$ and $N_1, N_2, \dots, N_s, \dots$ and a function $f(s)$ with an inverse $f^{-1}(\sigma)$ such that if $\sigma \leq f(s)$ or $\sigma \leq f^{-1}(\sigma) \leq s$, then the condition analogous to (1') and (2') holds. § When $s \geq f^{-1}(\sigma) \geq \sigma$ and $\nu_s/\tau < k \leq \nu_{s+1}/\tau$, then there will be exactly α points in I_k if and only if the digit $(k - \nu_s/\tau)$ of U_s is one. Hence, when $s \geq f^{-1}(\sigma) \geq \sigma$, the digits $(\nu_s/\tau) + 1$ to ν_{s+1}/τ of x_s are the same as the digits 1 to N_s/τ of U_s . We shall now show that this is true.

The numbers N_s referred to in §3 are chosen so that $\nu_s/(n\tau)$ is an integer, when $n \leq f(s)$, $\sigma \leq f^{-1}(\sigma) \leq s$ and $0 < \rho \leq 2^{2^{\sigma-1}}$, from which we see that $N_s/(n\tau)$ is an integer. Choosing α numbers from the set of equations (1) of §4, where

* Since it is possible to choose α intervals from β_s in $\beta_s C_\alpha$ ways, then for each choice it is possible to form $\beta_s C_\alpha$ numbers similar to the number which is given in the braces. The symbol

$$\sum^{\beta_s C_\alpha}$$

represents the symbolic sum (\vee) of these numbers.

† Copeland, *Admissible numbers in the theory of probability*, American Journal of Mathematics, vol. 50, Oct., 1928, Theorem 16, p. 550.

‡ See von Mises, *Vorlesungen aus dem Gebiete der angewandten Mathematik*, pp. 147-48.

§ See (3) and (4) of this section given below.

$\rho \cdot 2^{s-\sigma}$ has been replaced by β_s , we may form the symbolic product (\cdot) of these numbers and the negations of the remaining $(\beta_s - \alpha)$ numbers. This product (\cdot) is

$$(a) \quad \prod_{i=1}^{\alpha} \left(\frac{q_i}{\beta_s} \right) X_s \cdot \prod_{i=\alpha+1}^{\beta_s} \sim \left(\frac{q_i}{\beta_s} \right) X_s = \left(\prod_{i=1}^{\alpha} X_s^{(q_i)} \cdot \prod_{i=\alpha+1}^{\beta_s} (1 - X_s^{(q_i)}) \right), \\ \left(\prod_{i=1}^{\alpha} X_s^{(q_i + \beta_s)} \cdot \prod_{i=\alpha+1}^{\beta_s} (1 - X_s^{(q_i + \beta_s)}) \right), \dots$$

We now raise the question when will

$$(b) \quad \prod_{i=1}^{\alpha} X_s^{(q_i)} \cdot \prod_{i=\alpha+1}^{\beta_s} (1 - X_s^{(q_i)})$$

be one? In order that this product be one, every $X_s^{(q_i)}$ ($i=1, 2, \dots, \alpha$) must be one and every $X_s^{(q_i)}$ ($i=\alpha+1, \alpha+2, \dots, \beta_s$) must be zero. The only time this will be the case, is when there is a point in each of the α intervals, q_i/β_s ($i=1, 2, \dots, \alpha$), and not a point in the $(\beta_s - \alpha)$ intervals, q_i/β_s ($i=\alpha+1, \alpha+2, \dots, \beta_s$), for the first period of the s -stage; that is, of the period $I_{(\nu_s/\tau)+1}$. There are ${}_s C_{\alpha}$ such numbers as (a) which can be formed from (1) of §4, but if (a) has for its first digit one, the remaining $({}_s C_{\alpha} - 1)$ numbers have for their digit zero and hence U_s will have its first digit one. Hence the $[(\nu_s/\tau) + 1]$ th digit of x_r is the same as the first digit of U_s . This process may be continued for the N_s/τ digits of U_s .

We will now find the function $f(s)$ which has been referred to heretofore and the monotonic non-increasing sequence ϵ_s approaching zero. We will show that $\eta(\rho, k, \sigma, \alpha, s) < \epsilon_s/3$ for every k, ρ, α and σ for which $k, \alpha, \sigma \leq f(s)$ and $0 < \rho \leq 2^{2\sigma-1}$, where

$$(1) \quad \left| \left\{ {}_s C_{\alpha} (1 - e^{-\tau/\beta_s})^{\alpha} (e^{-\tau/\beta_s})^{\beta_s - \alpha} \right\}^k - \left\{ (\tau^{\alpha} e^{-\tau}) / \alpha! \right\}^k \right| = \eta(\rho, k, \sigma, \alpha, s).$$

If k, ρ, σ and α are fixed, then the

$$\lim_{s \rightarrow \infty} \eta(\rho, k, \sigma, \alpha, s) = 0.$$

For every σ , there is a finite number of integers ρ , hence we may choose $\eta(k, \sigma, \alpha, s)$ greater than the greatest of the numbers $\eta(\rho, k, \sigma, \alpha, s)$. Then the $\eta(k, \sigma, \alpha, s)$ sequence dominates the $\eta(\rho, k, \sigma, \alpha, s)$ sequence, but it is not necessarily monotonic. Let $\epsilon(k, \sigma, \alpha, s)$ equal the least upper bound of the sequence $\eta(k, \sigma, \alpha, s), \eta(k, \sigma, \alpha, s+1), \dots$, and now we have $\epsilon(k, \sigma, \alpha, s)$ dominating $\eta(k, \sigma, \alpha, s)$, which is a monotonic zero sequence. If for a given δ , there exists a z_{δ} such that $z_{\delta+1} > z_{\delta}$ and $\epsilon(k, \sigma, \alpha, z_{\delta}) \leq \epsilon(1, 1, 1, \delta)$, if $k \leq \delta, \sigma \leq \delta$, and $\alpha \leq \delta$. Let $f(s) = \delta$ if $z_{\delta} \leq s < z_{\delta+1}$. Hence $\epsilon(k, \sigma, \alpha, s) \leq \epsilon(1, 1, 1, f(s))$ if $k,$

$\sigma, \alpha \leq f(s)$. Let $\epsilon_s/3$ equal $\epsilon(1, 1, 1, f(s))$, from which it follows that $\eta(k, \sigma, \alpha, s) < \epsilon_s/3$. Hence we conclude that

$$(1') \quad \left| \{ \beta_s C_\alpha (1 - e^{-\tau/\beta_s})^\alpha (e^{-\tau/\beta_s})^{\beta_s - \alpha} \}^k - \{ (\tau^\alpha e^{-\tau}) / \alpha! \}^k \right| < \epsilon_s/3,$$

if $k, \sigma, \alpha \leq f(s)$ and $0 < \rho \leq 2^{2\sigma-1}$.

We are now in a position to define the function $f^{-1}(\sigma)$. Let $f^{-1}(\delta) = z_\delta$. According to the definition of z_δ , it follows that $f^{-1}(\delta+1) > f^{-1}(\delta)$ and since $f^{-1}(1) \geq 1$, then $f^{-1}(\delta) \geq \delta$. If $f^{-1}(\sigma) \leq s < f^{-1}(\sigma+1)$, then $f(s) = \sigma$ and if $f^{-1}(\sigma) \leq s$, then $s \geq f^{-1}(\sigma) \geq \sigma$. Hence the function $f^{-1}(\sigma)$ has been established.

We shall now give the formal proof of Theorem 3; that is to say, we may now put it in form so that the fundamental lemma may be applied. From the fact that U_s is an element of

$$A [\beta_s C_\alpha (1 - e^{-\tau/\beta_s})^\alpha (e^{-\tau/\beta_s})^{\beta_s - \alpha}],$$

we have

$$(2) \quad p \left\{ \prod_{i=1}^k \left(\frac{r_i}{n} \right) U_s \right\} = \{ \beta_s C_\alpha (1 - e^{-\tau/\beta_s})^\alpha (e^{-\tau/\beta_s})^{\beta_s - \alpha} \}^k.$$

It follows from (1') and (2) that we can select the two sets of integers $M_1, M_2, \dots, M_s, \dots$ and $N_1, N_2, \dots, N_s, \dots$, referred to above, such that

$$(3) \quad \left| p_N \left\{ \prod_{i=1}^k \left(\frac{r_i}{n} \right) U_{s+1} \right\} - \left\{ \frac{\tau^\alpha e^{-\tau}}{\alpha!} \right\}^k \right| < \frac{\epsilon_s}{3},$$

if $N \geq M_s/(n\tau)$, where the sequence M_s is chosen so that $M_s/(n\tau)$ is an integer, and

$$(4) \quad \left| p_N \left\{ \prod_{i=1}^k \left(\frac{r_i}{n} \right) U_s \right\} - \left\{ \frac{\tau^\alpha e^{-\tau}}{\alpha!} \right\}^k \right| + \frac{\nu_s + M_s}{N_s} < \frac{\epsilon_s}{3},$$

if $N \geq N_s/(n\tau)$, where $\nu_s = N_1 + N_2 + \dots + N_{s-1}$.

The inequalities (3) and (4) hold for every $k, n, \rho, \beta_s, f^{-1}(\sigma), \sigma, s, f(s)$ and α such that $k \leq n \leq f(s)$, $0 < \rho \leq 2^{2\sigma-1}$, $\beta_s = \rho \cdot 2^{2\sigma}$, $\sigma \leq f^{-1}(\sigma) \leq s$, $\sigma \leq f(s)$, α is less than or equal to the smaller of β_s or $f(s)$, and for every set of numbers r_1, r_2, \dots, r_k , such that $r_i \leq n$ and $r_i \neq r_j$ if $i \neq j$.

When $\sigma \leq f^{-1}(\sigma) \leq s$, we have shown that the digits $(\nu_s/\tau) + 1$ to ν_{s+1}/τ of x_r are the same as the digits 1 to N_s/τ of U_s . From the definition of x_r , we know that the digits 1 to $\nu_{f^{-1}(\sigma)}/\tau$ of x_r are determined from the intervals I_1 to I_K of the time axis, where $K = \nu_{f^{-1}(\sigma)}/\tau$. The digits

$$\frac{\nu_s}{n\tau} + 1 \quad \text{to} \quad \frac{\nu_{s+1}}{n\tau} \quad \text{of} \quad \left[\prod_{i=1}^k \frac{r_i}{n} x_r \right]$$

are the same as

$$1 \text{ to } \frac{N_s}{n\tau} \text{ of } \left[\prod_{i=1}^k \frac{r_i}{n} U_s \right].$$

In applying the lemma, the numbers

$$\left[\prod_{i=1}^k \frac{r_i}{n} U_s \right], \left[\frac{\tau^\sigma e^{-\tau}}{\alpha!} \right], \frac{M_s}{n\tau},$$

$$\frac{N_s}{n\tau}, \frac{v_s}{n\tau}, f^{-1}(\sigma) \text{ and } \left[\prod_{i=1}^k \frac{r_i}{n} x_r \right]$$

have taken the places of U_s , p , H_s , J_s , μ_s , s_0 and x respectively. Since the numbers N_s have been selected, the time series is determined so that x_r is a member of the set $A[(\tau^\sigma e^{-\tau})/\alpha!]$, for every p , σ and α .

8. We will now consider the period τ where sub-intervals of the form $2^{-\sigma+1}$ are omitted. With only slight modifications of the theorem in the preceding section, we can show that the laws of admissibility are satisfied for this case. The same type of time series will be used.

We shall consider m intervals of the form $2^{-\sigma+1}$ which repeat every ρ intervals. These intervals are of the same length as those intervals which make up that part of the time axis at the σ -stage. Let the intervals which are under consideration begin at $\rho_i \cdot 2^{-\sigma+1}$ ($i=1, 2, \dots, m$), where $\rho_m+1 \leq \rho$ and $m \leq \rho$. The sub-intervals of I_k will be defined as $(k-1)\tau + \rho_i \cdot 2^{-\sigma+1} < h \leq (k-1)\tau + (\rho_i+1)2^{-\sigma+1}$ ($i=1, 2, \dots, m$). We will define x_r^* such that its k th digit is one if exactly α points lie in the m intervals of I_k and zero otherwise. Since the same type of time series is used here, then the series will be defined when the numbers N_s are determined. Hence,

THEOREM 4. *If the conditions (H_1) of Theorem 1 hold, then the numbers $N_1, N_2, \dots, N_s, \dots$ can be chosen for every τ and τ' satisfying the conditions*

$$(H_2) \quad \begin{aligned} \tau &= \rho \cdot 2^{-\sigma+1}; \\ \tau' &= m \cdot 2^{-\sigma+1}; \end{aligned}$$

m , ρ and σ are positive integers, where $m \leq \rho$ and $0 < \rho \leq 2^{2\sigma-1}$, so that the corresponding number x_r^* is an element of $A[(\tau'^\sigma e^{-\tau'})/\alpha']$.

Let ρ , τ , τ' , β_s , m and α be defined such that $0 < \rho \leq 2^{2\sigma-1}$, $\tau = \rho \cdot 2^{-\sigma+1}$, $\tau' = m \cdot 2^{-\sigma+1}$, $\beta_s = m \cdot 2^{-\sigma}$, $m \leq \rho$ and $\alpha \leq \beta_s$. At the s -stage there are β_s intervals in the m intervals of I_k . Let us consider an event which succeeds if there

* The symbol x_r^* is used to denote the number $x(\alpha, \tau', t, \Delta)$, where $\tau' = m \cdot 2^{-\sigma+1}$, $t=0$ and $\Delta = \tau = \rho \cdot 2^{-\sigma+1}$.

is a point in each of α such intervals and no points in the remaining $(\beta_s - \alpha)$ intervals. The number corresponding to this situation is

$$W_s = \sum_{\beta_s C_\alpha} \left\{ \prod_{ik} \left(\frac{q_{ik}}{\rho \cdot 2^{s-\sigma}} \right) X_s \cdot \prod_{jn} \sim \left(\frac{q_{jn}}{\rho \cdot 2^{s-\sigma}} \right) X_s \right\} v,$$

where q_{ik} is of the form $2^{s-\sigma} \cdot \rho_i + k$ and q_{jn} is of the form $2^{s-\sigma} \cdot \rho_j + n$, and if $i=j$, then $k \neq n$, and if $k=n$, then $i \neq j$. The numbers i and j take values from 1 to m inclusive and the numbers k and n take values 1 to $2^{s-\sigma}$ inclusive.

This number W_s is admissible and is a member of the set

$$A [\beta_s C_\alpha (1 - e^{-\tau'/\beta_s})^\alpha (e^{-\tau'/\beta_s})^{\beta_s - \alpha}].$$

When $s \geq f^{-1}(\sigma) \geq \sigma$ and $\nu_s/\tau < k \leq \nu_{s+1}/\tau$, then there exist exactly α points in the m intervals of I_k if and only if the digit $(k - \nu_s/\tau)$ of W_s is one. Hence, when $s \geq f^{-1}(\sigma) \geq \sigma$, the digits $(\nu_s/\tau) + 1$ to ν_{s+1}/τ of x_r are the same as the digits 1 to N_s/τ of W_s . The numbers N_s are chosen so that $\nu_s/(n\tau)$ is an integer, when $n \leq f(s)$, $\sigma \leq f^{-1}(\sigma) \leq s$ and $0 < \rho \leq 2^{2\sigma-1}$, from which it follows that $N_s/(n\tau)$ is an integer.

As in Theorem 3, we can find a function $f(s)$ and a monotonic non-increasing sequence ϵ_s whose limit is zero, such that

$$(a) \quad \left| \beta_s C_\alpha (1 - e^{-\tau'/\beta_s})^\alpha (e^{-\tau'/\beta_s})^{\beta_s - \alpha} \right\}^k - \left\{ (\tau'^\alpha e^{-\tau'}) / \alpha! \right\}^k \right| < \epsilon_s/3,$$

where $k, \alpha, \sigma \leq f(s)$, $m \leq \rho$ and $0 < \rho \leq 2^{2\sigma-1}$.

Since W_s is a member of the set

$$A [\beta_s C_\alpha (1 - e^{-\tau'/\beta_s})^\alpha (e^{-\tau'/\beta_s})^{\beta_s - \alpha}],$$

we know that

$$(b) \quad p \left\{ \prod_{i=1}^k \left(\frac{r_i}{n} \right) W_s \right\} = \left\{ \beta_s C_\alpha (1 - e^{-\tau'/\beta_s})^\alpha (e^{-\tau'/\beta_s})^{\beta_s - \alpha} \right\}^k.$$

It follows from (a) and (b) that we can select two sets of integers $M_1, M_2, \dots, M_s, \dots$ and $N_1, N_2, \dots, N_s, \dots$ such that

$$(c) \quad \left| p_N \left\{ \prod_{i=1}^k \left(\frac{r_i}{n} \right) W_{s+1} \right\} - \left\{ \frac{\tau'^\alpha e^{-\tau'}}{\alpha!} \right\}^k \right| < \frac{\epsilon_s}{3},$$

when $N \geq M_s/(n\tau)$, where the sequence M_s has been chosen so that $M_s/(n\tau)$ is an integer, and

$$(d) \quad \left| p_N \left\{ \prod_{i=1}^k \left(\frac{r_i}{n} \right) W_s \right\} - \left\{ \frac{\tau'^\alpha e^{-\tau'}}{\alpha!} \right\}^k \right| + \frac{\nu_s + M_s}{N_s} < \frac{\epsilon_s}{3},$$

if $N \geq N_s/(n\tau)$, where $\nu_s = N_1 + N_2 + \dots + N_{s-1}$.

It is understood that conditions (c) and (d) hold for every $\rho_1, \rho_2, \dots, \rho_m$, $\rho, n, m, k, \sigma, s, f^{-1}(\sigma), \beta_s$ and α such that $k \leq n \leq f(s)$, $0 < \rho \leq 2^{2\sigma-1}$, $m \leq \rho$, $\sigma \leq f(s)$, $\rho_m + 1 \leq \rho$, $\sigma \leq f^{-1}(\sigma) \leq s$, $\beta_s = m \cdot 2^{s-\sigma}$, α is less than or equal to the smaller of β_s or $f(s)$, and for every set of numbers, r_1, r_2, \dots, r_k , such that $r_i \leq n$ and $r_i \neq r_j$ if $i \neq j$.

It follows from the definition of x_r that the digits $v_s/(n\tau) + 1$ to $v_{s+1}/(n\tau)$ of $[\prod_{i=1}^k (r_i/n)x_r]$ are the same digits as 1 to $N_s/(n\tau)$ of $[\prod_{i=1}^k (r_i/n)W_s]$.

Hence, in applying the lemma to this theorem, we see that the numbers N_s have been selected so that x_r is an element of the set $A[(\tau'^a e^{-r'})/\alpha!]$, for every m, ρ, σ and α , where $m \leq \rho$.

If $m = \rho$, then $\rho_1 = 0$ and this theorem becomes Theorem 3. Furthermore, if $\rho_1 = r$ and $\rho_m = r + m - 1$, the number x_r becomes $x(\alpha, \tau, t, \Lambda)$, where $t = r \cdot 2^{-\sigma+1}$, $\tau = m \cdot 2^{-\sigma+1}$ and $\Lambda = \rho \cdot 2^{-\sigma+1}$.

9. Since the sequence of numbers N_s has been found, we have constructed a time series which is consistent with the frequency theory of probability.

It may be noted that several theorems proved in Professor A. H. Copeland's work (American Journal of Mathematics, vols. 50, 51, and 53), can be proved by applying the fundamental lemma in this paper.

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INSCRIBED SEQUENCES OF SURFACES ASSOCIATED WITH GENERALIZED SEQUENCES OF LAPLACE*

BY
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1. INTRODUCTION

The theory of conjugate nets of curves on surfaces in a projective space of n dimensions was generalized by Bompiani† in his theory of systems of curves in conjugacy of type ν . Bompiani also generalized the theory of families of asymptotic curves on surfaces by a theory of families of curves in autoconjugacy of type ν .

A set of transformations for surfaces bearing systems of curves in conjugacy of type ν was offered by B. Segre‡, who also gave a system of transformations for surfaces bearing families of curves in autoconjugacy of type ν ($\nu > 1$). These transformations produce sequences of surfaces quite analogous to the classical sequences of Laplace. In fact the sequences of surfaces bearing systems of curves in conjugacy of type ν are generalizations of the classical sequences of Laplace. It is these generalized sequences to which we refer in the title.

It is the purpose of this paper to point out a large class of sequences associated with any given sequence of surfaces, and to examine the transformations which generate certain special sequences in that class. Several of the special sequences which we study in that class are associated with the above mentioned sequences of Segre.

In §2, we state the geometric basis for a class of sequences called *associated sequences*, and define *generalized inscribed sequences*. These generalized inscribed sequences form a subclass of the above associated sequences. A generalized inscribed sequence is generated by the same kind of transformation as generates the sequence in which it is inscribed. The existence of a large class of the general inscribed sequences is established in §3, and a *web of inscribed sequences* is defined. The existence theorem of §3 is applied in §4 where a study is made of two classes of sequences of surfaces which are inscribed in

* Presented to the Society, April 7, 1934; received by the editors December 15, 1933.

† E. Bompiani, *Sistemi coniugati sulle superficie degli iperspazi*, Rendiconti del Circolo Matematico di Palermo, vol. 46 (1922), p. 91.

‡ B. Segre, *Les systèmes conjugués et autoconjugués d'espèce ν et leur transformation de Laplace*, Annales Scientifiques de l'Ecole Normale Supérieure, (3), vol. 44 (1927), pp. 153-212.

a given *hyperbolic sequence of Segre*. Two additional types of sequences, inscribed in a given *parabolic sequence of Segre*, are considered in §5.

As a result of this investigation, the transformations of Laplace and Segre are made available for a much less restricted class of surfaces than the class to which they have been applied heretofore.

In the analytical considerations which follow, a point of a surface in a projective space of n dimensions is represented by $n+1$ coordinates x_i , denoted by the single symbol x . The x_i are functions of the curvilinear coordinates u and v , and they have as many partial derivatives with respect to u and v as are needed. Partial derivatives are denoted in accordance with the formula

$$(1.1) \quad \frac{\partial^{i+j} x}{\partial u^i \partial v^j} = x^{ij}.$$

2. INSCRIBED SEQUENCES IN GENERAL

Consider any sequence T of surfaces

$$\Sigma, \Sigma_1, \Sigma_2, \dots, \Sigma_i, \dots,$$

which is generated by repeated application of a definite transformation such that any surface Σ_{i+1} is a transform of the surface Σ_i by means of a one-to-one point correspondence. The surfaces of T have in common all of the properties of the initial surface Σ which are invariant under the generating transformation. In this sense, they will be called *mathematically equivalent* surfaces. Let ω_r be any osculant of the surface Σ_r at a general point P_r , or let ω_r be any osculant at P_r of a curve belonging to a family which lies on the surface Σ_r . In either case, let ω_{r+1} be the corresponding osculant pertaining to the point P_{r+1} of the surface Σ_{r+1} . Associated with the ∞^2 points of Σ_r , there are ∞^2 osculants ω_r , and we shall denote the doubly infinite set by Ω_r . Obviously, there is associated with the sequence of surfaces Σ_i a sequence of sets of osculants Ω_i .

Consider a surface Σ'_r , whose points are in a one-to-one correspondence with the ∞^2 osculants ω_r , pertaining to the surface Σ_r , in such a manner that each point of the surface Σ'_r is in united position with its corresponding osculant ω_r . Let Σ'_{r+1} be a similarly described surface whose points are in a one-to-one united correspondence with the osculants ω_{r+1} of the set Ω_{r+1} . The surface Σ'_{r+1} is a transform of the surface Σ'_r , by means of the indirect correspondence which connects them. The surfaces $\dots, \Sigma'_r, \Sigma'_{r+1}, \dots$ form a sequence T' of surfaces which will be referred to as an *associated sequence* of the sequence T .

DEFINITION 2.1. *If the points of a surface are in a one-to-one correspondence with a set of ∞^2 linear spaces of v dimensions, in such a manner that each point of the surface is in united position with its corresponding linear space, then and only then will the surface be said to be transversal to the set of linear spaces.*

DEFINITION 2.2. *Let T denote a sequence of surfaces in which the points of each surface Σ_i are joined in a one-to-one manner to the corresponding points of the adjacent surface Σ_{i+1} by a set Ω of ∞^2 linear spaces ω of v dimensions, where the spaces ω are osculating spaces at points of the surface Σ_i to the curves of a family, which lies on the surface Σ_i , or the spaces ω are osculants of the surface Σ_i itself. Let T' denote a sequence in which consecutive surfaces are connected in the same manner as those of the sequence T . Let the sequence T' be related to the sequence T so that each surface Σ'_i of the sequence T' is transversal to the set Ω_i of osculants which connect the points of the surfaces Σ_i and Σ_{i+1} of T . Under these conditions the sequence T' will be said to be inscribed in the sequence T . The sequence T will be said to circumscribe the sequence T' .*

The surface Σ'_i of the above inscribed sequence T' may belong to a more general class of surfaces than does the surface Σ_i of the sequence T . For this reason, the point differential equations which represent the definition of the surface Σ'_i will, in general, be of a higher order than the differential equations which represent the definition of the surface Σ_i of the sequence T . However, the surfaces of the two sequences are connected by correspondences of the same kind. The analytical forms of the transformations in the two sequences will be the same. By proving the existence of these generalized inscribed sequences associated with known sequences, we extend the application of known transformations to a more general class of surfaces and to their point differential equations.

It is observed that a classical inscribed sequence of Laplace furnishes a special example under Definition 2.2. The following article will establish the existence of inscribed sequences of great generality.

3. A WEB OF INSCRIBED SEQUENCES

Before defining a web of inscribed sequences, we shall establish the following basic

THEOREM 3.1. *Let T denote a sequence of surfaces in which the points of each surface Σ_{i+1} are joined in a one-to-one manner to the corresponding points of the preceding surface Σ_i , by a set Ω_i of ∞^2 osculating spaces of v dimensions belonging to the curves of a family on the surface Σ_i . Let Σ'_i be any surface which is transversal to the set Ω_i of osculating spaces ω_i . Then it follows that the transversal surface Σ'_i belongs to a sequence T' of surfaces which is inscribed in the given sequence T .*

Consider a surface Σ , and on it a family F of curves. Let λ denote the curve of the family which passes through the generating point P of the surface Σ . Denote by ω the osculating space of ν dimensions at the point P to the curve λ . Let Σ' and Σ'' denote two surfaces which are transversal to the set Ω of ∞^2 osculating spaces ω pertaining to the ∞^2 points of the curves in the family F . Let P' and P'' denote the points of intersection of the surfaces Σ' and Σ'' respectively with the osculating space ω . As the point P moves along the curve λ on the surface Σ , the points P' and P'' generate two curves, λ' and λ'' respectively, on the surfaces Σ' and Σ'' . Denote by ω' and ω'' the osculating spaces of ν dimensions to the curves λ' and λ'' at the respective points P' and P'' . We shall show that the osculants ω' and ω'' intersect in a point.

Let $x(u, v)$ be the coordinates of the generating point of the above surface Σ . Let the curves of the above family F be chosen as the parametric u -curves. Choose any other family of curves as the v -curves. The osculating space of ν dimensions ω to the u -curve at the point P is determined by $\nu+1$ points, whose coordinates are x and the first ν derivatives of x with respect to u . Since the generating point P' of the surface Σ' is in contact with the osculant ω , the coordinates y of the point P' can be expressed as

$$(3.1) \quad y = \sum_{i=0}^{\nu} \alpha_{i0} x^{i0}.$$

For a similar reason, the coordinates z of the generating point P'' of the surface Σ'' are

$$(3.2) \quad z = \sum_{i=0}^{\nu} \beta_{i0} x^{i0}.$$

The osculating spaces of ν dimensions ω' and ω'' to the u -curves at P' and P'' are determined by two sets of points, whose coordinates are y and the first ν derivatives of y , and z with the first ν derivatives of z with respect to u . We exhibit these as follows:

$$(3.3) \quad y^{i0} = \sum_{j=0}^{i+\nu} \alpha_{j0}^{(i0)} x^{j0} \quad (i = 0, 1, 2, \dots, \nu),$$

$$(3.4) \quad z^{i0} = \sum_{j=0}^{i+\nu} \beta_{j0}^{(i0)} x^{j0} \quad (i = 0, 1, 2, \dots, \nu).$$

The $2\nu+2$ coordinates y^{i0} and z^{i0} , on the left of equations (3.3) and (3.4), are expressed linearly in terms of the $2\nu+1$ functions x^{j0} . Hence there exists a linear relation among the z^{i0} and the y^{i0} . That relation will be indicated as

$$(3.5) \quad \sum_{j=0}^v \theta_{j0} y^{j0} = \sum_{j=0}^v \phi_{j0} z^{j0}.$$

Equation (3.5) indicates that the osculating space ω' intersects the osculating space ω'' in a point P'_1 , the coordinates of which are given by either the right or left member of (3.5). These coordinates are denoted as

$$(3.6) \quad y_1 = \sum_{j=0}^v \phi_{j0} z^{j0}.$$

The surface generated by the point P'_1 will be denoted by Σ'_1 .

The above facts justify

LEMMA 3.1. *Let ω be the osculating space of v dimensions at the point P to a curve λ of a family of curves on a surface Σ . Let Σ' and Σ'' be two surfaces which are transversal to the set Ω of ∞^2 osculants ω , pertaining to the ∞^2 points of the surface Σ . Let λ' and λ'' be the curves on Σ' and Σ'' respectively which correspond to the curve λ on Σ . Then it follows that the osculating space of v dimensions ω' to the curve λ' at a point P' of the surface Σ' , and the osculating space of v dimensions ω'' to the curve λ'' at the point P'' of the surface Σ'' , intersect in a point P'_1 which generates a surface Σ'_1 .*

From this lemma, Theorem 3.1 can be obtained directly by assuming that the surface Σ'' of the lemma is a transform Σ_1 of the surface Σ , and that the surfaces Σ and Σ_1 belong to a sequence of mathematically equivalent surfaces

$$(3.7) \quad \Sigma, \Sigma_1, \Sigma_2, \dots, \Sigma_i, \dots$$

Since the surface Σ'_1 bears the same relation to the assumed surface Σ_1 as the surface Σ' bears to the surface Σ , and since the surfaces Σ and Σ_1 are mathematically equivalent, it follows that the surfaces Σ' and Σ'_1 are also mathematically equivalent. That is, the surface Σ'_1 is a transform of the surface Σ' . Also, since the surface Σ_1 is transformable into the surface Σ_2 of the sequence (3.7), by a repetition of the above argument, it follows that the surface Σ'_1 is transformable in the same manner into a surface Σ'_2 of the sequence

$$(3.8) \quad \Sigma', \Sigma'_1, \Sigma'_2, \dots, \Sigma'_i, \dots$$

If in equations (3.2), (3.5) and (3.6) we replace the coordinates z by x_1 of the point P_1 of the surface Σ_1 , we obtain the following relations:

$$(3.9) \quad x_1 = \sum_{i=0}^v \beta_{i0} x^{i0},$$

$$(3.10) \quad \sum_{j=0}^v \theta_{j0} y^{j0} = \sum_{j=0}^v \phi_{j0} x_1^{j0},$$

$$(3.11) \quad y_1 = \sum_{j=0}^{\nu} \theta_{j0} y^{j0},$$

$$(3.12) \quad y_1 = \sum_{j=0}^{\nu} \phi_{j0} x_1^{j0}.$$

Equation (3.12) shows that the surface Σ'_1 , of point coordinates y_1 , is transversal to the ∞^2 osculating spaces of ν dimensions to the u -curves of the surface Σ_1 . This fact indicates that the sequence (3.8) is inscribed in the sequence (3.7). Equation (3.11) shows that corresponding points of the two surfaces Σ' and Σ'_1 , of the sequence (3.8), are joined in a one-to-one manner by the osculating spaces of ν dimensions of the u -curves on the surface Σ' . These facts complete the proof of the theorem.

Equation (3.11) shows that the transformation which generates the inscribed sequence (3.8) is of the same analytical form as the transformation (3.9) which generates the circumscribed sequence (3.7).

The inscribed sequence T' of Theorem 3.1 has all of the properties of T which are required by the hypothesis. As a consequence, the theorem is applicable to the sequence T' , and repeatedly, showing that there is in general an endless aggregate of sequences of surfaces

$$(3.13) \quad T', T'', T''', \dots,$$

successively inscribed in a given sequence T .

DEFINITION. *An aggregate of successively inscribed sequences of the type (3.13) will be called a web of inscribed sequences. The sequence T will be said to be circumscribed about the web.*

In a given web, the properties of the surfaces vary from one sequence to the next, but the transformations have the same form for the entire web.

4. SEQUENCES INSCRIBED IN A HYPERBOLIC SEQUENCE OF SEGRE

A surface Σ bearing a system of curves in conjugacy of type ν may be defined as an integral surface of a hyperbolic differential equation*

$$(4.1) \quad \sum_{i=0}^{\nu} \sum_{j=0}^1 A_{ij} x^{ij} = 0.$$

Segre's transformation of the First Kind† for the above surface Σ has the form

$$(4.2) \quad x_1 = \sum_{i=0}^{\nu} b_{i0} x^{i0},$$

* B. Segre, loc. cit., p. 161.

† B. Segre, loc. cit., p. 169.

for which the coefficients b_{i0} are determined, to within a proportionality factor, in terms of the A_{ij} of equation (4.1). It is evident from equation (4.2) that the surface Σ_1 , generated by the point having coordinates x_1 , is transversal to the osculating spaces of ν dimensions to the u -curves on the surface Σ . Corresponding points of the two surfaces Σ and Σ_1 are joined in a one-to-one manner by the osculating spaces of ν dimensions of the u -curves of the surface Σ . By repeated application of the transformation (4.2), a sequence T_+ of surfaces

$$(4.3) \quad \Sigma, \Sigma_1, \Sigma_2, \dots$$

is generated.

Since each surface of the sequence (4.3) is an integral surface of a hyperbolic differential equation of the type (4.1), we shall refer to the sequence as a *hyperbolic sequence of Segre*. That part of the entire sequence which is generated by the Segre transformation of the First Kind will be called the *forward* or *positive branch*.

From the foregoing remarks, we verify that the positive branch T_+ of the above hyperbolic sequence of Segre has all of the properties required by the hypothesis of Theorem 3.1. Consequently, the theorem is applicable to any surface which is transversal to the ∞^2 osculating spaces which connect corresponding points of any pair of consecutive surfaces in the sequence of Segre. From this fact we have

THEOREM 4.1. *The positive or forward branch T_+ of a hyperbolic sequence of Segre is circumscribed about a web of inscribed sequences*

$$(4.4) \quad T'_+, T''_+, T'''_+, \dots$$

We now consider a class of sequences of surfaces inscribed in the *inverse* or *negative branch* T_- of a hyperbolic sequence of Segre. The transformation which takes the above surface Σ into its transform Σ_{-1} of the negative or inverse branch of the sequence of Segre* is represented by the equation

$$(4.5) \quad x_{-1} = \sum_{i=0}^{\nu-1} \sum_{j=0}^1 C_{ij} x^{ij}.$$

The x_{-1} are the coordinates of the point P_{-1} which generates the surface Σ_{-1} . The coefficients C_{ij} are determined to within a proportionality factor, in terms of the coefficients A_{ij} of equation (4.1).

From the terms of equation (4.5) we observe that the point P_{-1} lies in the sum-space of $2\nu-1$ dimensions, formed by the osculating space of $\nu-1$ dimensions to the u -curve at the point P of Σ , and by the osculating space of

* B. Segre, loc. cit., p. 183.

$\nu-1$ dimensions to the u -curve through the point $(u, v+\Delta v)$, adjacent to P . We shall denote this sum-space by σ , and likewise the corresponding sum-spaces at the generating points of the surfaces $\Sigma_{-1}, \Sigma_{-2}, \dots$ by the corresponding symbols $\sigma_{-1}, \sigma_{-2}, \dots$.

Let Σ' denote a surface which is distinct from the two surfaces Σ and Σ_{-1} , but which is transversal to the ∞^2 osculating spaces σ of the surface Σ . The coordinates y of the generating point P' of the surface Σ' can be expressed as

$$(4.6) \quad y = \sum_{i=0}^{\nu-1} \sum_{j=0}^1 g_{ij} x^{ij},$$

in which the g_{ij} are arbitrary except that they are distinct from the C_{ij} of equation (4.5).

On examining the two osculating sum-spaces σ_{-1} and σ' of the surfaces Σ_{-1} and Σ' , we find, as will be shown analytically in the next paragraph, that they intersect in a point P'_{-1} . The point P'_{-1} generates a surface Σ'_{-1} which is transversal to the ∞^2 osculating spaces σ_{-1} . It is also transversal to the ∞^2 osculants σ' of the surface Σ' . From the mathematical equivalence of the surfaces Σ and Σ_{-1} and by the fact that the surface Σ'_{-1} bears the same relation to the surface Σ_{-1} as the surface Σ' bears to Σ , it follows that the two surfaces Σ' and Σ'_{-1} are mathematically equivalent. The surface Σ'_{-1} is a transform of the surface Σ' , and corresponding points of the two surfaces are joined by the ∞^2 osculants σ' of the u -curves on the surface Σ' . These facts justify

THEOREM 4.2. *Let Σ and Σ_{-1} denote two consecutive surfaces of the inverse or negative branch T_{-1} of a hyperbolic sequence of Segre. Let Σ' be any surface which is distinct from the surfaces Σ and Σ_{-1} , and which is transversal to the ∞^2 osculating sum-spaces σ joining corresponding points of Σ and Σ_{-1} . It follows that the surface Σ' belongs to a sequence T'_{-1} of surfaces which is inscribed in the branch T_{-1} of the given sequence of Segre.*

For the analytical justification of the above theorem, we exhibit the coordinates of the points which determine the osculating sum-spaces σ' and σ_{-1} at the points P' and P_{-1} of the surfaces Σ' and Σ_{-1} . By computing derivatives of equations (4.5) and (4.6), we obtain the desired coordinates as the left members of

$$(4.7) \quad \begin{aligned} x_{-1}^{\lambda\mu} &= \sum_{i=0}^{\nu-1+\lambda} \sum_{j=0}^{1+\mu} C_{ij}^{(\lambda\mu)} x^{ij} & (\lambda = 0, 1, \dots, \nu-1; \mu = 0, 1), \\ y^{\lambda\mu} &= \sum_{i=0}^{\nu-1+\lambda} \sum_{j=0}^{1+\mu} G_{ij}^{(\lambda\mu)} x^{ij}. \end{aligned}$$

The left members of (4.7) are 4ν functions expressed linearly in terms of the $6\nu-3$ functions x^{ij} ($i=0, 1, \dots, 2\nu-2; j=0, 1, 2$). By computing higher derivatives of equation (4.1), it is easily shown that there are $2\nu-2$ linear relations among the above functions x^{ij} . These relations are expressed by the equation

$$\sum_{i=0}^{\nu+\gamma} \sum_{j=0}^{1+\delta} A_{ij}^{(\gamma\delta)} x^{ij} = 0 \quad (\gamma = 0, 1, \dots, \nu-2; \delta = 0, 1).$$

Hence the above mentioned 4ν functions of the left members of (4.7) are ultimately expressed in terms of $4\nu-1$ of the $6\nu-3$ variables x^{ij} . The left members of (4.5) therefore satisfy a linear relation of the form

$$(4.8) \quad \sum_{i=0}^{\nu-1} \sum_{j=0}^1 \theta_{ij} y^{ij} = \sum_{i=0}^{\nu-1} \sum_{j=0}^1 \psi_{ij} x_{-1}^{ij}.$$

This equation indicates that the osculating sum-space σ' of the surface Σ' intersects the osculating sum-space σ_{-1} of Σ_{-1} in the point P'_{-1} , the coordinates y_{-1} of which are obtained from the left member of (4.6) as

$$(4.9) \quad y_{-1} = \sum_{i=0}^{\nu-1} \sum_{j=0}^1 \theta_{ij} y^{ij}.$$

Equation (4.9) exhibits the analytical form of the transformation which generates the inscribed sequence T'_{-1} indicated by the above theorem. This transformation is essentially of the same form as the transformation (4.5) which generates the negative branch T_{-1} of the hyperbolic sequence of Segre.

5. SEQUENCES INSCRIBED IN A PARABOLIC SEQUENCE OF SEGRE

A surface Σ bearing a family of curves in autoconjugacy of type ν may be defined as an integral surface of a parabolic differential equation of the type

$$(5.1) \quad \sum_{i=0}^{\nu+1} A_{i0} x^{i0} + \sum_{i=0}^{\nu-1} A_{i1} x^{i1} = 0.*$$

The transformation of Segre†, which takes the above surface Σ into the surface Σ_1 of the positive branch T_+ of a sequence, can be represented by the equation

$$(5.2) \quad x_1 = \sum_{i=0}^{\nu-1} b_{i0} x^{i0} \quad (\nu > 1).$$

* B. Segre, loc. cit., p. 159.

† B. Segre, loc. cit., p. 206.

The b_{i0} are determined, to within a factor, in terms of the coefficients of equation (5.1).

A Segre sequence of surfaces bearing families of curves in autoconjugacy of type ν will be referred to as a *parabolic sequence of Segre*.

Equation (5.2) shows that in the positive branch of a parabolic sequence of Segre the corresponding points of two adjacent surfaces Σ and Σ_1 are joined, in a one-to-one manner, by the ∞^2 osculating spaces of $\nu-1$ dimensions to u -curves of the surface Σ . On replacing the index ν by $\nu-1$ in Theorem 3.1, we have the resulting

THEOREM 5.1. *The positive branch T_+ of a parabolic sequence of Segre is circumscribed about a web of inscribed sequences of surfaces.*

Attention will now be given to a class of sequences inscribed in the inverse or negative branch T_- of a given parabolic sequence of Segre.

The transformation which carries a given surface Σ of a parabolic sequence of Segre* into its transform Σ_{-1} of the negative branch has the analytic form

$$(5.3) \quad x_{-1} = \sum_{i=0}^{\nu} C_{i0} x^{i0} + \sum_{i=0}^{\nu-2} C_{i1} x^{i1},$$

in which the C_{ij} are uniquely defined, except for a proportionality factor, in terms of the A_{ij} of (5.1).

Equation (5.3) shows that the point P_{-1} , which generates the surface Σ_{-1} , is in the sum-space formed by the osculating space of ν dimensions to the u -curve through the point P of Σ and by the osculating space of $\nu-2$ dimensions to the u -curve through the point $(u, v+\Delta v)$ of Σ . We shall denote this osculating sum-space by σ , and shall denote the corresponding sum-space of the surface Σ_{-1} by σ_{-1} , etc.

Let Σ' represent any surface, distinct from the surfaces Σ and Σ_{-1} , which is transversal to the osculating sum-spaces σ of the surface Σ . On examining the osculating sum-spaces σ' and σ_{-1} of the surfaces Σ' and Σ_{-1} we find, as will be demonstrated analytically later, that these two sum-spaces intersect in a point P'_{-1} . The point P'_{-1} generates a surface Σ'_{-1} which is transversal to the osculating sum-spaces σ_{-1} and σ' of the surfaces Σ_{-1} and Σ' . The surface Σ'_{-1} bears the same relation to the surface Σ_{-1} as the surface Σ' bears to the surface Σ . Since the surfaces Σ and Σ_{-1} are mathematically equivalent, it follows that the surfaces Σ' and Σ'_{-1} are mathematically equivalent. The surface Σ'_{-1} is a transform of the surface Σ' , and corresponding points of the

* B. Segre, loc. cit., p. 209.

two surfaces are joined by the osculating sum-spaces σ' at points of the surface Σ' . From these facts we have

THEOREM 5.2. *Let Σ and Σ_{-1} be any two consecutive surfaces in the inverse or negative branch T_- of a parabolic sequence of Segre. Let Σ' be any third surface which is transversal to the ∞^2 connecting sum-spaces σ pertaining to the surface Σ . Then it follows that the surface Σ' belongs to a branch T'_{-1} of a sequence of surfaces inscribed in the given sequence of Segre.*

To justify the above theorem analytically we exhibit the coordinates of the points which determine the osculating sum-space σ_{-1} at the point P_{-1} of the surface Σ_{-1} . By taking derivatives of the x_{-1} , as expressed in (5.3), we obtain

$$(5.4) \quad \begin{aligned} x_{-1}^{\lambda 0} &= \sum_{i=0}^{p+\lambda} C_{i0}^{(\lambda 0)} x^{i0} + \sum_{i=0}^{p-2+\lambda} C_{i1}^{(\lambda 0)} x^{i1} & (\lambda = 0, 1, \dots, \nu), \\ x_{-1}^{\mu 1} &= \sum_{i=0}^{p+\mu} \sum_{j=0}^1 g_{ij}^{(\mu 1)} x^{ij} + \sum_{i=0}^{p-2+\mu} \sum_{j=0}^2 h_{ij}^{(\mu 1)} x^{ij} & (\mu = 0, 1, \dots, \nu-2). \end{aligned}$$

In a similar manner the coordinates y of the point P' and the remaining points which determine the sum-space σ' at the point P' of the surface Σ' can be displayed in the form

$$(5.5) \quad \begin{aligned} y^{\lambda 0} &= \sum_{i=0}^{p+\lambda} M_{i0}^{(\lambda 0)} x^{i0} + \sum_{i=0}^{p-1+\lambda} M_{i1}^{(\lambda 0)} x^{i1} & (\lambda = 0, 1, \dots, \nu), \\ y^{\mu 1} &= \sum_{i=0}^{p+\mu} \sum_{j=0}^1 K_{ij}^{(\mu 1)} x^{ij} + \sum_{i=0}^{p-2+\mu} \sum_{j=0}^2 P_{ij}^{(\mu 1)} x^{ij} & (\mu = 0, 1, \dots, \nu-2). \end{aligned}$$

The left members of (5.4) and (5.5) are 4ν functions expressed linearly in terms of the $6\nu-3$ functions $x^{00}, x^{10}, \dots, x^{2\nu,0}; x^{01}, x^{11}, \dots, x^{2\nu-2,1}; x^{02}, \dots, x^{2\nu-4,2}$. But by means of equation (5.1) and its derivatives we have $2\nu-2$ linear relations among the above $6\nu-3$ functions. We exhibit the $2\nu-2$ relations as follows:

$$(5.6) \quad \begin{aligned} \sum_{i=0}^{p+1+\lambda} A_{i0}^{(\lambda 0)} x^{i0} + \sum_{i=0}^{p-1+\lambda} A_{i1}^{(\lambda 0)} x^{i1} &= 0 & (\lambda = 0, 1, \dots, \nu-1), \\ \sum_{i=0}^{p+1+\mu} \sum_{j=0}^1 B_{ij}^{(\mu 1)} x^{ij} + \sum_{i=0}^{p-1+\mu} \sum_{j=1}^2 D_{ij}^{(\mu 1)} x^{ij} &= 0 & (\mu = 0, 1, \dots, \nu-3). \end{aligned}$$

By means of the $2\nu-2$ relations (5.6), the 4ν left members of equations (5.4) and (5.5) are expressed linearly in terms of $4\nu-1$ derivatives of x . Hence the left members of (5.4) and (5.5) satisfy a linear relation of the form

$$(5.7) \quad \sum_{i=0}^p \theta_{i0} y^{i0} + \sum_{i=0}^{p-2} \theta_{i1} y^{i1} = \sum_{i=0}^p \phi_{i0} x_{-1}^{i0} + \sum_{i=0}^{p-2} \phi_{i1} x_{-1}^{i1}.$$

This equation shows that the osculating sum-spaces σ' and σ_{-1} meet in a point P_{-1} the coordinates y_{-1} of which are given by the left members, which we exhibit as

$$(5.8) \quad y_{-1} = \sum_{i=0}^p \theta_{i0} y^{i0} + \sum_{i=0}^{p-2} \theta_{i1} y^{i1}.$$

The transformation (5.8), which sends the surface Σ' into the surface Σ'_{-1} , is obviously of the same form as the transformation (5.3) which sends the surface Σ into the surface Σ_{-1} .

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THE GEOMETRY OF RIEMANNIAN SPACES*

BY

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The primary purpose of this paper is to expose, in as simple and clear a form as is possible, the fundamentals of the geometric structure of a Riemannian space.

It is a general truth that the methods which pierce most deeply into the heart of a geometric theory are invariant methods, that is, methods which are independent of the choice of the coordinates in terms of which the theory is expressed analytically. In the case of Riemannian geometry, these are the methods of tensor analysis.

As important, perhaps, as the use of invariant methods is the expression of the analytic theory, so far as possible, in terms of invariant quantities alone. For it is in this form that the theory becomes most illuminating and suggestive. But, in ordinary tensor analysis, the components of a tensor are not invariants. A first step toward our goal will be, then, to introduce for Riemannian geometry an intrinsic tensor analysis, that is, a form of tensor analysis in which the components of all tensors are invariants.

Any theory of the geometry of a Riemannian space presupposes that the space is referred to a certain ennuple of congruences of curves. In the ordinary theory, this ennuple consists of the parametric curves. In the intrinsic theory, it is an ennuple E , whose choice, as will presently be evident, is entirely arbitrary.

The ordinary components of a tensor, that is, the components in the ordinary theory, are referred to the differentials of the coordinates x^i pertaining to the ennuple of the parametric curves. The intrinsic components are referred to the differentials of arc, ds^i , of the curves of the ennuple E .

There is no need in the intrinsic theory of actual coordinates pertaining to the ennuple E ; the differentials of arc ds^i suffice. Accordingly, E can be chosen arbitrarily; it does not have to be a parametric ennuple, that is, an ennuple with which coordinates can be associated. It may be, and we shall ordinarily take it to be, an ennuple of general type, and hence, of course, not necessarily orthogonal.

Intrinsic components, referred to E , of the covariant derivative of a tensor are made possible by the introduction of invariant Christoffel symbols in

* An invited paper, presented to the Society, June 21, 1933, under a different title; received by the editors February 12, 1934.

place of the ordinary ones and the use of directional differentiation along the curves of E instead of partial differentiation with respect to the coordinates x^i .

It is perhaps well to emphasize the fact that we are not introducing a new tensor analysis or a new covariant differentiation, but simply new components for the usual tensors and their customary covariant derivatives.*

The fact that second directional derivatives are not, in general, independent of the order of differentiation has two consequences. On the one hand, it necessitates for directional differentiation conditions of integrability involving a set of invariants, B_{ij}^k , depending on three indices. On the other hand, it implies that the invariant Christoffel symbols, for example, those of the second kind, C_{ij}^k , are not symmetric in i and j . Actually, it turns out that $C_{ij}^k - C_{ji}^k = B_{ij}^k$.

In a previous paper,† the author discussed and compared two concepts bearing on two families of curves on a two-dimensional surface, namely, the concept of distantial spread, a measure of the deviation from equidistance of one of the families of curves with respect to the other, and the concept of angular spread, or associate curvature, a measure of the deviation from parallelism, in the sense of Levi-Civita, of the one family with respect to the other. These concepts, when generalized so as to apply to two congruences of curves in Riemannian space, give rise to a "distantial spread vector" of the two congruences, taken in a given order, and an "angular spread vector" of each

* The intrinsic absolute calculus which we employ may be described, with reference to the literature, from two different points of view. In the first place, it is a generalization to the case of an arbitrary ennuple of the intrinsic absolute calculus with respect to an orthogonal ennuple which gradually grew out of Ricci's theory of an orthogonal ennuple. The invariants associated with a tensor with respect to the orthogonal ennuple came to be known as the orthogonal coordinates of the tensor with respect to the ennuple and corresponding components of the covariant derivative of a tensor, based on Ricci's coefficients of rotation, were eventually introduced. The first complete account of this intrinsic absolute calculus with respect to an orthogonal ennuple seems to be in Berwald, *Differential-invarianten in der Geometrie, Riemannsche Mannigfaltigkeiten und ihre Verallgemeinerungen*, Encyklopädie der Mathematischen Wissenschaften, III, D, 11 (1923), pp. 141-143.

From another point of view, our intrinsic absolute calculus may be described as the result of employing the arcs of the curves of the arbitrary ennuple E as so-called nonholonomic parameters. Though he does not use the term, G. Hessenberg, in his *Vektorielle Begründung der Differentialgeometrie*, Mathematische Annalen, vol. 78 (1917), pp. 187-217, appears to be the first to employ the method of nonholonomic parameters. The Pfaffians in his theory are not differentials of arc, whereas it is essential for our purpose that they should be. Of recent years, nonholonomic parameters have been used by Cartan, Schouten, Vranceanu, Horák, and others, particularly in the study of affine and more general connections and of nonholonomic manifolds. For the formal results in the case of a general linear connection, see Z. Horák, *Die Formeln für allgemeine lineare Übertragung bei Benutzung von nichtholonomen Parametern*, Nieuw Archief, vol. 15 (1928), pp. 193-201.

† Graustein, *Parallelism and equidistance in classical differential geometry*, these Transactions, vol. 34 (1932), pp. 557-593.

congruence with respect to the other. The latter vector is the same as the associate curvature vector of Bianchi and becomes, when the two congruences are identical, the curvature vector of the single congruence.

The intrinsic contravariant components of the distantal spread vector of the i th and j th congruences of the ennuple E are precisely B_{ij}^k , and those of the angular spread vector of the i th congruence with respect to the j th are C_{ij}^k . Thus, the intrinsic Christoffel symbols and the invariants B_{ij}^k have geometric meanings of the first order of importance.

The results thus far described are given in §§1-5. In §6 are to be found applications of distantal spread vectors to questions of equidistance and to the problem of the inclusion of r linearly independent congruences of curves in a family of r -dimensional surfaces. Thereby further geometrical interpretations of the invariants B_{ij}^k are obtained.

In §7 special types of ennuples are discussed: parametric ennuples; particular parametric ennuples, designated as ennuples of Tchebycheff and characterized by the fact that the differentials of arc ds^i are all exact; and, finally, Cartesian ennuples, that is, Tchebycheff ennuples whose angles are all constant. The next section treats of ennuples Cartesian at a point and their relationship to coordinates geodesic at a point.

A digression is made in §10 to apply the methods previously developed to spaces with general metric connections. Geometric interpretations of the intrinsic components of the tensor of torsion are found, in terms of torsion vectors closely allied to the torsion vector of Cartan, and relations between these torsion vectors and the distantal and angular spread vectors are established. Application of the results is made to spaces admitting absolute parallelism.

In §11 the transformation from one ennuple of congruences to a second is discussed. Of course, the transformation from the intrinsic components of a tensor, referred to the one ennuple, to those of the same tensor, referred to the other ennuple, is found to obey the formal laws of tensor analysis. Moreover, the relations between the intrinsic Christoffel symbols for the two ennuples are patterned precisely after the equations of Christoffel. But these Christoffel symbols may be interpreted in terms of the curvature and associate curvature vectors of the congruences of the two ennuples, as already noted. Thus, Christoffel's equations, expressed in terms of invariants, are simply a generalization to Riemannian geometry of the fundamental formula of Liouville for geodesic curvatures on a two-dimensional surface.

In §12 the general problem of the determination of the family of surfaces of lowest dimensionality in which lie all the congruences of an arbitrarily chosen set of congruences of curves is discussed. The problem is, of course,

identical with that of the determination of the maximum number of functionally independent integrals of a system of linear homogeneous partial differential equations of the first order. It is considered here from a geometric point of view and is shown to depend, for its solution, on the consideration of a sequence of sets of vectors such that the vectors of each set are distasteful spread vectors of congruences determined by the vectors of the preceding sets. Application of the results is made to nonholonomic manifolds.

1. **Oblique ennuple of congruences.** Let there be given in a Riemannian space V_n , referred to coordinates (x^1, x^2, \dots, x^n) , an arbitrarily chosen ennuple, E , consisting of n ordered linearly independent congruences of directed curves, and let the contravariant components of the field of unit vectors tangent to the curves C_i of the i th congruence, and directed in the same senses as these curves, be $\bar{a}_i^j, j=1, 2, \dots, n$.

Suppose that \bar{a}^i_j is the cofactor of \bar{a}_i^j in the determinant $|\bar{a}_i^j|$, divided by the determinant: $\bar{a}_k^i \bar{a}^j_k = \delta_k^j$, $\bar{a}_i^i \bar{a}^i_k = \delta_k^i$. Then \bar{a}^i_j , for i fixed and $j=1, 2, \dots, n$, are the covariant components of a vector-field, or, more simply, a vector, which is perpendicular to the tangent vectors of all n congruences except that of the curves C_i . The n vectors thus determined are known as the vectors conjugate (or reciprocal) to the n unit vectors tangent to the curves of E .

If $\partial/\partial s^i$ denotes directional differentiation in the positive direction of an arbitrary curve C_i ,

$$(1a) \quad \bar{a}_i^j = \frac{\partial x^j}{\partial s^i}.$$

Suppose that we write also, purely as a matter of notation,

$$(1b) \quad \bar{a}^i_j = \frac{\partial s^i}{\partial x^j}.$$

Then the relations between \bar{a}_i^j and \bar{a}^i_j become

$$(2) \quad \frac{\partial x^j}{\partial s^k} \frac{\partial s^i}{\partial x^j} = \delta_k^i, \quad \frac{\partial x^i}{\partial s^j} \frac{\partial s^j}{\partial x^k} = \delta_k^i \quad (i, k = 1, 2, \dots, n).$$

The first set of these equations says that the Pfaffian $\bar{a}^i_j dx^j$ has the value zero for every curve of E except a curve C_i and for a curve C_i is equal to the differential of arc of the curve, measured in the positive direction along it. Thus, the relations between the differentials of arc ds^i of the curves C_i and the differentials dx^i are

$$(3) \quad ds^i = \frac{\partial s^i}{\partial x^j} dx^j, \quad dx^i = \frac{\partial x^i}{\partial s^j} ds^j \quad (i = 1, 2, \dots, n).$$

The relations between the directional derivatives $\partial/\partial s^i$ and the partial derivatives $\partial/\partial x^i$ are obviously

$$(4) \quad \frac{\partial f}{\partial s^i} = \frac{\partial x^j}{\partial s^i} \frac{\partial f}{\partial x^j}, \quad \frac{\partial f}{\partial x^i} = \frac{\partial s^j}{\partial x^i} \frac{\partial f}{\partial s^j} \quad (i = 1, 2, \dots, n).$$

From (2), (3), and (4) it follows that

$$(5) \quad df = \frac{\partial f}{\partial s^i} ds^i.$$

Conditions of integrability. The fundamental relations

$$\frac{\partial}{\partial x^q} \frac{\partial f}{\partial x^p} - \frac{\partial}{\partial x^p} \frac{\partial f}{\partial x^q} = 0 \quad (p, q = 1, 2, \dots, n),$$

when expressed in terms of directional derivatives, take the form

$$(6) \quad \frac{\partial}{\partial s^j} \frac{\partial f}{\partial s^i} - \frac{\partial}{\partial s^i} \frac{\partial f}{\partial s^j} = B_{ij}^k \frac{\partial f}{\partial s^k} \quad (i, j = 1, 2, \dots, n),$$

where

$$(7a) \quad B_{ij}^k = \frac{\partial x^r}{\partial s^j} \frac{\partial}{\partial s^i} \frac{\partial s^k}{\partial x^r} - \frac{\partial s^r}{\partial s^i} \frac{\partial}{\partial s^j} \frac{\partial s^k}{\partial x^r},$$

or

$$(7b) \quad B_{ij}^k = \frac{\partial s^k}{\partial x^r} \left(\frac{\partial}{\partial s^j} \frac{\partial x^r}{\partial s^i} - \frac{\partial}{\partial s^i} \frac{\partial x^r}{\partial s^j} \right).$$

The expression in (7b) follows from that in (7a) by virtue of the relations obtained by directional differentiation of the first set of equations (2).

THEOREM 1. *A necessary and sufficient condition that $f_i ds^i$, where $f_i = f_i(x^1, x^2, \dots, x^n)$, be an exact differential is that*

$$(8) \quad \frac{\partial f_i}{\partial s^j} - \frac{\partial f_j}{\partial s^i} = B_{ij}^k f_k \quad (i, j = 1, 2, \dots, n).$$

The theorem follows directly from (6) inasmuch as, according to (5), $f_i ds^i$ is an exact differential if and only if there exists a function f such that $\partial f / \partial s^i = f_i$, $i = 1, 2, \dots, n$.

2. Intrinsic tensor analysis. The geometric basis, or system of reference, for ordinary tensor analysis is the system of parametric hypersurfaces $x^i = c_i$, $i = 1, 2, \dots, n$, or the corresponding ennuple of congruences of parametric curves. This ennuple is evidently of very special type.

As the system of reference for our intrinsic tensor analysis, we take the arbitrarily chosen ennuple E of the preceding section.

In the ordinary theory, the basic differentials are the differentials dx^i and the basic derivatives are the partial derivatives $\partial/\partial x^i$. In the intrinsic theory it is the differentials of arc, ds^i , and the directional derivatives, $\partial/\partial s^i$, which are fundamental.

Whereas the ordinary components of a tensor, that is, the components in the ordinary theory, are referred to dx^i and $\partial/\partial x^i$, the intrinsic components are to be referred to ds^i and $\partial/\partial s^i$. For example, if \bar{b}_{ij}^{kl} are the ordinary components of a tensor of the fourth order, that is, if $\bar{b}_{ij}^{kl} dx^i dx^j (\partial f/\partial s^k) (\partial \phi/\partial s^l)$, where f and ϕ are invariant functions, is an invariant, the intrinsic components, b_{ij}^{kl} , of the tensor are to be such that $b_{ij}^{kl} ds^i ds^j (\partial f/\partial s^k) (\partial \phi/\partial s^l)$ is the new form of this invariant:

$$(9) \quad \bar{b}_{ij}^{kl} dx^i dx^j \frac{\partial f}{\partial x^k} \frac{\partial \phi}{\partial x^l} = b_{ij}^{kl} ds^i ds^j \frac{\partial f}{\partial s^k} \frac{\partial \phi}{\partial s^l}.$$

To obtain the transformation from the ordinary to the intrinsic components of a tensor, we should substitute for dx^i , δx^i , \dots , $\partial f/\partial x^i$, $\partial \phi/\partial x^i$, \dots in an equation such as (9) their values in terms of ds^i , δs^i , \dots , $\partial f/\partial s^i$, $\partial \phi/\partial s^i$, \dots , as given by (3) and (4). But equations such as (9) and the transformations (3) and (4) have the same form as the analogous equations and transformations associated with a change from the coordinates x^i to new coordinates y^i . Hence, *the transformation from the ordinary to the intrinsic components of a tensor obeys the standard formal laws of tensor analysis*. If, in the transformation of the components of a tensor which is the result of a change from the coordinates x^i to coordinates y^i , y^i is replaced by s^i , the transformation becomes that from the ordinary components to the intrinsic components.

Thus, if \bar{g}_{ij} and \bar{g}^{ij} are the ordinary covariant and contravariant components, and g_{ij} and g^{ij} the corresponding intrinsic components, of the fundamental tensor, we have

$$(10a) \quad g_{ij} = \bar{g}_{kl} \frac{\partial x^k}{\partial s^i} \frac{\partial x^l}{\partial s^j}, \quad \bar{g}_{ij} = g_{kl} \frac{\partial s^k}{\partial x^i} \frac{\partial s^l}{\partial x^j},$$

$$(10b) \quad g^{ij} = \bar{g}^{kl} \frac{\partial s^i}{\partial x^k} \frac{\partial s^j}{\partial x^l}, \quad \bar{g}^{ij} = g^{kl} \frac{\partial x^i}{\partial s^k} \frac{\partial x^j}{\partial s^l}.$$

The invariant form of the linear element, $ds^2 = \bar{g}_{ij} dx^i dx^j$, is

$$(11) \quad ds^2 = g_{ij} ds^i ds^j.$$

Since \bar{g}_{ij} and \bar{g}^{ij} are symmetric, so also are g_{ij} and g^{ij} .

Again, the relations between the ordinary contravariant and covariant

components, \bar{a}^i and \bar{a}_i , of a vector and the corresponding intrinsic components, a^i and a_i , are

$$(12a) \quad a^i = \bar{a}^i \frac{\partial s^i}{\partial x^i}, \quad \bar{a}^i = a^i \frac{\partial x^i}{\partial s^i},$$

$$(12b) \quad a_i = \bar{a}_j \frac{\partial x^j}{\partial s^i}, \quad \bar{a}_i = a_j \frac{\partial s^j}{\partial x^i}.$$

Inasmuch as $\partial x^j / \partial s^i$ and $\partial s^i / \partial x^j$, for i fixed and $j = 1, 2, \dots, n$, are respectively the contravariant and covariant components of vectors, namely, the i th tangent and the i th conjugate vector associated with E , it follows from equations such as the first sets in (10) and (12) that *the intrinsic components of a tensor are actually invariants*.*

Components of the vectors pertaining to E . If $\bar{a}_h|^i$ and $\bar{a}_h|_i$ are the ordinary contravariant and covariant components, and $a_h|^i$ and $a_h|_i$ the corresponding intrinsic components, of the field of unit vectors tangent to the curves C_h of the h th congruence of E , we have

$$(13a) \quad \bar{a}_h|^i = \frac{\partial x^i}{\partial s^h}, \quad \bar{a}_h|_i = g_{hi} \frac{\partial s^i}{\partial x^i},$$

$$(13b) \quad a_h|^i = \delta_h^i, \quad a_h|_i = g_{hi}.$$

Formulas (13b) follow from (13a) by means of (12) and (2), and the second equation in (13a) follows from the first by virtue of (10a) and (2).

Denoting the ordinary covariant and contravariant components of the h th conjugate vector-field by $\bar{a}^h|_i$ and $\bar{a}^h|^i$, and the corresponding intrinsic components by $a^h|_i$ and $a^h|^i$, we have†

$$(14a) \quad \bar{a}^h|_i = \frac{\partial s^h}{\partial x^i}, \quad \bar{a}^h|^i = g^{hi} \frac{\partial x^i}{\partial s^i},$$

$$(14b) \quad a^h|_i = \delta_i^h, \quad a^h|^i = g^{hi}.$$

Geometric interpretations. The first of the formulas (10a) says that

$$(15) \quad g_{ii} = 1, \quad g_{ij} = \cos \omega_{ij} \quad (i, j = 1, 2, \dots, n),$$

where ω_{ij} is the angle‡ at $P: (x^1, x^2, \dots, x^n)$ between the directed curves C_i and C_j which pass through P .

* From the usual point of view, the first sets of equations in (10) and (12) define invariants pertaining to the given tensors with respect to E , and the second sets express the ordinary components of the tensors in terms of these invariants and the components of the vectors pertaining to E . See the long footnote in the introduction and, for example, Eisenhart, *Riemannian Geometry*, p. 97.

† It is to be noted, from (13a), (14a), and (11), that $\bar{a}_i|^i = \bar{a}^i|_i$ and $\bar{a}^i|^i = \bar{a}_i|_i$.

‡ The angle ϕ between two vectors at a point shall be restricted to lie in the interval $0 \leq \phi \leq \pi$.

Inasmuch as

$$\sum_i a^h | a^h |^i = \sum_i \delta_i^h \delta_h^i = 1,$$

we conclude that the length of the conjugate vector $a^h |$ at the point $P: (x^i)$ is equal to $\sec \theta_h$, where θ_h is the angle at P between the vector $a^h |$ and the tangent vector $a_h |$ at P . It follows that $0 \leq \theta_h < \pi/2$.

The geometric meaning of the first of the formulas (10b) is now clear:

$$(16) \quad g^{ii} = \sec^2 \theta_i, \quad g^{ij} = \cos \Omega_{ij} \sec \theta_i \sec \theta_j \quad (i, j = 1, 2, \dots, n),$$

where Ω_{ij} is the angle at P between the conjugate vectors $a^i |$ and $a^j |$.

We now introduce in the flat space, S_n , tangent to V_n at P , the Cartesian coordinate system with respect to which the intrinsic contravariant components X^i of an arbitrary vector V at P are the coordinates (X^1, X^2, \dots, X^n) of the "terminal point" Q of V . The axes of this system, which we shall call the intrinsic contravariant coordinate system at P , are the directed tangents to the curves of E which pass through P . Furthermore, inasmuch as the tangent vector $a_h |^i = \delta_h^i$ is of unit length, the unit of measure on each axis, relative to measurement in V_n , is actually unity.

We also introduce in S_n the intrinsic covariant coordinate system, with respect to which the intrinsic covariant components X_i of the vector V are the coordinates (X_1, X_2, \dots, X_n) of the point Q . The axes of this system coincide in direction and sense with the conjugate vectors at P . The unit of measurement on the h th axis is not unity, but $\sec \theta_h$; for the h th conjugate vector $a^h |_i = \delta_i^h$ is of length $\sec \theta_h$.

THEOREM 2. *The intrinsic covariant (contravariant) components of a vector V at a point P are, on the one hand, the orthogonal projections of V on the axes of the intrinsic contravariant (covariant) system of coordinates at P , and, on the other hand, the parallel projections of V on the axes of the covariant (contravariant) system of coordinates at P .*

The second part of the theorem amounts to the previous identification of the components of V as Cartesian coordinates. The first part follows from the relations $a_h |^i X_i = X^h$, $a^h |_i X^i = X^h$. In particular, we have in $a_h |_i = g_{hi}$ and $a^h |^i = g^{hi}$ new interpretations of g_{hi} and g^{hi} .

Case of an orthogonal ennuple. If each two congruences of E cut at right angles, the tangent vectors $a_h |$ form an orthogonal ennuple of vectors, the conjugate vectors $a^h |$ become unit vectors coincident with the corresponding tangent vectors, the intrinsic contravariant and covariant coordinate systems of Theorem 2 coincide in a rectangular system, parallel and orthogonal projections on the axes of this system are identical, and the intrinsic contra-

variant and covariant components of a vector are the same. Furthermore, inasmuch as now $g_{ii} = g^{ii} = 1$, $g_{ij} = g^{ij} = 0$, $i \neq j$, any two tensors which are associate to one another in that they are obtainable from one another by raising subscripts or lowering superscripts by means of the fundamental tensor, have the same components.

3. *Intrinsic covariant differentiation.* We now introduce, for the ennuple E , invariant Christoffel symbols, C_{ijk} and C_i^j , to take the place of the ordinary Christoffel symbols,

$$\bar{C}_{ijk} = [ij, k], \quad \bar{C}_i^j = \left\{ \begin{matrix} k \\ ij \end{matrix} \right\}.$$

Inasmuch as we shall assume that $C_{ijk} = g_{hk} C_{ij}^h$, it suffices to define C_{ij}^k . We first write the formula for the transformation of \bar{C}_{ij}^k induced by a change from the coordinates x^i to coordinates y^i . In this formula we replace the first partial derivatives of x^i with respect to y^i by the corresponding directional derivatives and the single second partial derivative by a specific one of the two corresponding directional derivatives. Thus we get

$$(17) \quad C_{ij}^k \frac{\partial x^i}{\partial s^k} = \bar{C}_{pq}^k \frac{\partial x^p}{\partial s^i} \frac{\partial x^q}{\partial s^j} + \frac{\partial}{\partial s^i} \frac{\partial x^i}{\partial s^j}.$$

Since second directional derivatives are not, in general, independent of the order of differentiation, C_{ij}^k is not, in general, symmetric in i and j . In fact, we have, from (17) and (7a), that

$$(18) \quad C_{ij}^k - C_{ji}^k = B_{ij}^k.$$

Intrinsic covariant differentiation. Formula (17) enables us to find the intrinsic components of the covariant derivative of any tensor in terms of the intrinsic components of the tensor.

Consider, for example, the vector of equations (12). The ordinary components of the covariant derivative of this vector are

$$(19) \quad \bar{a}_{i,j} = \frac{\partial \bar{a}_i}{\partial x^j} - \bar{a}_h \bar{C}_{ij}^h, \quad \bar{a}^i_{,j} = \frac{\partial \bar{a}^i}{\partial x^j} + \bar{a}^h \bar{C}_{hj}^i.$$

The intrinsic components, which we shall denote by $a_{i,j}$ and $a^i_{,j}$ respectively, are expressible in terms of the ordinary components, according to the definitions of §2, by the formulas

$$(20) \quad a_{i,j} = \bar{a}_{k,l} \frac{\partial x^k}{\partial s^i} \frac{\partial x^l}{\partial s^j}, \quad a^i_{,j} = \bar{a}^k_{,l} \frac{\partial s^i}{\partial x^k} \frac{\partial x^l}{\partial s^j}.$$

When the values of $\bar{a}_{i,j}$ and $\bar{a}^i_{,j}$, rewritten in terms of directional derivatives

and the intrinsic components a_i and a^i of the vector, are substituted in (20), these formulas take on, by virtue of (17), the following desired forms*:

$$(21) \quad a_{i,j} = \frac{\partial a_i}{\partial s^j} - a_h C_{ij}^h, \quad a^i{}_{,j} = \frac{\partial a^i}{\partial s^j} + a^h C_{hj}^i.$$

In comparing (21) with (19), it must be borne in mind that C_{ij}^k , unlike \bar{C}_{ij}^k , is not symmetric in the two subscripts. In (21) and, in fact, in all similar formulas for the intrinsic components of covariant derivatives of tensors, it will be found that the second subscript on the C indicates the component of the covariant derivative in question.† In the corresponding formulas for the ordinary components of covariant derivatives, the second subscript on the \bar{C} may be, and usually is, made to play the same role. The two sets of formulas have, then, the same forms.

It is evident that, if the ordinary components of a tensor are all zero, so also are the intrinsic components. Thus, since $\bar{g}_{ij,k} = 0$ and $\bar{g}^{ij}{}_{,k} = 0$, it follows that $g_{ij,k} = 0$ and $g^{ij}{}_{,k} = 0$.

Similarly, it follows that, if $f_{,ij}$ are the intrinsic components of the covariant derivative of the gradient, $f_{,i} = \partial f / \partial s^i$, of an invariant function f , then $f_{,ij} = f_{,ji}$. Hence, a necessary and sufficient condition that the vector with the intrinsic covariant components a_i be the gradient of a function is that $a_{i,j}$ be a symmetric tensor.

A little consideration shows that this last proposition should be simply a restatement of Theorem 1. As a matter of fact, it is readily proved that

$$(22) \quad a_{i,j} - a_{j,i} \equiv \frac{\partial a_i}{\partial s^j} - \frac{\partial a_j}{\partial s^i} - B_{ij}^k a_k.$$

Invariant form of C_{ijk} . Setting

$$(23) \quad C_{ijk} = g_{kh} C_{ij}^h, \quad B_{ijk} = g_{kh} B_{ij}^h,$$

we get, from (18),

$$(24) \quad C_{ijk} - C_{jik} = B_{ijk}.$$

Since $g_{ik,j} = 0$, we also have

$$(25) \quad C_{ijk} + C_{kji} = \frac{\partial g_{ik}}{\partial s^j}.$$

* We might have started with these forms, with C_{ij}^k unknown, and then derived (17) from them.

† This corresponds to the fact that in (17) the second subscript in C_{ij}^k indicates the second differentiation in the formation of the last term.

Equations (24) and (25) are n^3 in number and yield a unique solution for C_{ijk} , namely*

$$(26) \quad C_{ijk} = \frac{1}{2} \left[\frac{\partial g_{ik}}{\partial s^j} + \frac{\partial g_{jk}}{\partial s^i} - \frac{\partial g_{ij}}{\partial s^k} + B_{ijk} + B_{jki} - B_{kij} \right].$$

The corresponding expression for C_{ij}^k is readily obtained.

4. Geometric interpretations of invariant Christoffel symbols. These have to do with the curvature and associate curvature vectors of the congruences of the given ennuple.

If a^i are the intrinsic contravariant components of the field of unit vectors tangent to the directed curves C of a given congruence, the vector

$$(27a) \quad c^r = a^r_{;j} a^j = \left(\frac{\partial a^r}{\partial s^j} + C_{ij}^r a^i \right) a^j$$

is the curvature vector of the curves C and its identical vanishing is the condition that the curves C be geodesics.

The vector

$$(27b) \quad c^r = b^r_{;j} a^j = \left(\frac{\partial b^r}{\partial s^j} + C_{ij}^r b^i \right) a^j,$$

where b^i are the intrinsic components of an arbitrary field of unit vectors, is known as the associate curvature vector of this vector field with respect to the curves C . It is identically the null vector when and only when the vectors of the field are parallel, in the sense of Levi-Civita, with respect to the curves C .

If the vector-field b^i originates as the field of unit vectors tangent to the directed curves K of a certain congruence, we call the vector (27b) the associate curvature vector of the curves K with respect to the curves C and say that the curves K are parallel with respect to the curves C when and only when it is identically null.

Curvature vectors of the given congruences. For the curves C_h of the ennuple E , $a^r = a_h^r|' = \delta_h^r$. Hence (27a) becomes $c^r = C_{hh}^r$.

* The corresponding formula in Hessenberg, loc. cit., p. 211, has the same form, though Hessenberg employs, instead of ds^i and $\partial/\partial s^i$, the Pfaffians $du^i = ds^i/\rho_i$ and the corresponding derivatives $\partial/\partial u^i = \rho_i(\partial/\partial s^i)$, where ρ_i are invariants. This is, of course, to be expected, inasmuch as equations (24) and (25) are obviously unchanged by this change of Pfaffians.

The change of Pfaffians still leaves the space referred to the ennuple E . The transformations which it effects on the fundamental quantities are readily found to be $g_{ij}^* = \rho_i \rho_j g_{ij}$ and

$$\rho_k B_{ij}^{*k} = \rho_i \rho_j B_{ij}^k + \rho_i \delta_i^k \frac{\partial \rho_j}{\partial s^i} - \rho_j \delta_j^k \frac{\partial \rho_i}{\partial s^i}, \quad \rho_k C_{ij}^{*k} = \rho_i \rho_j C_{ij}^k + \rho_i \delta_i^k \frac{\partial \rho_j}{\partial s^i}.$$

It will be clear from these relations, after the perusal of the next two sections, why it is essential that we use ds^i as the basic Pfaffians.

THEOREM 3. *The intrinsic components, $c_{hh}|^i$ and $c_{hh}|_i$, of the curvature vector of the curves C are respectively C_{hh}^i and C_{hhi} .**

$$(28) \quad c_{hh}|^i = C_{hh}^i, \quad c_{hh}|_i = C_{hhi} \quad (i = 1, 2, \dots, n).$$

According to Theorem 2, C_{hh}^i and C_{hhi} are respectively the orthogonal and parallel (parallel and orthogonal) projections of the curvature vector of the curves C_h on the i th axis of the intrinsic contravariant (covariant) coordinate system at $P:(x^i)$. Since the curvature vector of a curve is perpendicular to the curve, C_{hhh} should be zero and this is the case.

COROLLARY. *A necessary and sufficient condition that the curves C_h be geodesics is that $C_{hh}^i = 0$ or $C_{hhi} = 0$, $i = 1, 2, \dots, n$.*

Associate curvature vectors. When we set $b^r = a_h|r$ and $a^j = a_k|j$, (27b) becomes $c^r = C_{hk}^r$.

THEOREM 4. *The intrinsic components, $c_{hk}|^i$ and $c_{hk}|_i$, of the associate curvature vector of the curves C_h with respect to the curves C_k are respectively C_{hk}^i and C_{hki} :*

$$(29) \quad c_{hk}|^i = C_{hk}^i, \quad c_{hk}|_i = C_{hki} \quad (i = 1, 2, \dots, n).$$

In the sense of Theorem 2, C_{hki} and C_{hk}^i are respectively the orthogonal and parallel projections, on the curves C_i , of the associate curvature vector of the curves C_h with respect to the curves C_k . In particular, $C_{hkh} = 0$; this vector is perpendicular to C_h .

COROLLARY. *The curves C_h are parallel with respect to the curves C_k if and only if $C_{hk}^i = 0$ or $C_{hki} = 0$, $i = 1, 2, \dots, n$.*

Geometric interpretation of intrinsic covariant differentiation. The geometric significance of the first of formulas (21) is now clear.

THEOREM 5. *The (i, j) th intrinsic component of the covariant derivative of a covariant tensor is equal to the directional derivative, along C_j , of the i th intrinsic component of the vector, minus the scalar product of the vector with the associate curvature vector of the curves C_i with respect to the curves C_j .*

In particular, the (i, j) th component reduces to $\partial a_i / \partial s^j$ if and only if the vector is always perpendicular to the associate curvature vector in question.

We conclude also: (a) for every vector a , but for a fixed i , $a_{i,j}$ reduces to $\partial a_i / \partial s^j$ for $j = 1, 2, \dots, n$ when and only when the curves C_i are parallel with respect to all the curves C_j , $j = 1, 2, \dots, n$; and (b) for every vector a ,

* For the case of an orthogonal ennuple, this theorem is known; see Levi-Civita, *The Absolute Differential Calculus*, p. 275.

but for a fixed j , $a_{i,j}$ reduces to $\partial a_i / \partial s^j$ for $i = 1, 2, \dots, n$ if and only if all the curves C_i , $i = 1, 2, \dots, n$, are parallel with respect to the curves C_j . In particular, the curves C_i in case (a), and the curves C_j in case (b), must be geodesics.

From the geometric interpretation of $a_{i,j}$ we may pass to one for $a^i_{,j}$ by means of the relation $a^i_{,j} = g^{ir} a_{r,j}$.

Geodesics and parallelism. Suppose that there is given a directed curve C : $x^i = x^i(s)$, expressed parametrically in terms of the arc s , and let \bar{a}^i and a^i be the ordinary and intrinsic contravariant components of the unit vector tangent to C at an arbitrary point P of C .

For the curve C , $dx^i = \bar{a}^i ds$. But $dx^i = (\partial x^i / \partial s^j) ds^j$. Thus, $(\partial x^i / \partial s^j) ds^j = \bar{a}^i ds$. Hence, since $\bar{a}^i = a^i (\partial x^i / \partial s^j)$, it follows that $ds^i = a^i ds$. For the curve C , we have, then,

$$(30) \quad \bar{a}^i = \frac{dx^i}{ds}, \quad a^i = \frac{ds^i}{ds} \quad (i = 1, 2, \dots, n).$$

Applying these results to the right-hand side of (27a), set equal to zero, we obtain, as the conditions that the curve C be a geodesic:

$$\frac{d^2 s^r}{ds^2} + C_{ir} \frac{ds^i}{ds} \frac{ds^j}{ds} = 0 \quad (r = 1, 2, \dots, n),$$

where $d^2 s^r / ds^2$ stands for $(d/ds)(ds^r/ds)$.

From (27b) we obtain similar conditions that unit vectors b in the points of the curve C be parallel with respect to the curve C .

Coefficients of rotation. The coefficients of rotation of the ennuple E , patterned after Ricci's coefficients of rotation for an orthogonal ennuple, are of two kinds, namely,

$$\gamma_{ihk} = a_i |_{i,j} a_h |^j a_k |^i, \quad \gamma^i_{hk} = a^i |_{i,j} a_h |^j a_k |^i.$$

It is readily shown that

$$\gamma_{ihk} = -C_{hkl} + \frac{\partial g_{hl}}{\partial s^k} = C_{ikh}, \quad \gamma^i_{hk} = -C_{hk}^i,$$

and hence, in case E is an orthogonal ennuple, that

$$\gamma_{ihk} = \gamma^i_{hk} = -C_{hkl} = -C_{hk}^i.$$

Thus, the coefficients of rotation of any ennuple E are identical, essentially, with the invariant Christoffel symbols formed for the ennuple. Theorems 3 and 4 furnish, then, simple geometric interpretations of the coefficients of rotation.

5. The distantial spread vector. The associate curvature vector of the congruence of curves C_h with respect to the congruence of curves C_k is based on angle. Since its length is a measure of the deviation from parallelism of the first congruence with respect to the second, we may call it, in accordance with a terminology employed in a previous paper,* the angular spread vector of the curves C_h with respect to the curves C_k .

We shall now introduce for the two congruences of curves a "spread vector" which is based on distance and which we shall call the *distantial spread vector* of the two congruences, in a given order.

Let P be an arbitrarily chosen point in V_n and let C_h and C_k be the curves of the two congruences which pass through P . Mark on C_h the point P_1 at the directed distance Δs^h from P and on C_k the point P_2 at the directed distance Δs^k from P . On the curve of the k th congruence through P_1 mark the point Q_1 at the directed distance Δs^k from P_1 , and on the curve of the h th congruence through P_2 mark the point Q_2 at the directed distance Δs^h from P_2 , and draw the vector $\overline{Q_1 Q_2}$ joining Q_1 to Q_2 . Then, the limit of the ratio $\overline{Q_1 Q_2}/(\Delta s^h \Delta s^k)$, when Δs^h and Δs^k approach zero, exists and is defined as the distantial spread vector, at P , of the congruences of curves C_h and C_k , in this order.

If $\bar{b}_{hk}|^i$ are the ordinary contravariant components of this vector, the definition says that

$$\bar{b}_{hk}|^i = \lim_{\Delta s^h, \Delta s^k \rightarrow 0} \frac{w^i - z^i}{\Delta s^h \Delta s^k},$$

where (z^i) and (w^i) are respectively the coordinates of Q_1 and Q_2 .

It is readily shown that

$$z^i = x^i + \frac{\partial x^i}{\partial s^h} h + \frac{\partial x^i}{\partial s^k} k + \frac{1}{2!} \left(\frac{\partial^2 x^i}{\partial s^{h2}} h^2 + 2 \frac{\partial}{\partial s^h} \frac{\partial x^i}{\partial s^k} h k + \frac{\partial^2 x^i}{\partial s^{k2}} k^2 \right) + \dots,$$

where $h = \Delta s^h$ and $k = \Delta s^k$ and the coefficients are evaluated for P . From these coordinates for Q_1 we obtain the coordinates (w^i) of Q_2 by interchanging h and k and $\partial/\partial s^h$ and $\partial/\partial s^k$. Thus we find that

$$\bar{b}_{hk}|^i = \frac{\partial}{\partial s^k} \frac{\partial x^i}{\partial s^h} - \frac{\partial}{\partial s^h} \frac{\partial x^i}{\partial s^k}.$$

It follows from this result and (7b) that $b_{hk}|^i = B_{hk}^i$, where $b_{hk}|^i$ are the intrinsic components of the distantial spread vector.

THEOREM 6. *The intrinsic components, $b_{hk}|^i$ and $b_{hk}|_i$, of the distantial spread vector of the curves C_h and the curves C_k are respectively B_{hk}^i and B_{hki} :*

$$(31) \quad b_{hk}|^i = B_{hk}^i, \quad b_{hk}|_i = B_{hki}.$$

* Graustein, loc. cit., p. 559.

We thus have simple geometric interpretations of the invariants B .

It is evident that $b_{kk}|^i = -b_{kk}|^i$. In particular, $b_{kk}|^i = 0$.

By virtue of (28), (29), and (31), relations (18) and (24) become

$$(32) \quad c_{kk}|^i - c_{kk}|^i = b_{kk}|^i, \quad c_{kk}|^i - c_{kk}|^i = b_{kk}|^i.$$

Thus, the difference between the angular spread vector of the curves C_k with respect to the curves C_k and that of the curves C_k with respect to the curves C_k is equal to the distantal spread vector of the curves C_k and C_k . In particular:

THEOREM 7. *If two of the three spread vectors of two congruences are null, the third is also.*

Further interpretations of the distantal spread vector are given in §§6, 12.

Some identities. We note, for future use, the identical relations

$$(33) \quad \frac{\partial}{\partial s^i} B_{jk}^m + \frac{\partial}{\partial s^j} B_{ki}^m + \frac{\partial}{\partial s^k} B_{ij}^m = B_{ir}^m B_{jk}^r + B_{jr}^m B_{ki}^r + B_{kr}^m B_{ij}^r,$$

which are readily established by substituting for the B 's their values from (7a) and making use of the integrability conditions (6).

The corresponding relations for the B_{ijk} are obtained from these by multiplying by g_{pm} , summing over m , and applying (25). They are

$$(34) \quad \frac{\partial}{\partial s^i} B_{jkp} + \frac{\partial}{\partial s^j} B_{kip} + \frac{\partial}{\partial s^k} B_{ijp} = D_{irp} B_{jk}^r + D_{jrp} B_{ki}^r + D_{krp} B_{ij}^r,$$

where

$$D_{irp} = B_{irp} + \frac{\partial g_{ip}}{\partial s^r} = C_{irp} + C_{pir}.$$

6. Applications and interpretations of distantal spread vectors. In this connection we shall first discuss the question of the inclusion of the curves of two or more congruences in subspaces of V_n , and show that it finds its answer in conditions on the distantal spread vectors of the congruences.

A family of r -dimensional surfaces ($r=2, \dots, n-1$) consists of the ∞^{n-r} r -dimensional surfaces defined by $n-r$ equations of the form $\phi^i(x^1, x^2, \dots, x^n) = c_i$, $i=1, 2, \dots, n-r$, where $\phi^1, \phi^2, \dots, \phi^{n-r}$ are functionally independent and c_1, c_2, \dots, c_{n-r} are arbitrary constants. A family of $(n-1)$ -dimensional surfaces is called a family of hypersurfaces.

A family of r -dimensional surfaces is said to contain a congruence of curves, or the congruence is said to lie in it, if each surface of the family contains ∞^{r-1} curves of the congruence.

THEOREM 8. *The family of hypersurfaces $\phi(x^1, x^2, \dots, x^n) = \text{const.}$ contains a given congruence of curves if and only if the directional derivative of ϕ along the curves of the congruences is identically zero.*

The theorem is self-evident.

THEOREM 9. *The curves of r linearly independent congruences of curves lie in a family of r -dimensional surfaces if and only if the distantal spread vectors of the congruences, taken in pairs, are linear combinations of the r tangent vectors of the congruences.**

Without loss of generality, we may assume that the given congruences are the first r congruences of the ennuple E . According to Theorem 8, the family of hypersurfaces $\phi = \text{const.}$ contains these r congruences if and only if $\partial\phi/\partial s^i = 0$, $i = 1, 2, \dots, r$. Hence, the r congruences lie in a family of r -dimensional surfaces if and only if the system of differential equations

$$\frac{\partial\phi}{\partial s^i} = 0 \quad (i = 1, 2, \dots, r)$$

is completely integrable. But the conditions of integrability, as obtained from (6), reduce to

$$\sum_{k>r} B_{ij}{}^k \frac{\partial\phi}{\partial s^k} = 0 \quad (i, j = 1, 2, \dots, r),$$

and hence are identically satisfied when and only when

$$B_{ij}{}^k = 0 \quad (i, j = 1, 2, \dots, r; k = r+1, \dots, n).$$

But, by (31) and (13b), these equations constitute necessary and sufficient conditions that the distantal spread vectors of the first r congruences, taken in pairs, are linearly dependent on the tangent vectors of these congruences.

The theorem is of the greatest interest in the cases $r=2$ and $r=n-1$. We shall discuss these cases in detail, with the purpose of bringing out the bearing of distantal spread vectors on equidistance. In this discussion, indices a, b, c, \dots are fixed, whereas indices i, j, k, \dots vary from 1 to n , except as otherwise stated.

Case $r=2$. A typical instance of this case is the following.

THEOREM 10. *The two congruences consisting of the curves C_a and C_b of the ennuple E lie in a family of two-dimensional surfaces if and only if*

$$(35) \quad B_{ab}{}^k = 0 \quad (k \neq a, b).$$

* For this theorem, expressed in terms of associate curvature vectors, see Struik, *Grundzüge der mehrdimensionalen Differentialgeometrie*, p. 53.

The pair of differential equations $\partial\phi/\partial s^a=0$, $\partial\phi/\partial s^b=0$ have, then, $n-2$ functionally independent solutions, ϕ^k , $k \neq a, b$, and the $n-2$ equations $\phi^k=c_k$, $k \neq a, b$, define the family of two-dimensional surfaces containing the curves C_a and C_b .

The individual equations $\partial\phi/\partial s^a=0$ and $\partial\phi/\partial s^b=0$ each have $n-1$ independent solutions, $n-2$ of which may be taken in each case as ϕ^k , $k \neq a, b$. Let the $(n-1)$ st be ϕ^a in the case of the equation $\partial\phi/\partial s^b=0$, and ϕ^b , in the case of the equation $\partial\phi/\partial s^a=0$. Then, the congruence of curves C_a is represented by the equations $\phi^k=c_k$, $k \neq a$, and that of the curves C_b by the equations $\phi^k=c_k$, $k \neq b$.

The conditions of Theorem 10 demand the vanishing of all the components of the distantal spread vector of the congruences of curves C_a and C_b except B_{ab}^a and B_{ab}^b . These two components have geometric interpretations in terms of the concept of the distantal spread of the one congruence with respect to the other formulated by R. M. Peters* as a generalization of a corresponding concept for the case $n=2$.†

To define this concept, we note first that on an arbitrary but fixed surface, S_2 , of the family of two-dimensional surfaces containing the given congruences, there are ∞^1 curves C_a , defined by the equation $\phi^b=\text{const.}$, and ∞^1 curves C_b , defined by the equation $\phi^a=\text{const.}$ Restricting ourselves for the moment to these curves C_a and C_b , we form the logarithmic directional derivative in the positive direction of the curve C_a , of the distance, measured along an arbitrary curve C_b , between the curve C_a : $\phi^b=\phi_0^b$ and a neighboring curve C_a : $\phi^b=\phi_0^b+\Delta\phi^b$. Then the limit of this derivative, when $\Delta\phi^b$ approaches zero, namely,

$$-\frac{\partial}{\partial s^a} \log \left| \frac{\partial\phi^b}{\partial s^b} \right|,$$

is the *distantal spread of the congruence of curves C_a with respect to the congruence of curves C_b* . It vanishes identically when and only when on each surface S_2 the curves C_a are equidistant with respect to the curves C_b in that each two of them cut segments of equal length from the curves C_b .

The components B_{ab}^a and B_{ab}^b of the distantal spread vector of the congruences of curves C_a and C_b are essentially the distantal spreads of the two congruences with respect to one another. For we conclude from the identity

$$\frac{\partial}{\partial s^b} \frac{\partial\phi^b}{\partial s^a} - \frac{\partial}{\partial s^a} \frac{\partial\phi^b}{\partial s^b} = B_{ab}^k \frac{\partial\phi^b}{\partial s^k},$$

* Peters, *Parallelism and equidistance in Riemannian geometry*, offered to the American Journal of Mathematics.

† Graustein, loc. cit., p. 561.

inasmuch as $\partial\phi^b/\partial s^k = 0$, $k \neq b$, that

$$(36a) \quad B_{ab}{}^b = - \frac{\partial}{\partial s^a} \log \left| \frac{\partial\phi^b}{\partial s^a} \right|,$$

and, in a similar fashion, find that

$$(36b) \quad B_{ab}{}^a = \frac{\partial}{\partial s^b} \log \left| \frac{\partial\phi^a}{\partial s^a} \right|.$$

These results, together with the conclusions they imply, are summarized, in general form, in the following theorems.

THEOREM 11. *Two congruences of curves C and K lie in a family of two-dimensional surfaces, S_2 , if and only if their distantial spread vector lies always in the plane of their tangent vectors. The component, then, in the direction of the curves C , of the distantial spread vector of the curves C and K (in this order) is the negative of the distantial spread of the curves K with respect to the curves C , and the component in the direction of the curves K is equal to the distantial spread of the curves C with respect to the curves K . A necessary and sufficient condition that on each of the surfaces S_2 the curves of one congruence be equidistant with respect to those of the second is that the distantial spread vector of the congruences lie along the tangent vector of the first-named congruence.*

THEOREM 12. *The distantial spread vector of two congruences is identically the null vector if and only if (a) the two congruences lie in a family of two-dimensional surfaces S_2 , and (b) on each surface S_2 the curves of each congruence are equidistant with respect to those of the other, that is, clothe the surface in the sense of Tchebycheff.*

Case $r = n - 1$. Here we have the following typical result.

THEOREM 13. *A necessary and sufficient condition that the $n - 1$ congruences of curves C_i , $i \neq a$, of the ennuple E lie in a family of hypersurfaces is that*

$$(37) \quad B_{ij}{}^a = 0 \quad (i, j \neq a).$$

The $n - 1$ differential equations $\partial\phi/\partial s^i = 0$, $i \neq a$, have, then, a solution, ϕ^a , other than a constant, and the equation $\phi^a = \text{const.}$ defines the hypersurfaces, S_{n-1} , in which the $n - 1$ congruences lie.

Since each curve C_a of the a th congruence meets each of the hypersurfaces S_{n-1} in just one point, ϕ^a is a parameter common to all the curves C_a . In terms of this parameter, the differential of arc of curves C_a has the value

$$(38) \quad ds^a = \left(\frac{\partial\phi^a}{\partial s^a} \right)^{-1} d\phi^a.$$

For, inasmuch as $\partial\phi^a/\partial s^i = 0$, $i \neq a$,

$$(39) \quad \frac{\partial\phi^a}{\partial s^i} = \frac{\partial\phi^a}{\partial s^a} \delta_a^i \quad (i = 1, 2, \dots, n),$$

and hence

$$(40) \quad d\phi^a = \frac{\partial\phi^a}{\partial s^a} ds^a.$$

In this case we shall find useful another concept developed by Peters,* namely that of the *distantial spread, in the direction of the curves C_i , of the hypersurfaces S_{n-1} with respect to the curves C_a , $i \neq a$* . If d is the distance, measured along an arbitrary curve C_a , between the hypersurface $\phi^a = \phi_0^a$ and a neighboring hypersurface $\phi^a = \phi_0^a + \Delta\phi^a$, this distantial spread is the limit, when $\Delta\phi^a$ approaches zero, of the logarithmic derivative of d in the positive direction of the curves C_i , $i \neq a$. Its value, as obtained from (38), is found to be

$$-\frac{\partial}{\partial s^i} \log \left| \frac{\partial\phi^a}{\partial s^a} \right|.$$

As noted by Peters, a necessary and sufficient condition that the hypersurfaces S_{n-1} be equidistant with respect to the curves C_a in that each two of them cut segments of equal length from all the curves C_a is that the distantial spreads of the hypersurfaces S_{n-1} with respect to the curves C_a in the directions of the curves C_i , $i \neq a$, all vanish.

To express the fact that the distantial spread, in the direction of the curves C_b , of the hypersurfaces S_{n-1} with respect to the curves C_a vanishes, we shall say that the hypersurfaces are equidistant with respect to the curves C_a in the direction of the curves C_b .

The conditions of Theorem 13 demand the vanishing of the a th components of the distantial spread vectors of each two of the $n-1$ given congruences. The a th components of the distantial spread vectors of each of these congruences taken with the congruence of curves C_a are precisely the distantial spreads just defined. For, employing the method of the preceding case, we readily find that

$$(41) \quad B_{ia}^a = -\frac{\partial}{\partial s^i} \log \left| \frac{\partial\phi^a}{\partial s^a} \right| \quad (i \neq a).$$

The results we have obtained may be summarized as follows.

* Loc. cit.

THEOREM 14. *The curves of $n-1$ linearly independent congruences lie in a family of hypersurfaces S_{n-1} if and only if the distantal spread vectors of each two of the congruences are linearly dependent on the tangent vectors of the congruences. If the congruences are those of the ennuple E other than that of the curves C_a , the component, in the direction of the curves C_a , of the distantal spread vector of the i th and a th congruences ($i \neq a$) is the distantal spread, in the direction of the curves C_i , of the hypersurfaces S_{n-1} with respect to the curves C_a . A necessary and sufficient condition that this component vanish is that the hypersurfaces S_{n-1} be equidistant with respect to the curves C_a , in the direction of the curves C_i .*

THEOREM 15. *The distantal spread vectors of each two congruences of an ennuple are linear combinations of the tangent vectors of $n-1$ of the congruences if and only if these $n-1$ congruences lie in a family of hypersurfaces and the hypersurfaces are equidistant with respect to the curves of the n th congruence.**

Returning now to the analytic discussion, we note the equivalence of equations (39) and (40), and hence conclude:

THEOREM 16. *The differential of arc ds^a of the curves C_a of an ennuple E possesses an integrating factor if and only if the other curves of E lie in a family of hypersurfaces. If $\phi^a = \text{const.}$ is an equation of these hypersurfaces, then $\partial\phi^a/\partial s^a$ is an integrating factor of ds^a .*

It will be instructive to look at this question from another point of view. According to Theorem 1, $Id s$ is an exact differential if and only if $B_{ij}^a = 0$, $i, j \neq a$, and

$$(42) \quad \frac{\partial \log I}{\partial s^i} = -B_{ia}^a \quad (i \neq a).$$

But Theorem 16 and equation (37) guarantee that the conditions $B_{ij}^a = 0$, $i, j \neq a$, are sufficient that ds^a possess an integrating factor. Hence, the differential equations (42) must be compatible. Now, on the one hand, equations (42) are equivalent, by Theorem (16), to equations (41), and, on the other hand, the conditions for their complete integrability are

$$\frac{\partial}{\partial s^j} B_{ia}^a - \frac{\partial}{\partial s^i} B_{ja}^a = B_{ij}^a B_{ra}^a \quad (i, j \neq a),$$

and these equations, by virtue of (37), are special cases of the identity (33).

7. Parametric, Tchebycheff, and Cartesian ennuples of congruences. *Parametric ennuples.* If the n congruences of curves of an ennuple are the intersections of n families of hypersurfaces, taken $n-1$ at a time, we shall call the

* For applications of distantal spreads in the case of a parametric ennuple, see Peters, loc. cit.

ennuple parametric. If $\phi^i = c_i$ is the equation of the i th family of hypersurfaces, S_{n-1}^i , and it is the curves C_i of the i th congruence which do not lie in the hypersurfaces S_{n-1}^i , $i = 1, 2, \dots, n$, then ϕ^i is a parameter for the curves C_i and $\phi^1, \phi^2, \dots, \phi^n$ may be used as parameters (coordinates) in V_n .

It is evident geometrically that an ennuple is parametric when and only when each $n-1$ congruences belonging to it lie in a corresponding family of hypersurfaces, or, what amounts to the same thing, according to Theorem 16, if and only if the differential of arc of the curves of each congruence has an integrating factor. Theorem 13 and the subsequent developments lead, then, to the following conclusions.

THEOREM 17. *A necessary and sufficient condition that the ennuple E be parametric is that*

$$(43) \quad B_{ij}^k = 0 \quad (k \neq i, j; i, j, k = 1, 2, \dots, n).$$

If, then, ϕ^i is a parameter for the curves C_i ,

$$(44) \quad B_{ik}^k = -\frac{\partial}{\partial \phi^i} \log \left| \frac{\partial \phi^k}{\partial s^k} \right| \quad (k \neq i; i, k = 1, 2, \dots, n).$$

Returning to Theorem 10, we note that equations (35), when a and b , as well as k , vary from 1 to n , are identical with (43). Hence:

THEOREM 18. *An ennuple is parametric if and only if each two of its congruences lie in a corresponding family of two-dimensional surfaces.**

From Theorems 11 and 14 we get two interpretations of the quantities B_{ik}^k of (44):

THEOREM 19. *If the ennuple E is parametric, then B_{ik}^k is equal to the distastial spread of the curves C_i with respect to the curves C_k , and also to the distastial spread, in the direction of the curves C_i , of the hypersurfaces S_{n-1}^k with respect to the curves C_k .*

From (43) and (18) we conclude:

THEOREM 20. *A necessary and sufficient condition that the ennuple E be parametric is that $C_{ij}^k = C_{ji}^k$, $k \neq i, j$, that is, that the angular spread vectors of each two congruences of E with respect to one another have the same intrinsic contravariant components external to the plane of the tangent vectors of the two congruences.*

In case E is an orthogonal ennuple, it is readily shown that the equations

* This condition, expressed in analytic form, is to be found in Bortolotti, *Reti di Cebiceff e sistemi coniugati nelle V_n riemanniane*, Rendiconti della Accademia dei Lincei, (6), vol. 5 (1927), pp. 741-745.

$C_{ij}^k = C_{ji}^k$, $k \neq i, j$, are equivalent, by virtue of (25), to the equations $C_{ij}^k = 0$ ($i, j, k \neq$).

COROLLARY. *An orthogonal ennuple is parametric, that is, each of its congruences is normal, if and only if the angular spread vectors of each two of its congruences with respect to one another lie in the plane of the tangent vectors of the two congruences.*

Inasmuch as $B_{ij}^k = 0$, $k \neq i, j$, it follows that, if we set

$$\frac{\nabla_i F}{\nabla s^i} = \frac{\partial F}{\partial s^i} + B_{ij}^i F,$$

the conditions of integrability (6) take the form

$$\frac{\nabla_i}{\nabla s^i} \frac{\partial f}{\partial s^i} = \frac{\nabla_j}{\nabla s^j} \frac{\partial f}{\partial s^j} \quad (i, j = 1, 2, \dots, n)$$

and the conditions (8) that $f ds^i$ be an exact differential become $\nabla_i f / \nabla s^i = \nabla_j f / \nabla s^j$, $i, j = 1, 2, \dots, n$.

The expression $\nabla_i F / \nabla s^i$ may be called the *modified directional derivative of F in the direction of the curves C_i with respect to the curves C_j* . It is a generalization of the modified directional derivative employed, to good effect, in the theory of ordinary surfaces.* In comparison with the covariant derivative, it has the advantage that it involves, besides F , only the B 's. On the other hand, it does not have tensor character, and cannot be extended to apply to an arbitrary ennuple of congruences.

Tchebycheff ennuples. If each differential of arc, ds^k , of the ennuple E is an exact differential of a function s^k of the x 's, E is a parametric ennuple with the variables s^k as parameters. The linear element, referred to these parameters, is

$$(45) \quad ds^2 = g_{ij} ds^i ds^j, \quad g_{ij} = \cos \omega_{ij}.$$

Inasmuch as s^k is the common arc of all the curves C_k , $k = 1, 2, \dots, n$, the ennuple is a generalization of a Tchebycheff system of curves clothing a two-dimensional surface and may appropriately be called a Tchebycheff ennuple.

According to Theorem 1, ds^k is an exact differential if and only if $B_{ij}^k = 0$, $i, j = 1, 2, \dots, n$. Hence:

THEOREM 21. *An ennuple of congruences is a Tchebycheff ennuple if, and only if, for each two congruences of the ennuple, the distantal spread vector is a null vector, or the angular spread vector of the one with respect to the other is identical with that of the second with respect to the first.*

* Graustein, loc. cit., p. 575. The generalization was first discovered by Ruth M. Peters, in another connection.

It is to be noted that a parametric ennuple is, in particular, a Tchebycheff ennuple if and only if the k th family of hypersurfaces is an equidistant family with respect to the k th congruence of curves, $k = 1, 2, \dots, n$; see Theorem 15.

From (24) and (25) we conclude, inasmuch as $C_{kk} = 0$, that

$$C_{kk} = \frac{\partial g_{kk}}{\partial s^k} - B_{kkk}.$$

From these relations we may obtain interesting conditions under which the curves C_k are geodesics. In particular, we have

THEOREM 22. *The curves of a Tchebycheff ennuple are all geodesics if and only if the angle between the curves of each two congruences is constant along the curves of both congruences.**

It follows that the linear element (45), where s^i are actual coordinates and g_{ij} is independent of s^i and s^j , $i, j = 1, 2, \dots, n$, is characteristic of a space which contains a Tchebycheff ennuple of geodesics.

Cartesian ennuples. When E is a Tchebycheff ennuple and g_{ij} are constants, the linear element (45) characterizes V_n as a euclidean space referred to the ennuple of congruences of coordinate curves of a Cartesian coordinate system in which the unit of measure for each coordinate is the unit distance of the space. Accordingly, we shall call a Tchebycheff ennuple for which the g_{ij} are constant a Cartesian ennuple. We may then state

THEOREM 23. *A necessary and sufficient condition that V_n be euclidean is that it contain a Cartesian ennuple of congruences.*

This type of ennuple may be characterized analytically and geometrically as follows.

THEOREM 24. *An ennuple of congruences is a Cartesian ennuple if and only if*

$$(46) \quad B_{ij}^k = 0, \quad \frac{\partial g_{ij}}{\partial s^k} = 0 \quad (i, j, k = 1, 2, \dots, n),$$

or

$$(47) \quad C_{ij}^k = 0 \quad (i, j, k = 1, 2, \dots, n),$$

* The counterpart of this theorem, to the effect that the curves of each congruence of a Tchebycheff ennuple are parallel with respect to those of every other congruence if and only if the angle between the curves of each two congruences is constant along the curves of the remaining congruences, is given by Bortolotti, loc. cit., p. 741. It is to be noted that Bortolotti's conception of a Tchebycheff ennuple differs from the one here used.

that is, if and only if it is a Tchebycheff ennuple each two of whose congruences intersect under a constant angle or has the property that the curves of each congruence are geodesics and are parallel with respect to the curves of every other congruence.

The equivalence of equations (46) and (47) follows from previous relations. If $\partial g_{ij}/\partial s^k = 0$ and $B_{ijk} = 0$, then (26) says that $C_{ijk} = 0$; and, if $C_{ijk} = 0$, (25) and (24) tell us that $\partial g_{ij}/\partial s^k = 0$ and $B_{ijk} = 0$. But the equations $B_{ijk} = 0$ and $B_{ij}^k = 0$ are equivalent, and also the equations $C_{ijk} = 0$ and $C_{ij}^k = 0$.

8. Ennuples Cartesian at a point. We shall say that an ennuple of congruences is Cartesian at a particular point P if it behaves at P like a Cartesian ennuple, that is, if it has at P the analytic or geometric characteristics described in Theorem 23.

THEOREM 25. *If x^i are geodesic coordinates at P , the ennuple E is Cartesian at P if and only if the ordinary components, $\bar{a}_h|_i$, of the unit tangent vectors of E behave like constants at P .*

Since x^i are geodesic coordinates at P , $\bar{C}_{ij}^k = 0$ at P . Hence, according to (17), $C_{ij}^k = 0$ at P if and only if $\bar{a}_h|_i$ behave like constants at P . Thus, the theorem is proved.

Inasmuch as, when x^i are geodesic coordinates at P , \bar{g}_{ij} behave like constants at P , it follows that, if the ordinary components of a vector field behave like constants at P , so also do the ordinary components of the corresponding field of unit vectors. Hence, we conclude:

THEOREM 26. *There exist infinitely many ennuples which are Cartesian at a given point P and have at P prescribed tangent vectors.*

If $\bar{a}_h|_i$ behave like constants at P , so also do $\bar{a}^h|_i$. Hence, if x^i are geodesic coordinates at P : $(0, 0, \dots, 0)$ and the ennuple E is Cartesian at P ,

$$ds^h = \bar{a}^h|_i dx^i = (\bar{a}^h|_i)_0 dx^i + c^h_{jrs} x^r x^s dx^i + \dots$$

Consequently, if terms in x^i of the second degree and higher are neglected, the differentials of arc ds^h become exact differentials and we may write $s^h = (\bar{a}^h|_i)_0 x^i$.

Comparison with geodesic coordinates. We assume now that the ennuple E is a parametric ennuple and inquire whether, if parameters for E are geodesic at a point P , E is Cartesian at P , and conversely.

We may assume that E is the parametric ennuple corresponding to the basic coordinates x^i . According to Theorem 17, we have, then,

$$(48) \quad B_{ij}^i = \frac{\partial}{\partial s^j} \log \left| \frac{\partial x^i}{\partial s^i} \right| \quad (i \neq j; i, j = 1, 2, \dots, n),$$

whereas $B_{ij}^k = 0$ for $k \neq i, j$.

Since in this case

$$\frac{\partial x^i}{\partial s^j} = 0 \quad (i \neq j; i, j = 1, 2, \dots, n),$$

and hence

$$\frac{\partial s^i}{\partial x^j} = 0, \quad \frac{\partial s^i}{\partial x^i} = \left(\frac{\partial x^i}{\partial s^i} \right)^{-1} \quad (i \neq j; i, j = 1, 2, \dots, n).$$

equation (17) becomes

$$(49a) \quad C_{ij}^k = \bar{C}_{ij}^k \frac{\partial x^i}{\partial s^i} \frac{\partial x^j}{\partial s^j} \frac{\partial s^k}{\partial x^k} + \delta_i^k \frac{\partial}{\partial s^j} \log \left| \frac{\partial x^i}{\partial s^i} \right|,$$

whence

$$(49b) \quad C_{ii}^i = \bar{C}_{ii}^i \frac{\partial x^i}{\partial s^i} + \frac{\partial}{\partial s^i} \log \left| \frac{\partial x^i}{\partial s^i} \right|.$$

Furthermore, we have, from (10a), since $g_{ii} = 1$,

$$\bar{g}_{ii} = \left(\frac{\partial x^i}{\partial s^i} \right)^{-2}.$$

If the coordinates x^i are geodesic at P , $\bar{C}_{ij}^k = 0$ at P ; and \bar{g}_{ii} , and hence $\partial x^i / \partial s^i$, act like constants at P . It follows, then, from (49a), that $C_{ij}^k = 0$ at P , that is, that E is Cartesian at P .

If, conversely, E is Cartesian at P , $C_{ij}^k = 0$ and $B_{ij}^k = 0$ at P . Hence, from (49a) and (48), we find that $\bar{C}_{ij}^k = 0$ at P except when $i = j = k$, whereas from (49b) we get

$$\frac{\partial x^i}{\partial s^i} \bar{C}_{ii}^i + \frac{\partial}{\partial s^i} \log \left| \frac{\partial x^i}{\partial s^i} \right| = 0 \text{ at } P.$$

Thus, the coordinates x^i are not necessarily geodesic at P ; they are geodesic if and only if $\partial^2 x^i / \partial s^{i2} = 0$ at P . However, it is evident that there exist coordinates X^i for E which are geodesic at P ; we have merely to choose $X^i = X^i(x^i)$, $i = 1, 2, \dots, n$, so that $\partial^2 X^i / \partial s^{i2} = 0$ at P .*

We have now answered the proposed question.

THEOREM 27. *There exist, for a parametric ennuple, coordinates which are geodesic at a given point P if and only if the ennuple is Cartesian at P .*

* It would appear from this discussion that a completely geometric characterization of geodesic coordinates is impossible. In this connection, see Levi-Civita, *The Absolute Differential Calculus*, p. 168.

It is evident from this discussion that the concept of an ennuple Cartesian at a point is more fundamental than the concept of geodesic coordinates. Furthermore it is also more general, in that an ennuple Cartesian at a point does not need to be a parametric ennuple at all.

That an ennuple Cartesian at a point serves all the purposes for which geodesic coordinates are ordinarily employed is guaranteed by the following proposition.

THEOREM 28. *When the ennuple of reference is Cartesian at a point P , then at P intrinsic covariant differentiation becomes directional differentiation and second cross directional derivatives are independent of the order of differentiation.*

From (6) and Theorem 21 it follows that second cross directional derivatives are independent of the order of differentiation at a given point P if and only if the ennuple of reference behaves like a Tchebycheff ennuple at P .

9. **Intrinsic components of the Riemann tensors.** It may be shown in various ways that the intrinsic components, referred to the ennuple E , of the Riemann tensor with the ordinary components

$$(50) \quad \bar{R}^l_{ijk} = \frac{\partial}{\partial x^j} \bar{C}_{ik}^l - \frac{\partial}{\partial x^k} \bar{C}_{ij}^l + \bar{C}_{ik}^m \bar{C}_{mj}^l - \bar{C}_{ij}^m \bar{C}_{mk}^l,$$

are

$$(51) \quad R^l_{ijk} = \frac{\partial}{\partial s^j} C_{ik}^l - \frac{\partial}{\partial s^k} C_{ij}^l + C_{ik}^m C_{mj}^l - C_{ij}^m C_{mk}^l + C_{im}^l B_{jk}^m.$$

From $R_{hijk} = g_{hl} R^l_{ijk}$, it follows that

$$(52) \quad R_{hijk} = \frac{\partial}{\partial s^j} C_{ikh} - \frac{\partial}{\partial s^k} C_{ijh} + C_{ij}^m C_{hkm} - C_{ik}^m C_{hjm} + C_{im}^h B_{jk}^m$$

are the intrinsic components of the Riemann tensor whose ordinary components are $\bar{R}_{hijk} = \bar{g}_{hl} \bar{R}^l_{ijk}$.

The identities satisfied by the covariant Riemann tensor have, of course, the same form in terms of its intrinsic components as in terms of its ordinary components. In this connection, it is of interest to note that the identities $R_{hijk} + R_{hjki} + R_{khij} = 0$ are equivalent to the identities (34).*

The conditions of integrability for covariant differentiation also have the same forms in terms of intrinsic components as in terms of ordinary components. Thus, if a_i , $a_{i,j}$, and $a_{i,jk}$ are respectively the intrinsic components of a covariant vector and its first and second covariant derivatives, then

* See Dei, *Sulle relazioni differenziali che legano i coefficienti di rotazione del Ricci*, Rendiconti della Accademia dei Lincei, (5), vol. 32 (1923), pp. 474-478, where this conclusion is reached in essence, though not in form, in the case of an orthogonal ennuple.

$$(53) \quad a_{i,jk} - a_{i,kj} = a_l R^l_{ijk}.$$

Again, in the case of a contravariant vector, we have

$$(54) \quad a^i_{,jk} - a^i_{,kj} = -a^l R^i_{ljk}.$$

Setting $a^i = a_h |^i = \delta^i_h$ in (54) and $a_i = a_h |^i = g_{hi}$ in (53), we obtain the following expressions for R^h_{ijk} and R_{hijk} :

$$R^h_{ijk} = a_i |^h_{,kj} - a_i |^h_{,jk}, \quad R_{hijk} = a_h |_{i,jk} - a_h |_{i,kj}$$

in terms of the derivatives of the unit vectors tangent to the curves of E . On the other hand, formulas (51) and (52), in light of (29) and (31), furnish expressions for R^h_{ijk} and R_{hijk} in terms of the curvature and angular and distant spread vectors of E , namely,

$$R^h_{ijk} = c_{ik} |^h_{,j} - c_{ij} |^h_{,k} + c_{im} |^h b_{jk} |^m,$$

$$R_{hijk} = c_{ik} |_{h,j} - c_{ij} |_{h,k} + c_{im} |_{h} b_{jk} |^m.$$

To these relations may be adjoined the simple expressions

$$c_{hk} |^i = a_h |^i_{,k}, \quad c_{hk} |^i = a_h |^i_{,k}$$

for the curvature and angular spread vectors of E in terms of the tangent vectors.*

The Ricci tensor. From (50) we find as the components of the Ricci tensor, $R_{ij} = R^k_{ijk}$:

$$R_{ij} = \frac{\partial}{\partial s^j} C_{ik}{}^k - \frac{\partial}{\partial s^k} C_{ij}{}^k + C_{ik}{}^m C_{mj}{}^k - C_{ij}{}^m C_{mk}{}^k + C_{imk} B_{jk}{}^m.$$

Adding to the right-hand side of this equation the expression

$$\frac{1}{2} \left(\frac{\partial}{\partial s^i} B_{jk}{}^k - \frac{\partial}{\partial s^j} B_{ik}{}^k + \frac{\partial}{\partial s^k} B_{ij}{}^k + B_{ij}{}^m B_{mk}{}^k \right),$$

whose value is readily shown to be zero by (33), and making use of (18) and the relation $\partial(\log g^{1/2})/\partial s^i = C_{ik}{}^k$, we find the following symmetric expression for R_{ij} :

$$R_{ij} = (\log g^{1/2})_{,ij} + \frac{1}{2} \frac{\partial}{\partial s^i} B_{jk}{}^k + \frac{1}{2} \frac{\partial}{\partial s^j} B_{ik}{}^k - \frac{1}{2} \frac{\partial}{\partial s^k} (C_{ij}{}^k + C_{ji}{}^k) \\ - \frac{1}{2} (C_{ij}{}^k + C_{ji}{}^k) B_{km}{}^m + C_{ik}{}^m C_{jm}{}^k.$$

Geometric interpretations. We note, without going into detail, that the

* It is to be noted that none of the relations in this paragraph are invariant in form with respect to a change from intrinsic to ordinary components.

intrinsic components of the Riemann and Ricci tensors, especially in case E is an orthogonal ennuple, have interesting geometrical interpretations in terms of the Riemannian and mean curvatures.

10. Metric connections. The torsion vector. The most general connection which possesses a metric based on the tensor \bar{g}_{ij} is obtained by employing, instead of the Christoffel symbols $\bar{\Gamma}_{ij}^k$, arbitrary coefficients of connection, Γ_{ij}^k , such that

$$(55) \quad \bar{g}_{ij;k} \equiv \frac{\partial \bar{g}_{ij}}{\partial x^k} - \bar{g}_{ir}\Gamma_{jk}^r - \bar{g}_{rj}\Gamma_{ik}^r = 0 \quad (i, j, k = 1, 2, \dots, n).$$

The skew-symmetric part of Γ_{ij}^k , namely

$$(56) \quad \bar{S}_{ij}^k = \frac{1}{2}(\Gamma_{ij}^k - \Gamma_{ji}^k),$$

has tensor character. It is called the *torsion tensor* of the connection.

When the torsion tensor \bar{S}_{ij}^k is known, the connection Γ_{ij}^k is completely determined. For it is readily found, from (55) and (56), that

$$(57) \quad \Gamma_{ijk} = \bar{\Gamma}_{ijk} + (\bar{S}_{ijk} + \bar{S}_{jki} - \bar{S}_{kij}),$$

where, for example, $\Gamma_{ijk} = \bar{g}_{hk}\Gamma_{ij}^h$. In particular, the connection is Riemannian if and only if the torsion tensor is null.

By replacing C_{ij}^k and \bar{C}_{pq}^i in (17) by Γ_{ij}^k and $\bar{\Gamma}_{pq}^i$, we obtain equations which define the invariant coefficients of connection, Γ_{ij}^k , referred to the ennuple E . In the same way, we obtain from (19) and (21) the formulas for covariant differentiation with respect to the connection, and, from (27), the expressions for the new angular spread or curvature vectors. From the latter, it follows that Γ_{hk}^i are the intrinsic contravariant components of the angular spread vector, $\gamma_{hk}|^i$, of the congruence of the curves C_h with respect to the congruence of curves C_k . On the other hand, the components B_{hk}^i of the distant spread vector, $b_{hk}|^i$, of the curves C_h and C_k remain the same as before; this vector is dependent only on the components, \bar{g}_{ij} , of the metric.

From the laws of transformation of Γ_{ij}^k and \bar{C}_{ij}^k into Γ_{ij}^k and C_{ij}^k , it follows that $\Gamma_{ij}^k - \bar{C}_{ij}^k$ is a tensor whose intrinsic components are $\Gamma_{ij}^k - C_{ij}^k$. Hence, we conclude from (57) that

$$(58) \quad \Gamma_{ijk} = C_{ijk} + (S_{ijk} + S_{jki} - S_{kij}),$$

where S_{ijk} are the intrinsic components of the tensor \bar{S}_{ijk} .

From these relations we obtain, by virtue of (24), the following expressions for the intrinsic components of the torsion tensor*:

* An equation of the same form as this holds for the general linear connection; see Horák, loc. cit., p. 197.

$$(59) \quad 2S_{ij}{}^k = (\Gamma_{ij}{}^k - \Gamma_{ji}{}^k) - B_{ij}{}^k.$$

Torsion vectors. In a given oriented planar element at a point P choose two ordered infinitesimal vectors \overline{PP}_1 and \overline{PP}_2 such that the direction of rotation about P from the first vector to the second is that of the given orientation. Transport each vector along the other by the displacement determined by $\Gamma_{ij}{}^k$, thus obtaining the new vectors $\overline{P_1Q_1}$ and $\overline{P_2Q_2}$. Then the limit of the ratio of the vector $\overline{Q_1Q_2}$ to the area of the parallelogram determined by the vectors \overline{PP}_1 and \overline{PP}_2 is a vector at P which is independent of these vectors, provided merely that they are chosen as described, and so pertains simply to the given oriented planar element. This vector is due to Cartan* and is called by him *the torsion vector at P for the given oriented planar element*.

If the vectors \overline{PP}_1 and \overline{PP}_2 are respectively $d_h x^i$ and $d_k x^i$, the coordinates of Q_1 are, to within terms of higher order,

$$x^r + d_h x^r + d_k x^r + d_h d_k x^r + \Gamma_{ij}{}^r d_h x^i d_k x^j.$$

Hence, the torsion vector at P for the given oriented planar element has the components

$$2 \csc \phi \bar{S}_{ij}{}^r \bar{a}_h |^i \bar{a}_k |^j,$$

where $\bar{a}_h |^i$ and $\bar{a}_k |^j$ are the unit vectors in the directions of \overline{PP}_1 and \overline{PP}_2 and ϕ is the angle between them.

We shall find it convenient to employ, instead of the torsion vector of Cartan, a *torsion vector for two ordered directions at a point*. This we define by the equations

$$(60) \quad \bar{s}_{hk} |^r = -2 \bar{S}_{ij}{}^r \bar{a}_h |^i \bar{a}_k |^j,$$

where $\bar{a}_h |^i$ and $\bar{a}_k |^j$ are the unit vectors in the two directions. In particular, if $\bar{a}_h |^i$ and $\bar{a}_k |^j$ are the fields of unit vectors tangent to the curves of the h th and k th congruences of the ennuple E , we shall call $\bar{s}_{hk} |^r$ the torsion vector of these congruences, in the order given.

A geometric interpretation of the torsion vector (60) is obtained by rephrasing the definition of the torsion vector of Cartan. A second interpretation, more useful to us, is the following: *If, in the definition of the distantial spread vector of the congruences of curves C_h and C_k , we redefine Q_1 and Q_2 as the terminal points of the vectors at P_1 and P_2 which are parallel respectively, according to the connection $\Gamma_{ij}{}^k$, to the vectors \overline{PP}_2 and \overline{PP}_1 , the definition becomes a description of the torsion vector of the congruences, in the order given.*

* *Sur les variétés à connexion affine et la théorie de la relativité généralisée*, Annales de l'École Normale, (3), vol. 40 (1923), pp. 325-412.

To establish this interpretation, it suffices to note that the coordinates of the new point Q_1 are, to within terms of higher order,

$$x^r + d_h x^r + d_k x^r - \Gamma_{ij}^r d_h x^i d_k x^j.$$

It follows from the interpretation that a sufficient condition that the torsion vector of two congruences be identical with their distantial spread vector is that the curves of each congruence be parallel, according to the connection Γ_{ij}^k , with respect to those of the other.* This is not a necessary condition, as we shall see shortly.

Whereas the distantial spread vector of the two congruences depends only on the metric, the torsion vector depends also on the connection. If the connection is Riemannian, the torsion vector is always null.

In terms of intrinsic components, referred to the ennuple E , (60) becomes

$$s_{hk}|^r = -2S_{ij}^r a_h|^i a_k|^j = -2S_{hk}^r.$$

Thus we have simple geometric interpretations of the intrinsic components of the tensor of torsion.

THEOREM 29. *The intrinsic components, $s_{hk}|^r$ and $s_{hk}|_r$, of the torsion vector of the curves C_h and C_k are respectively $-2S_{hk}^r$ and $-2S_{hkr}$:*

$$s_{hk}|^r = -2S_{hk}^r, \quad s_{hk}|_r = -2S_{hkr}.$$

We may now interpret the important relations (59), by rewriting them in the form

$$(61) \quad s_{ij}|^r = b_{ij}|^r - (\gamma_{ij}|^r - \gamma_{ji}|^r).$$

THEOREM 30. *The difference between the distantial spread vector and the torsion vector of two ordered congruences is equal to the difference between the angular spread vectors of the two congruences with respect to one another.*

It is now clear that a necessary and sufficient condition that the torsion vector of two ordered congruences coincide with their distantial spread vector is that the angular spread vectors of the congruences with respect to one another be identical.

We shall say that the connection Γ_{ij}^k is symmetric with respect to the ennuple E if $\Gamma_{ij}^k = \Gamma_{ji}^k$ for $i, j, k = 1, 2, \dots, n$. From (59) or (61) we infer, then, the following proposition:

THEOREM 31. *The distantial spread and torsion vectors of each two congruences of the ennuple E are identical if and only if the connection is symmetric with respect to E .*

* The analytic content of this statement, in a different form, is to be found in Bortolotti, *Parallelismi assoluti nelle V_n riemanniane*, Atti del Reale Istituto Veneto, vol. 86 (1927), pp. 455-465.

For a given metric with the fundamental tensor \bar{g}_{ij} , there is a unique metric connection which is symmetric with respect to E . For it follows from $g_{ij;k} = 0$, or from (58), (59), and (26), that, if $\Gamma_{ij}^k = \Gamma_{ji}^k$, then

$$(62) \quad \Gamma_{ij}^k = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial s^j} + \frac{\partial g_{jk}}{\partial s^i} - \frac{\partial g_{ij}}{\partial s^k} \right).$$

*Spaces admitting absolute parallelism.** The given space is said to admit (complete) absolute parallelism if there exist n linearly independent fields of absolutely parallel vectors, or, what amounts to the same thing, if there exists an ennuple of congruences such that the curves of each congruence are parallel with respect to all n congruences. It is evident from the geometric interpretation of Γ_{ij}^k that this is true of the ennuple E if and only if $\Gamma_{ij}^k = 0$ for $i, j, k = 1, 2, \dots, n$, and hence, according to (62), if and only if Γ_{ij}^k is symmetric in i, j and g_{ij} are constants.

THEOREM 32. *A metric connection admits absolute parallelism if and only if there exists an ennuple of congruences, E , which has constant angles and with respect to which the connection is symmetric.*

Since $\Gamma_{ij}^k = 0$, it follows that the intrinsic components, referred to E , of the covariant derivative of a tensor are simply the directional derivatives, along the curves of E , of the intrinsic components of the tensor.

The intrinsic components of the curvature tensor of the connection are obtained by replacing C_{ij}^k by Γ_{ij}^k in (51). Hence, when the connection admits absolute parallelism, the curvature tensor is actually zero, as it should be.

11. Transformation from one ennuple of congruences to a second. We return now to the study of Riemannian space, and assume that there is given, in addition to E , a second ennuple, E' , consisting of the congruences of directed curves $C'_i, i = 1, 2, \dots, n$.

We shall distinguish by primes symbols referred to the ennuple E' . For example, we shall denote by g'_{ij} the (intrinsic) covariant components of the fundamental tensor, referred to E' .

The components, referred to E , of the tangent and conjugate vectors and the distantal and angular spread vectors of the congruences of E we have denoted by a_h, a^h, b_{hk}, c_{hk} . The components, referred to E' , of the corresponding fundamental vectors connected with the congruences of E' we shall designate by $\alpha'_h, \alpha'^h, \beta'_{hk}, \gamma'_{hk}$. According to the foregoing convention, $\alpha_h, \alpha^h, \beta_{hk}, \gamma_{hk}$ then denote the components, referred to E , of the

* For a review of this subject, see Eisenhart, *Spaces admitting complete absolute parallelism*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 217-226.

fundamental vectors pertaining to E' , and $a'_h, a'^h, b'_{hk}, c'_{hk}$ the components, referred to E' , of the fundamental vectors pertaining to E .

In §4 we noted that

$$(63) \quad \bar{a}^i = \frac{dx^i}{ds}, \quad a^i = \frac{ds^i}{ds}$$

are respectively the ordinary components and the components referred to E of the unit contravariant vector tangent to the directed curve C . It follows, then, that

$$(64) \quad \frac{dx^i}{ds} = \frac{ds^j}{ds} \frac{\partial x^i}{\partial s^j}, \quad \frac{ds^i}{ds} = \frac{dx^j}{ds} \frac{\partial s^i}{\partial x^j}, \quad \frac{df}{ds} = \frac{\partial f}{\partial s^j} \frac{ds^j}{ds}.$$

Similar formulas hold when C is referred to E' instead of to E .

The second of formulas (63) suggests that we denote by $\partial s^i / \partial s'^h$ the contravariant components, referred to E , of the unit vector tangent to the general curve C'_h of the h th congruence of E' :

$$(65) \quad \alpha_h |^i = \frac{\partial s^i}{\partial s'^h},$$

and by $\partial s'^i / \partial s^h$ the contravariant components, referred to E' , of the unit vector tangent to the general curve C_h of the h th congruence of E :

$$(66) \quad a'_h |^i = \frac{\partial s'^i}{\partial s^h}.$$

From the first two formulas in (64) and the corresponding formulas mentioned in connection with them, we have

$$(67) \quad \begin{aligned} \frac{\partial x^i}{\partial s'^h} &= \frac{\partial s^j}{\partial s'^h} \frac{\partial x^i}{\partial s^j}, & \frac{\partial s^i}{\partial s'^h} &= \frac{\partial x^j}{\partial s'^h} \frac{\partial s^i}{\partial x^j}, \\ \frac{\partial x^i}{\partial s^h} &= \frac{\partial s'^j}{\partial s^h} \frac{\partial x^i}{\partial s'^j}, & \frac{\partial s'^i}{\partial s^h} &= \frac{\partial x^j}{\partial s^h} \frac{\partial s'^i}{\partial x^j}, \end{aligned}$$

where $\partial s'^i / \partial s^j$ has the same significance with respect to E' as has $\partial s^i / \partial x^j$ with respect to E .

Using (67) in conjunction with (2) and similar equations in $\partial x^i / \partial s'^i$ and $\partial s'^i / \partial x^j$, we readily establish the fundamental relations

$$(68) \quad \frac{\partial s^j}{\partial s'^h} \frac{\partial s'^i}{\partial s^i} = \delta^i_h, \quad \frac{\partial s^i}{\partial s'^j} \frac{\partial s'^i}{\partial s^k} = \delta^i_k,$$

which state that the systems of quantities $\partial s^j / \partial s'^i$ and $\partial s'^i / \partial s^j$ are conjugate to one another.

According to the first of these equations, the Pfaffian $(\partial s'^i / \partial s^i) ds^i$ is zero for every curve of the ennuple E' except a curve C'_i and for a curve C'_i is equal to the differential of arc, ds'^i , of the curve. Thus we obtain, as the relations between ds^i and ds'^i :

$$(69) \quad ds'^i = \frac{\partial s'^i}{\partial s^i} ds^i, \quad ds^i = \frac{\partial s^i}{\partial s'^i} ds'^i.$$

Applying the last of the equations (64), we find

$$(70) \quad \frac{\partial f}{\partial s'^i} = \frac{\partial f}{\partial s^i} \frac{\partial s^i}{\partial s'^i}, \quad \frac{\partial f}{\partial s^i} = \frac{\partial f}{\partial s'^i} \frac{\partial s'^i}{\partial s^i},$$

as the relations between the directional derivatives in the positive directions of the curves C_i and those in the positive directions of the curves C'_i .

Formulas (69) and (70) guarantee that the transformation from the components of a tensor, referred to E , to the components of the tensor, referred to E' , obeys the formal laws of tensor analysis. The relations between the two sets of components for the fundamental tensor are, for example,

$$(71) \quad \begin{aligned} g'^{ij} &= g^{kl} \frac{\partial s^k}{\partial s'^i} \frac{\partial s^l}{\partial s'^j}, & g_{ij} &= g'_{kl} \frac{\partial s'^k}{\partial s^i} \frac{\partial s'^l}{\partial s^j}, \\ g'^{ij} &= g^{kl} \frac{\partial s'^i}{\partial s^k} \frac{\partial s'^j}{\partial s^l}, & g_{ij} &= g'_{kl} \frac{\partial s^k}{\partial s'^i} \frac{\partial s^l}{\partial s'^j}, \end{aligned}$$

and the corresponding relations in the case of an arbitrary vector are

$$(72) \quad \begin{aligned} a'^i &= a^j \frac{\partial s'^i}{\partial s^j}, & a^i &= a'^j \frac{\partial s^i}{\partial s'^j}, \\ a'_i &= a_j \frac{\partial s^j}{\partial s'^i}, & a_i &= a'_j \frac{\partial s'^j}{\partial s^i}. \end{aligned}$$

Tangent and conjugate vector-fields of E and E' . The components, referred respectively to E and E' , of the field of unit vectors tangent to the curves C_h of the ennuple E are

$$(73a) \quad a_h |^i = \delta^i_h, \quad a'_h |^i = \frac{\partial s'^i}{\partial s^h},$$

$$(73b) \quad a_h |_i = g_{hi}, \quad a'_h |_i = g_{hi} \frac{\partial s^i}{\partial s'^h} = g'_{hi} \frac{\partial s'^i}{\partial s^h}.$$

The formulas on the left are identical with (16b); those on the right follow from them by means of (72), (71), and (68).

Similarly, we find, as the components of the h th field of conjugate vectors associated with the ennuple E ,

$$(74a) \quad a^h|_i = \partial_i^h, \quad a'^h|_i = \frac{\partial s^h}{\partial s'^i},$$

$$(74b) \quad a^h|i = g^{hi}, \quad a'^h|i = g^{hj} \frac{\partial s'^j}{\partial s^i} = g'^{ji} \frac{\partial s^h}{\partial s'^j}.$$

The components, referred respectively to E' and E , of the field of unit vectors tangent to the curves C'_h of the ennuple E' are

$$(75a) \quad \alpha'_h|i = \delta_h^i, \quad \alpha_h|i = \frac{\partial s^i}{\partial s'^h},$$

$$(75b) \quad \alpha'_h|i = g'_{hi}, \quad \alpha_h|i = g'_{hj} \frac{\partial s'^j}{\partial s^i} = g_{ji} \frac{\partial s^j}{\partial s'^h},$$

while those of the h th field of conjugate vectors associated with E' are

$$(76a) \quad \alpha'^h|i = \delta_i^h, \quad \alpha^h|i = \frac{\partial s'^h}{\partial s^i},$$

$$(76b) \quad \alpha'^h|i = g'^{hi}, \quad \alpha^h|i = g'^{hj} \frac{\partial s^j}{\partial s'^i} = g^{ji} \frac{\partial s'^j}{\partial s^i}.$$

From the relations

$$(77) \quad \alpha_i|j = a'^j|i = \frac{\partial s^j}{\partial s'^i}, \quad \alpha^i|_j = a_i'|^i = \frac{\partial s'^i}{\partial s^j},$$

it follows that, when i is fixed and $j=1, 2, \dots, n$, $\partial s^j/\partial s'^i$ and $\partial s'^i/\partial s^j$ are respectively components, referred to E , of the i th tangent and conjugate vectors associated with E' , whereas, when j is fixed and $i=1, 2, \dots, n$, they are respectively components, referred to E' , of the j th conjugate and tangent vectors associated with E .

Interpretations in terms of angles. If ϕ_{hi} is the angle which the h th tangent vector of E' makes with the i th conjugate vector of E and ϕ'_{hi} is the angle which the h th tangent vector of E makes with the i th conjugate vector of E' , it follows, either directly or by virtue of Theorem 2, that

$$(78) \quad \alpha_h|i = \frac{\partial s^i}{\partial s'^h} = \sec \theta_i \cos \phi_{hi}, \quad \alpha'_h|i = \frac{\partial s'^i}{\partial s^h} = \sec \theta'_i \cos \phi'_{hi},$$

where θ_i is the angle between the i th tangent and conjugate vectors of E and θ'_i is the angle between the i th tangent and conjugate vectors of E' .

If E is an orthogonal ennuple, the angles θ_i are all zero, and $\alpha_h|_i = \partial s^i / \partial s'^h = \cos \phi_{hi}$, so that $\partial s^i / \partial s'^h$ are direction cosines. If E' is also orthogonal, then $\phi_{ij} = \phi'_{ji}$ and hence $\partial s^i / \partial s'^i = \partial s'^i / \partial s^i$ for all i, j .

Returning to the general case, we remark that, inasmuch as we now have interpretations in terms of angles of $\partial s^i / \partial s'^i$, $\partial s'^i / \partial s^i$, and g_{ij} , g'^{ij} , g'_{ij} , g'^{ii} (see §2), we may write all of the formulas (71) and (73)–(76) in terms of angles. For example, if ψ_{hk} is the angle between the h th tangent vector of E and the k th tangent vector of E' , the second formulas in (73b) and (75b) both become, by application of Theorem 2,

$$(79) \quad \cos \psi_{hk} = \sum_j \cos \omega_{hj} \sec \theta_j \cos \phi_{kj} = \sum_j \cos \omega'_{kj} \sec \theta'_j \cos \phi'_{hj}.$$

Similarly, if χ_{hk} is the angle between the h th conjugate vector of E and the k th conjugate vector of E' , we obtain, from the second formula in either (74b) or (76b),

$$\cos \chi_{hk} = \sum_j \cos \Omega_{hj} \sec \theta_j \cos \phi'_{jk} = \sum_j \cos \Omega'_{kj} \sec \theta'_j \cos \phi_{jh}.$$

Here, ω_{ij} and Ω_{ij} are the angles defined in §2 for E , and ω'_{ij} and Ω'_{ij} are the corresponding angles for E' .

Transformation of Christoffel symbols and the B 's. From equations (17) and similar equations for the ennuple E' , we readily obtain the equations of the transformation,

$$(80) \quad C'_{ij}{}^k \frac{\partial s^r}{\partial s'^k} = C_{pq}{}^r \frac{\partial s^p}{\partial s'^i} \frac{\partial s^q}{\partial s'^j} + \frac{\partial}{\partial s'^i} \frac{\partial s^r}{\partial s'^j},$$

of the Christoffel symbols $C_{ij}{}^k$ for the ennuple E into the Christoffel symbols $C'_{ij}{}^k$ for the ennuple E' .

From these equations follow directly those of the transformation of the symbols $B_{ij}{}^k$ for E into the symbols $B'_{ij}{}^k$ for E' , namely

$$(81) \quad B'_{ij}{}^k \frac{\partial s^r}{\partial s'^k} = B_{pq}{}^r \frac{\partial s^p}{\partial s'^i} \frac{\partial s^q}{\partial s'^j} + \left(\frac{\partial}{\partial s'^j} \frac{\partial s^r}{\partial s'^i} - \frac{\partial}{\partial s'^i} \frac{\partial s^r}{\partial s'^j} \right).$$

If we apply the result of differentiating the first of equations (68) to the last term in (81), we find that (81) may be rewritten in the form

$$B'_{ij}{}^k = \left(\frac{\partial}{\partial s^p} \frac{\partial s'^k}{\partial s^q} - \frac{\partial}{\partial s^q} \frac{\partial s'^k}{\partial s^p} + B_{pq}{}^r \frac{\partial s'^k}{\partial s^r} \right) \frac{\partial s^p}{\partial s'^i} \frac{\partial s^q}{\partial s'^j},$$

and hence, by virtue of (22) and the relations $\partial s'^k / \partial s^q = \alpha^k|_q$ and $\alpha'^k|_j = \delta_j^k$, in the form

$$\alpha^k|_{i,j} - \alpha'^k|_{j,i} = (\alpha^k|_{p,q} - \alpha^k|_{q,p}) \frac{\partial s^p}{\partial s'^i} \frac{\partial s^q}{\partial s'^j},$$

where the components of the covariant derivatives are, as indicated, referred to E' .*

Transformation of angular and distantial spread vectors. Appealing to the equations for E' corresponding to (28) and (29), we conclude that the components, referred to E , of the angular spread and curvature vectors of the congruences of E' are $\gamma_{ij}|^r = C'_{ij}{}^k(\partial s^r/\partial s'^k)$. It follows, then, from (80), that the equations of the transformation from the angular spread and curvature vectors of E to those of E' , expressed in terms of components referred to E , are

$$(82) \quad \gamma_{ij}|^r = c_{pq}|^r \frac{\partial s^p}{\partial s'^i} \frac{\partial s^q}{\partial s'^j} + \frac{\partial}{\partial s'^i} \frac{\partial s^r}{\partial s'^j}.$$

The inverse transformation, when expressed in terms of the components referred to E' , that is, $\gamma'_{ij}|^k$ and $c'_{ij}|^k$, has the same form.

Equations (82) constitute the generalizations to Riemannian geometry of the most general form of the fundamental relation of Liouville for geodesic curvatures on a two-dimensional surface.† But equations (82) are Christoffel's equations in invariant form. Thus, we have a striking geometric interpretation of Christoffel's famous formulas.

By means of (31) and the corresponding equations for E' , we readily deduce from (81) the equations of the transformation from the distantial spread vectors of the pairs of congruences of E to those of the pairs of congruences of E' . Written in terms of components referred to E' , they are

$$(83) \quad \beta_{ij}|^r = b_{pq}|^r \frac{\partial s^p}{\partial s'^i} \frac{\partial s^q}{\partial s'^j} + \frac{\partial}{\partial s'^i} \frac{\partial s^r}{\partial s'^j} - \frac{\partial}{\partial s'^i} \frac{\partial s^r}{\partial s'^j}.$$

Since $\partial s^r/\partial s'^i = \alpha_i|^r$, we infer from these equations the following theorem.‡

THEOREM 33. *If E is an ennuple of Tchebycheff, a necessary and sufficient condition that E' be an ennuple of Tchebycheff is that*

$$\frac{\partial \alpha_i|^r}{\partial s'^j} = \frac{\partial \alpha_j|^r}{\partial s'^i} \quad (i, j, r = 1, 2, \dots, n).$$

* The first of these equations has a simple geometric interpretation. According to Theorem 21, $B'_{ij}{}^k=0$ characterizes ds'^k as an exact differential; but, by Theorem 1, the vanishing of the quantities in the parenthesis is precisely the condition that $(\partial s'^k/\partial s^r) ds^r = ds'^k$ be exact.

† Graustein, loc. cit., p. 570.

‡ A generalization of Theorem 20 in Graustein, loc. cit., p. 580. This theorem is for the case $n=2$ and assumes that E is an orthogonal ennuple.

An obvious solution of these equations is $\alpha_i|^\tau$ constant. But $\alpha_i|^\tau$ must, in any case, satisfy the n equations $\bar{g}_{kl}\alpha_i|^\tau\alpha_i|^\tau=1$, and these equations cannot be satisfied by constants, in general.

Applications to a sheaf of congruences. A totality of ∞^{n-1} congruences which has the property that each two congruences cut under a constant angle we shall call a sheaf of congruences.

THEOREM 34. *If E is an ennuple from a sheaf of congruences, E' is an ennuple from the sheaf if and only if the n^2 quantities $\partial s^i/\partial s'^i$ are constants.*

Since E belongs to the sheaf, the angles ω_{ij} , θ_i , Ω_{ij} pertaining to E are all constant. Evidently, the i th congruence of E' belongs to the sheaf if and only if the angles ψ_{ri} ($r=1, 2, \dots, n$) which it makes with the n congruences of E are constant. But this is the case, according to (79), if and only if the angles ϕ_{ir} ($r=1, 2, \dots, n$) are constants, and hence, by (78), if and only if $\partial s_r/\partial s'^i$ ($r=1, 2, \dots, n$) are constants. Thus, the theorem is proved.

It follows that, if E and E' are ennuples from the same sheaf, formulas (82) and (83) become

$$(84) \quad \gamma_{ij}|^\tau = c_{pq}|^\tau \frac{\partial s^p}{\partial s'^i} \frac{\partial s^q}{\partial s'^j}, \quad \beta_{ij}|^\tau = b_{pq}|^\tau \frac{\partial s^p}{\partial s'^i} \frac{\partial s^q}{\partial s'^j}.$$

These relations between the curvature and spread vectors of E and those of E' reflect the fact, evident from (80), that in a transformation from one ennuple of a sheaf to a second the Christoffel symbols behave like the components of tensors.

The relations embody various results. To begin with, we note

THEOREM 35. *If one ennuple from a sheaf of congruences has constant distastial spread vectors, and hence constant angular spread and curvature vectors, so has every ennuple from the sheaf.*

In particular, if one ennuple is an ennuple of Tchebycheff and therefore Cartesian, so is every ennuple.

From (84) and (71) we conclude:

$$g'^{ij}\gamma_{ij}|^\tau = g^{ij}c_{ij}|^\tau.$$

Hence the vector $g^{ij}c_{ij}|^\tau$ is the same for every ennuple of the sheaf and is, then, in this sense, an invariant vector of the sheaf. In particular, if E is orthogonal,

$$g^{ij}c_{ij}|^\tau = \sum_i c_{ii}|^\tau.$$

Thus, the sum of the curvature vectors of an orthogonal ennuple of a sheaf is the same for every orthogonal ennuple of the sheaf.*

Employing (84) and (71), we can construct other invariants of the sheaf. For example, the tensors

$$g^{ik}g^{jl}c_{ij}|^rc_{kl}|^t, \quad g^{ik}g^{jl}b_{ij}|^rb_{kl}|^t$$

are the same for every ennuple of the sheaf.

When we multiply each of these tensors by g_{rt} and sum over r, t , we obtain two scalar invariants of the sheaf. The values of these scalars for an orthogonal ennuple are $\sum(1/c_{ij}^2)$, $\sum(1/b_{ij}^2)$, where $1/c_{ij}$ and $1/b_{ij}$ are the lengths of the vectors $c_{ij}|$ and $b_{ij}|$. Thus, the sum of the squares of the lengths of all the curvature and angular spread vectors,† and the sum of the squares of the lengths of all the distastial spread vectors, of an orthogonal ennuple of a sheaf are the same for every orthogonal ennuple of the sheaf.

12. Inclusion of congruences of curves in families of surfaces. In §6 we found the conditions under which r linearly independent congruences lie in a family of r -dimensional surfaces. The purpose of this section is to treat the most general problem of this type, namely that of determining the family of surfaces of lowest dimensionality in which lie all the congruences of an arbitrarily chosen set of congruences.

The problem is not a simple one and much preliminary work is needed. To begin with, we shall show, by means of the following theorem, that we may restrict ourselves to sets of linearly independent congruences.

THEOREM 36. *If in a set of congruences there are r , and no more than r , linearly independent congruences and an arbitrarily chosen, but fixed, subset of r linearly independent congruences lies in a family of k -dimensional surfaces S_k and in no family of surfaces of lower dimensionality, all the congruences of the set lie in the surfaces S_k and in no family of surfaces of lower dimensionality.*

To establish the theorem, it suffices to prove that, if r linearly independent congruences lie in a family of k -dimensional surfaces S_k , any congruence which is a linear combination of them lies in the surfaces S_k . But this proposition is easily established.

The solution of the proposed problem is going to depend, not only on the distastial spread vectors of the given congruences, but also on the distastial spread vectors of the congruences determined by the distastial spread vectors

* Bortolotti, *Stelle di congruenze e parallelismo assoluto*, Rendiconti della Accademia dei Lincei, (6), vol. 9 (1929), pp. 530-538, gives this theorem. He approaches the subject indirectly, through the study of a metric connection with absolute parallelism, and is concerned with invariants which are the same simply for every orthogonal ennuple, not for every ennuple, of the sheaf.

† This result is also to be found in the paper by Bortolotti just cited.

of the given congruences. Accordingly, we shall find it convenient to introduce the following terminology.

DEFINITION 1. *The distantal spread vector of the two congruences determined by two ordered vector-fields shall be called the distantal spread vector of the two vector-fields, in the given order.*

We next prove a theorem for the distantal spread vectors of a given set of vector-fields analogous to Theorem 36 for congruences.

THEOREM 37. *If in a set of vector-fields there are r , and no more than r , linearly independent vector-fields, say V_1, V_2, \dots, V_r , and if V_1, V_2, \dots, V_r and their distantal spread vectors are linearly dependent on k linearly independent vector-fields $V_1, V_2, \dots, V_r, V_{r+1}, \dots, V_k$, then the distantal spread vector of any two vector-fields of the given set is a linear combination of V_1, V_2, \dots, V_k .*

Let the vector-fields V_1, V_2, \dots, V_r serve as the first r tangent vector-fields of an ennuple E , let r linearly independent linear combinations of V_1, V_2, \dots, V_r serve as the first r tangent vector-fields of a second ennuple E' , and assume that the remaining tangent vector fields of E and E' are identical. Then

$$\alpha_k |^i = \frac{\partial s^i}{\partial s'^k} = 0 \quad (k = 1, 2, \dots, r; i = r + 1, \dots, n).$$

Hence, by (83),

$$(85) \quad \beta_{ij} |^t = \sum_{p,q} b_{pq} |^t \frac{\partial s^p}{\partial s'^i} \frac{\partial s^q}{\partial s'^j} \quad (i, j = 1, 2, \dots, r; t = r + 1, \dots, n).$$

Without loss of generality we may assume that V_1, V_2, \dots, V_k are the first k tangent vector-fields of E . It follows, then, by hypothesis, that $b_{pq} |^t = 0$ for $p, q = 1, 2, \dots, r; t = k + 1, \dots, n$. Hence, $\beta_{ij} |^t$ is a linear combination of V_1, V_2, \dots, V_k and the theorem follows.

Suppose, now, that we have given a set, T_0 , of linearly independent vectors, that is, vector-fields. In solving the proposed problem for the congruences determined by the vectors of T_0 , we shall have to consider all the vectors obtainable from those of T_0 by repeated application of the process of finding distantal spread vectors. To systematize the repetition of this process, we adopt the following definition.

DEFINITION 2. *A distantal spread vector of T_0 of order $k (\geq 1)$ shall be a distantal spread vector of two distantal spread vectors of T_0 of orders lower than k , at least one of which is of order $k - 1$. A distantal spread vector of T_0 of order zero shall be a vector of T_0 .*

The totality of distantial spread vectors of T_0 of order i we shall denote by D_i and the totality of those of orders $0, 1, \dots, i$ we shall designate by T_i . Evidently,

$$T_i = T_{i-1} + D_i = D_0 + D_1 + \dots + D_i \quad (i \geq 1).$$

THEOREM 38. *The set of vectors $T_i (i \geq 1)$ consists of the vectors of T_0 and the distantial spread vectors of the vectors of T_{i-1} .*

The theorem follows directly from the definition.

For our purposes, the essential aspect of the set of vectors T_i is the *maximum number of linearly independent vectors contained in T_i* . We shall call this the *dimension number of T_i* , and denote it by n_i . Inasmuch as T_i contains T_{i-1} , it is clear that $n_i \geq n_{i-1}$, $i \geq 1$.

The sequence n_0, n_1, n_2, \dots we shall refer to as the *sequence of dimension numbers of T_0* . Corresponding to it we may choose, in various ways, a sequence of sets of vectors

$$(86) \quad V_1, \dots, V_{n_0}; V_{n_0+1}, \dots, V_{n_1}; V_{n_1+1}, \dots, V_{n_2}; \dots,$$

such that V_1, \dots, V_{n_i} are n_i linearly independent vectors in T_i . From the definition of n_i , it is evident that all the vectors of T_i are linear combinations of V_1, \dots, V_{n_i} , and that the vectors, $V_{n_{i-1}+1}, \dots, V_{n_i}$, of the $(i+1)$ st set of the sequence belong to D_i .

Since n_i can never exceed n , the sequence (86) is finite. We can, however, say more than this. It is true that, if a certain group in the sequence is empty, all succeeding groups are empty. In other words:

THEOREM 39. *If two consecutive numbers in the sequence of dimension numbers of T_0 are equal, all subsequent ones are equal to them:*

$$n_0 < n_1 < \dots < n_{k-1} < n_k = n_{k+l} \leq n \quad (l = 1, 2, \dots).$$

We are to prove that, if $n_{k+1} = n_k$, then $n_{k+2} = n_k$. Since $n_{k+1} = n_k$, the vectors of T_{k+1} , as well as the vectors of T_k , are linearly dependent on V_1, \dots, V_{n_k} . The distantial spread vectors of V_1, \dots, V_{n_k} , since they belong, by Theorem 38, to T_{k+1} , are then linearly dependent on V_1, \dots, V_{n_k} . Hence, by Theorem 37, the distantial spread vectors of all the vectors of T_{k+1} are linearly dependent on V_1, \dots, V_{n_k} . But these distantial spread vectors, together with the vectors of T_0 , are precisely the vectors of T_{k+2} . Thus the vectors of T_{k+2} are linear combinations of V_1, \dots, V_{n_k} , and therefore $n_{k+2} = n_k$.

By a *reduced set* of a given set of vectors we shall mean a set of linearly independent vectors of the given set on which all the vectors of the given set are linearly dependent; and by a *reduced sequence* of the sequence D_0 ,

D_1, D_2, \dots of successive distantial spread vectors of T_0 we shall mean a sequence of sets of vectors D'_0, D'_1, D'_2, \dots such that the vectors of T'_i , where $T'_i = T'_{i-1} + D'_i$, constitute a reduced set of the set of vectors T_i .

The sequence of groups of vectors (86) is a reduced sequence of the sequence D_0, D_1, D_2, \dots . But this sequence is perhaps lacking in an important property of the sequence D_0, D_1, D_2, \dots , namely, the property that every vector in the set D_i is a distantial spread vector of two vectors belonging to preceding sets. We shall call this the *property of cohesion*.

It is conceivable that a sequence of successive distantial spread vectors of T_0 cannot be rendered both reduced and cohesive. That this is not the case we shall prove by actually defining a sequence D'_0, D'_1, D'_2, \dots which has both properties.

DEFINITION 3. *The vectors of D'_0 shall be identical with those of D_0 (or T_0). The vectors of D'_i ($i \geq 1$) shall be chosen from the distantial spread vectors of the vectors of T'_{i-1} so that the vectors of T'_i constitute a reduced set for the vectors of T_0 and the distantial spread vectors of the vectors of T'_{i-1} .*

It is evident that the sequence D'_0, D'_1, D'_2, \dots thus defined is cohesive. To show that it is a reduced sequence of D_0, D_1, D_2, \dots , we must prove (a) that D'_i is a subset of D_i , and (b) that the dimension number of T'_i is the same as that of T_i .

A. D'_i is a subset of D_i . It is obvious from the definition that the theorem is true for $i=0, 1$. Suppose that $i \geq 2$. It follows from the definition that the distantial spread vectors of the vectors of T'_{i-1} , from which the vectors of D'_i are chosen, are distantial spread vectors of T_0 of orders not greater than i . Hence, if a vector of D'_i does not belong to D_i , it is a distantial spread vector of T_0 of order less than i , and so is included among the distantial spread vectors of vectors of T'_{i-2} . But these are, by definition, linear combinations of the vectors of T'_{i-1} . Thus, the vector of D'_i in question is linearly dependent on the vectors of T'_{i-1} , and this contradicts the demand that the vectors of $T'_i = T'_{i-1} + D'_i$ be linearly independent.

Since D'_i is a subset of D_i , T'_i is a subset of T_i , $i=0, 1, 2, \dots$. According to the definition, the vectors of T'_i are linearly dependent. It remains then to prove that the number of them, n'_i , is equal to the maximum number, n_i , of linearly independent vectors in T_i .

B. $n'_i = n_i$. Inasmuch as $n'_0 = n_0$, it suffices to show that, if $n'_i = n_i$, then $n'_{i+1} = n_{i+1}$. Since $n'_i = n_i$, the vectors of T_i are linear combinations of the vectors of T'_i . But the distantial spread vectors of the vectors of T'_i are linearly dependent on the vectors of $T'_{i+1} = T'_i + D'_{i+1}$. Hence, by Theorem 37, the distantial spread vectors of the vectors of T_i are linearly dependent on

the vectors of T'_{i+1} . But this means, according to Theorem 38, that the vectors of T_{i+1} are linear combinations of the vectors of T'_{i+1} , and hence $n'_{i+1} = n_{i+1}$.

The result thus established may be stated as follows.

THEOREM 40. *Any sequence of groups of vectors formed as described in Definition 3 is a cohesive, reduced sequence of the sequence D_0, D_1, D_2, \dots of the successive distantal spread vectors of T_0 .*

We return now to the proposed problem, restricting ourselves, as is permitted by Theorem 36, to a set of linearly independent congruences.

THEOREM 41. *If m is the largest number in the sequence of dimension numbers of the set, T_0 , of vector-fields tangent to r linearly independent congruences, the r congruences lie in a family of m -dimensional surfaces and in no family of surfaces of lower dimensionality.*

By hypothesis,

$$(87) \quad r = n_0 < n_1 < n_2 < \dots < n_{p-1} < n_p = m = n_{p+l} \quad (l = 1, 2, \dots).$$

Without loss of generality, we may take the given congruences as the first n_0 congruences of an ennuple E . The congruences lie, then, in the family of hypersurfaces $\phi = \text{const.}$ if and only if

$$(88) \quad \frac{\partial \phi}{\partial s^i} = 0 \quad (i = 1, 2, \dots, n_0),$$

and this system of equations is compatible if and only if

$$(89) \quad \sum_{k=n_0+1}^n B_{ij}^k \frac{\partial \phi}{\partial s^k} = 0 \quad (i \neq j; i, j = 1, 2, \dots, n_0).$$

The vectors $b_{ij}|^k = B_{ij}^k$, $i, j = 1, 2, \dots, n_0$, are the distantal spread vectors of the vectors of T_0 , or T'_0 . From them we choose, according to the prescriptions of Definition 3, the vectors of D'_1 . In $T'_1 = T'_0 + D'_1$ we have then n_1 linearly independent vectors, on which all the distantal spread vectors of T'_0 are linearly dependent.

The vectors of T'_0 are the first n_0 tangent vectors of E and those of D'_1 may be thought of as the next $n_1 - n_0$ tangent vectors of E . Equations (89) are, then, equivalent to the equations $\partial \phi / \partial s^k = 0$, $k = n_0 + 1, \dots, n_1$, and, when adjoined to equations (88), give rise to the extended system of equations

$$(90) \quad \frac{\partial \phi}{\partial s^i} = 0 \quad (i = 1, 2, \dots, n_1).$$

The procedure now repeats itself. The conditions of compatibility of (90), namely,

$$(91) \quad \sum_{k=n_1+1}^n B_{ij}^k \frac{\partial \phi}{\partial s^k} = 0 \quad (i \neq j; i, j = 1, 2, \dots, n_1),$$

involve the distastial spread vectors of the vectors of T'_1 . From these we choose, following Definition 3, the vectors of D'_2 , thus obtaining the n_2 linearly independent vectors, $T'_2 = T'_1 + D'_2$, on which the vectors involved in (91) are linearly dependent. Assuming that the vectors of D'_2 are the "next" $n_2 - n_1$ tangent vectors of E , we find, then, that equations (90) and (91) may be replaced by the equivalent system $\partial \phi / \partial s^i = 0, i = 1, 2, \dots, n_2$.

After this procedure has been carried out p times, we obtain the system of equations

$$(92) \quad \frac{\partial \phi}{\partial s^i} = 0 \quad (i = 1, 2, \dots, n_p),$$

as necessary and sufficient condition that the family of hypersurfaces $\phi = \text{const.}$ contain the r given congruences. Since, by (87), $n_p = m$, this system of equations is completely integrable. For the distastial spread vectors of the vectors of T'_p are, by hypothesis, linear combinations of the vectors of T'_p , and since these are the first n_p vectors of E , $B_{ij}^k = 0$ for $i, j = 1, 2, \dots, n_p$, $k = n_{p+1}, \dots, n$, so that the conditions of integrability are identically satisfied.

It follows, now, that the r given congruences lie in the family of m -dimensional surfaces defined by $n - m$ functionally independent solutions of the system of equations (92), and in no family of surfaces of lower dimensionality.

The fact that equations (92) are completely integrable means that the first $n_p (= m)$ congruences of E lie in a family of m -dimensional surfaces. The tangent vectors of these congruences are precisely the vectors of T'_p and, by hypothesis, T'_p is a reduced set of vectors for the set consisting of T_0 and the distastial spread vectors of T_0 of all orders. Hence:

THEOREM 42. *A necessary and sufficient condition that r linearly independent congruences lie in a family of m -dimensional surfaces and in no family of surfaces of lower dimensionality is that m be the minimum number of linearly independent vectors on which the tangent vectors of the congruences and their distastial spread vectors of all orders are linearly dependent. The curves of m congruences whose tangent vectors constitute such a set of linearly independent vectors lie, then, in a family of m -dimensional surfaces, and it is this family of surfaces in which the given congruences are contained.*

*Application to nonholonomic spaces.** Let there be given in V_n a system of $n-r$ ($r > 1$) linearly independent total differential equations

$$(93) \quad \bar{A}^i dx^i = 0 \quad (i = r+1, \dots, n),$$

where \bar{A}^i are functions of x^1, x^2, \dots, x^n .

If the system has no integral whatsoever, it is said to represent in V_n a single r -dimensional nonholonomic space V'_r .

In case the system has precisely $n-m$ ($m \geq r$) independent integrals, we shall say that it represents a nonholonomic manifold V_n which lies in a family of m -dimensional (metric) spaces. Actually, the system represents in this case ∞^{n-m} single nonholonomic spaces V'_m , one in each of the m -dimensional (metric) spaces determined by the $n-m$ integrals. In particular, if the system is completely integrable ($m=r$), the ∞^{n-r} nonholonomic spaces are the ∞^{n-r} (metric) space themselves.

We proceed to show how the discussion of the nonholonomic manifold (93) may be brought within the scope of our general theory and to deduce geometric conditions that the manifold lie in a family of m -dimensional (metric) spaces.

We think of the coefficients \bar{A}^i in (93) as defining $n-r$ linearly independent covariant vector-fields \bar{A}^i , $i=r+1, \dots, n$, and select r other covariant vector-fields \bar{A}^i , $i=1, 2, \dots, r$, so that the n fields \bar{A}^i , $i=1, 2, \dots, n$, are linearly independent. We then set $\bar{a}^i = \rho^i \bar{A}^i$, choosing the n functions ρ^i so that the n fields of contravariant vectors \bar{a}^i which are conjugate to the n fields of covariant vectors \bar{A}^i consist of unit vectors. Thereby, we obtain in V_n an ennuple of congruences, E , with reference to which the equations (93) of the nonholonomic manifold take the form

$$(94) \quad ds^i \equiv \bar{a}^i dx^i = 0 \quad (i = r+1, \dots, n).$$

It follows that the congruences of curves which lie in the nonholonomic manifold are precisely the congruences which are linearly dependent on the first r congruences of the ennuple E . Furthermore, ϕ is an integral of (94) if and only if $\partial\phi/\partial s^i = 0$, $i=1, 2, \dots, r$, that is, if and only if the family of hypersurfaces $\phi = \text{const.}$ contains the first r congruences of E . Hence, the conditions of Theorem 42, applied to these r congruences, are precisely the conditions under which the nonholonomic manifold lies in a family of m -dimensional (metric) spaces.

* For an extended treatment of nonholonomic spaces, see Vranceanu, *Studio geometrico dei sistemi anolonomi*, Annali di Matematica, (4), vol. 6 (1928), pp. 9-43.

SOME POINTS IN THE THEORY OF TRIGONOMETRIC AND POWER SERIES*

BY
ANTONI ZYGMUND

I. ON THE CHARACTER OF OSCILLATION OF THE PARTIAL SUMS OF FOURIER SERIES

1. The fundamental theorem. Completing a well known result of Kolmogoroff [12], Marcinkiewicz [15] has recently constructed a function integrable L , whose Fourier series possesses partial sums oscillating finitely almost everywhere. It is, therefore, natural to ask what may be said about the relative position of the interval of oscillation of $s_n(x)$ and the value $f(x)$, beyond the well known fact that the said interval contains $f(x)$. The result proved in this note is a first attempt in this direction.

THEOREM. *If the partial sums $s_n(x)$ of the Fourier series of a function $f(x)$ integrable L ,*

$$(1) \quad f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

satisfy an inequality

$$(2) \quad s_n(x) \geq -\phi(x) \quad (0 \leq x \leq 2\pi; n = 0, 1, 2, \dots),$$

with $\phi(x) \geq 0$ integrable L , then, for almost every x ,

$$(3) \quad f(x) = \frac{1}{2} \left[\limsup_{n \rightarrow \infty} s_n(x) + \liminf_{n \rightarrow \infty} s_n(x) \right].$$

2. Statement of lemmas. The proof of our theorem is based on three lemmas which will be stated in this section and the proof of which will be given in the next section. Let $\sigma_n(x)$, $\tilde{\sigma}_n(x)$, $\tilde{s}_n(x)$ denote, respectively, the first arithmetic means of the series (1), of the conjugate series

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx),$$

and the partial sums of the latter series. We have [20, 24, 18, 30]

* Presented to the Society, March 31, 1934; received by the editors November 5, 1933. The six notes constituting this paper are independent of one another, although they treat related topics. The numbers in square brackets refer to the Bibliography at the end of the paper.

$$s_n(x) - \sigma_n(x) = \tilde{s}_n'(x)/(n+1),$$

$$(4) \quad \frac{\tilde{s}_n'(x)}{n+1} = \frac{2}{\pi} \int_{-\pi}^{\pi} s_n(u) \cos(n+1)(u-x) K_n(u-x) du,$$

where

$$K_n(u) = \frac{1}{2(n+1)} \left[\frac{\sin \frac{n+1}{2} u}{\sin \frac{u}{2}} \right]^2$$

is the well known Fejér's kernel.

LEMMA 1. Let H be a measurable set contained in $(-\pi, \pi)$ and having $x=0$ as a point of zero density. Then the function

$$(5) \quad L_n(t) = \int_H K_n(u) K_n(u-t) du$$

satisfies the relations*

$$(6) \quad L_n(t) \leq \frac{Cn}{1+n^2 t^2},$$

$$(7) \quad L_n(t) = o(n) \quad (\text{uniformly in } t).$$

LEMMA 2. Under the hypothesis of the theorem we have

$$(8) \quad \left| \frac{\tilde{s}_n'(x)}{n+1} \right| = |s_n(x) - \sigma_n(x)| \leq \tau_n(x)$$

where $\tau_n(x)$ are the Fejér means of an integrable function $\psi(x) \geq 0$.

LEMMA 3. If, under the hypothesis of the theorem, we have for every x belonging to a set E of positive measure

$$(9) \quad s_n(x) - \sigma_n(x) = \tilde{s}_n'(x)/(n+1) \geq -M \quad (n \geq n_0, M > 0),$$

then, for almost every x in E ,

$$(10) \quad \limsup_{n \rightarrow \infty} [s_n(x) - \sigma_n(x)] \leq M.$$

3. Proof of lemmas. To prove Lemma 1, let $Q_n(u) = n/(1+n^2 u^2)$. Since the kernel $K_n(u)$ is $O(n)$ for $0 \leq u \leq 1/n$, and $O(1/(nu^2))$ for $1/n \leq u \leq 3\pi/2$, it is easy to see that $K_n(u) \leq CQ_n(u)$ (cf. Fejér [3]). From (5) we deduce that

* In the following we use C as a generic notation for an absolute constant, not necessarily the same in all formulas where it occurs.

$$L_n(t) \leq \int_{-\pi}^{\pi} K_n(u) K_n(u-t) du.$$

Since the integral is an even function of t , we may restrict ourselves to the values $0 \leq t \leq \pi$. Break up the integral into four, extended over the intervals $(-\pi, -\pi/2)$, $(-\pi/2, 0)$, $(0, t/2)$, $(t/2, \pi)$, and denoted respectively by $U_n^{(1)}(t)$, $U_n^{(2)}(t)$, $U_n^{(3)}(t)$, $U_n^{(4)}(t)$. Since $Q_n(u)$ is decreasing in the interval $0 \leq u \leq 3\pi/2$, it is readily seen that

$$U_n^{(1)}(t) = O(n^{-1}) \int_{-\pi}^{-\pi/2} K_n(u-t) du = O(n^{-1}) \leq CQ_n(t),$$

$$U_n^{(2)}(t) \leq CQ_n(t) \int_{-\pi/2}^0 K_n(u) du \leq CQ_n(t),$$

$$U_n^{(3)}(t) \leq CQ_n(t/2) \int_0^{t/2} K_n(u) du \leq CQ_n(t),$$

$$U_n^{(4)}(t) \leq CQ_n(t/2) \int_{t/2}^{\pi} K_n(u-t) du \leq CQ_n(t).$$

Adding these inequalities together we obtain (6).

To obtain (7), it is sufficient to replace in the integral (5) the function $K_n(u-t)$ by its upper bound (which is $O(n)$) and to notice that the remaining integral represents the Fejér means, at $x=0$, of the characteristic function of the set H , and so, by Lebesgue's well known criterion, tends to 0 with $1/n$.

To prove Lemma 2, replace, in the right-hand side of (4), $s_n(u)$ by $s_n(u) + \phi(u) - \phi(u)$. Then in view of (2), the first term in (8) does not exceed

$$\begin{aligned} & \frac{2}{\pi} \int_{-\pi}^{\pi} [s_n(u) + \phi(u)] K_n(u-x) du + \frac{2}{\pi} \int_{-\pi}^{\pi} \phi(u) K_n(u-x) du \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \psi(u) K_n(u-x) du \end{aligned}$$

where $\psi(u) = 2f(u) + 4\phi(u)$.

We pass on to the proof of Lemma 3. First, we have, for almost every x in the interval $(-\pi, \pi)$, the relation

$$(11) \quad \frac{\tilde{s}_n'(x)}{n+1} = \frac{2}{\pi} \int_{-\pi}^{\pi} [s_n(u) - \sigma_n(u) + M] \cos(n+1)(u-x) K_n(u-x) du + o(1).$$

For we may replace $s_n(u)$ by $s_n(u) + M$ under the sign of integral in (4), without changing its value. If we replace there s_n by σ_n , we obtain $\tilde{\sigma}_n'/(n+1)$, which represents the difference, multiplied by $(n+2)/(n+1)$, of the first and second arithmetic means of the series (1). This follows from the formula

$$\frac{1}{A_n^{(1)}} \sum_{r=0}^n A_{n-r}^{(1)} c_r - \frac{1}{A_n^{(2)}} \sum_{r=0}^n A_{n-r}^{(2)} c_r = \sum_{r=0}^n \frac{\nu c_r (n - \nu + 1)}{(n+1)(n+2)} = \frac{\tilde{\sigma}'_n(x)}{n+2},$$

where c_n is the general term of series (1) and

$$A_n^{(k)} = \binom{n+k}{k}.$$

This difference obviously tends to zero for almost every x

Let now $0 < r < 1$ be fixed and let

$$P_r(u) = \frac{1}{2} + r \cos u + r^2 \cos 2u + \dots$$

Since

$$\begin{aligned} \sum_{r=0}^n \left[\left(1 - \frac{\nu}{n+1} \right) - \left(1 - \frac{\nu}{n+1} \right)^2 \right] c_r \\ = \frac{\tilde{\sigma}'_n(x)}{n+1} = \frac{1}{\pi} \int_{-\pi}^{\pi} [s_n(u) - \sigma_n(u)] K_n(u-x) du, \end{aligned}$$

we have

$$(12) \quad \frac{1}{\pi} \int_{-\pi}^{\pi} [s_n(u) - \sigma_n(u)] K_n(u-x) du \rightarrow 0$$

for almost every x , and

$$\begin{aligned} r \frac{\tilde{\sigma}'_n(x)}{n+1} - M &= \frac{2}{\pi} \int_{-\pi}^{\pi} [s_n(u) - \sigma_n(u) + M] \\ &\quad \times \left[-\frac{1}{2} + r \cos(n+1)(u-x) \right] K_n(u-x) du + o(1) \\ &= -\frac{2}{\pi} \int_{-\pi}^{\pi} [s_n(u) - \sigma_n(u) + M] P_r\{(n+1)(u-x) + \pi\} K_n(u-x) du + o(1). \end{aligned}$$

Break up the last integral into two, extended over E and its complement H , and denote the corresponding expressions by J_1 and J_2 . $P_r(u)$ and $K_n(u)$ are non-negative, hence by (9), $J_1 \leq 0$ and it remains to show that $J_2 \rightarrow 0$ almost everywhere in E . It is sufficient to show that $J_2 \rightarrow 0$ at every point x where E has density 1 and where the integral of ψ (see Lemma 2) has a finite derivative. Suppose for simplicity that $x=0$ is such a point and let $M_r = \max P_r(u)$, $0 \leq u \leq 2\pi$. Then (see Lemma 2)

$$(13) \quad |J_2| \leq \frac{2}{\pi} M_r \int_H [\tau_n(u) + M] K_n(u) du = \frac{2}{\pi} M_r \int_H \tau_n(u) K_n(u) du + o(1).$$

Expressing $\tau_n(u)$ as a Fejér's integral and interchanging the order of integration, we see that the integral of the right-hand member of (13) is equal to

$$\frac{2}{\pi^2} M_r \int_{-\pi}^{\pi} \psi(t) L_n(t) dt = \frac{2}{\pi^2} M_r \left(\int_{-\pi}^0 + \int_0^{\pi} \right)$$

where $L_n(t)$ is given by formula (5). Let β be a fixed positive number. We have

$$(14) \quad \int_0^{\pi} \psi(t) L_n(t) dt = \int_0^{\beta/n} \dots + \int_{\beta/n}^{\pi}.$$

Let

$$\Psi(t) = \int_0^t \psi(u) du, \quad \Psi(t) \leq \gamma t \quad (t \geq 0),$$

γ being a constant (the inequality is implied by the fact of existence of a finite derivative of $\Psi(t)$ at $t=0$). By (7) the first integral on the right in (14) is $o(n)\Psi(\beta/n) = o(1)$. The second integral is less than

$$\begin{aligned} \frac{C}{n} \int_{\beta/n}^{\pi} \psi(t) t^{-2} dt &\leq \frac{C}{n} \left[\frac{\Psi(\pi)}{\pi^2} + 2 \int_{\beta/n}^{\pi} \Psi(t) t^{-3} dt \right] \\ &\leq o(1) + \frac{2C\gamma}{n} \int_{\beta/n}^{\infty} t^{-2} dt \leq \epsilon \quad (n \geq n_0(\epsilon)), \end{aligned}$$

where $\epsilon > 0$ is arbitrarily small, if only β is sufficiently large. An analogous discussion is applied to the integral $\int_{-\pi}^{\pi} \psi(t) L_n(t) dt$. It follows that $J_2 \rightarrow 0$ for every $0 < r < 1$. Since we may take r as near to 1 as we please, the truth of the lemma follows.

4. Proof of the theorem. Let now F and G denote the sets of points at which, respectively,

$$(15) \quad \begin{aligned} \limsup_{n \rightarrow \infty} [s_n(x) - f(x)] &> \limsup_{n \rightarrow \infty} [f(x) - s_n(x)], \\ \limsup_{n \rightarrow \infty} [s_n(x) - f(x)] &< \limsup_{n \rightarrow \infty} [f(x) - s_n(x)]. \end{aligned}$$

To prove that the set F is of measure 0, it is enough to show that the set F_1 of points for which

$$\limsup_{n \rightarrow \infty} [s_n(x) - \sigma_n(x)] > \limsup_{n \rightarrow \infty} [\sigma_n(x) - s_n(x)]$$

is of measure 0. If it were not, we could find two numbers $N > M > 0$ and a set $F_2 \subset F_1$, $\text{meas } F_2 > 0$, such that

$$(16) \quad \limsup_{n \rightarrow \infty} [s_n(x) - \sigma_n(x)] > N > M > \limsup_{n \rightarrow \infty} [\sigma_n(x) - s_n(x)].$$

From the last inequality we conclude the existence of an integer n_0 and of a set $E \subset F_2$, $\text{meas } E > 0$, such that

$$\sigma_n(x) - s_n(x) \leq M, \quad n > n_0, \quad x \in E,$$

and hence, by Lemma 3, we have

$$\limsup_{n \rightarrow \infty} [s_n(x) - \sigma_n(x)] \leq M < N$$

almost everywhere in E , contrary to the first of inequalities (16). The theorem is, therefore, established.

5. Additional remarks. (i) Under the hypothesis of our theorem we may prove also that the relation

$$(17) \quad \tilde{f}(x) = \frac{1}{2} \left[\limsup_{n \rightarrow \infty} \tilde{s}_n(x) + \liminf_{n \rightarrow \infty} \tilde{s}_n(x) \right],$$

where $\tilde{f}(x)$ is the function conjugate to $f(x)$, holds almost everywhere in $(-\pi, \pi)$. The proof is exactly the same as before, except that, instead of (4), we use the formula

$$\tilde{\sigma}_n(x) - \tilde{s}_n(x) = \frac{s'_n(x)}{n+1} = \frac{2}{\pi} \int_{-\pi}^{\pi} s_n(u) \sin(n+1)(u-x) K_n(u-x) du.$$

(ii) It is not difficult to see that the results above may be localized; if we suppose that (2) is satisfied in an interval (a, b) , the relations (3) and (17) are true almost everywhere in (a, b) . This follows from general localization theorems for trigonometric series.

(iii) The hypothesis that the trigonometric series considered in the theorem is a Fourier-Lebesgue series is superfluous and may be omitted. In fact, inequality (2) implies that the sequence $\{ \int_{-\pi}^{\pi} |s_n(x)| dx \}$ is bounded, and so the series (1) is a Fourier-Stieltjes series. The arguments which we have used in the proof may be, without any difficulty, adapted to this new, slightly more general, case. (See for instance [30].)

II. ON THE ABSOLUTE CONVERGENCE OF FOURIER SERIES

1. It has been proved that if $f(x)$, $0 \leq x \leq 2\pi$, is a periodic function of bounded variation, satisfying a Lipschitz condition of positive order, the Fourier series of $f(x)$ converges absolutely [28, 8].

In the same way it is possible to prove the following, more precise, theorem [26].*

* For a similar problem see also O. Szász [23].

THEOREM 1. If $f(x)$ is of bounded variation and satisfies a Lipschitz condition of order α , and if

$$(1) \quad f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

then the series

$$(2) \quad \sum_{n=1}^{\infty} \rho_n^k, \quad \rho_n^2 = a_n^2 + b_n^2,$$

converges for every $k > 2/(2+\alpha)$.

The main purpose of this note is to show that the condition imposed on k is the best possible. More precisely, we may state

THEOREM 2. For every value $0 < \alpha < 1$ there exists a function of bounded variation, satisfying a Lipschitz condition of order α , and such that the series (2) diverges when $k = 2/(2+\alpha)$.

2. For the sake of completeness we begin by proving Theorem 1.

Let N be a positive integer and $j = 1, 2, \dots, 2N$. By Parseval's identity,

$$(3) \quad \begin{aligned} & \int_0^{2\pi} \left[f\left(x + \frac{2\pi j}{2N}\right) - f\left(x + \frac{2\pi(j-1)}{2N}\right) \right]^2 dx \\ &= \int_0^{2\pi} \left[f\left(x + \frac{\pi}{2N}\right) - f\left(x - \frac{\pi}{2N}\right) \right]^2 dx = 4\pi \sum_{n=1}^{\infty} \rho_n^2 \sin^2 \frac{n\pi}{2N}. \end{aligned}$$

Let V denote the absolute variation of $f(x)$ in the interval $(0, 2\pi)$ and $\omega(\delta)$ the modulus of continuity of $f(x)$, i.e., $\omega(\delta) = \max |f(x_1) - f(x_2)|$ for $|x_1 - x_2| \leq \delta$. In our case $\omega(\delta) = O(\delta^\alpha)$.

For every x we have the inequality

$$(4) \quad \begin{aligned} & \sum_{j=1}^{2N} \left[f\left(x + \frac{2\pi j}{2N}\right) - f\left(x + \frac{2\pi(j-1)}{2N}\right) \right]^2 \\ & \leq \omega\left(\frac{\pi}{N}\right) \sum_{j=1}^{2N} \left| f\left(x + \frac{2\pi j}{2N}\right) - f\left(x + \frac{2\pi(j-1)}{2N}\right) \right| \leq \omega\left(\frac{\pi}{N}\right) V. \end{aligned}$$

Integrating this over the range $(0, 2\pi)$, and taking into account (3), we get successively

$$\begin{aligned} 2N \sum_{n=1}^{\infty} \rho_n^2 \sin^2 \frac{n\pi}{2N} &= O(N^{-\alpha}), & \sum_{n=2^{p-1}}^{2^p-1} \rho_n^2 \sin^2 \frac{n\pi}{2N} &= O(2^{-p(1+\alpha)}), \quad N = 2^p, \\ & & \sum_{n=2^{p-1}}^{2^p-1} \rho_n^2 &= O(2^{-p(1+\alpha)}). \end{aligned}$$

We may suppose that $k < 2$, the convergence of $\sum \rho_n^k$ being obvious. Then, by Hölder's inequality,

$$\sum_{n=2^{p-1}}^{2^p-1} \rho_n^k = O(2^{-p(1+\alpha)k/2} \cdot 2^{p(1-k/2)}).$$

It follows that the series

$$\sum_{n=1}^{\infty} \rho_n^k = \sum_{p=1}^{\infty} \sum_{n=2^{p-1}}^{2^p-1} \rho_n^k$$

converges, if only $k > 2/(\alpha+2)$.

3. To prove Theorem 2 we shall consider power series of the form $\sum b_n \exp(2\pi i n^\alpha) z^n$, where $0 < \alpha < 1$ and b_n are real and very regularly tend to zero. We shall study these series by means of the following lemmas due to van der Corput.*

LEMMA 1. Let $a(u)$ be a real function of u , $|a'(u)| \leq 1 - \delta$. Then

$$\left| \sum_{\alpha < \nu \leq \beta} \exp 2\pi i a(\nu) - \int_{\alpha}^{\beta} \exp 2\pi i a(u) du \right| < A_{\delta}.$$

where A_{δ} depends only on δ .

LEMMA 2. Let $a'(u)$ be positive and decreasing. Then†

$$\left| \int_{\alpha}^{\beta} \exp 2\pi i a(u) du \right| \leq \frac{C}{a'(\beta)}.$$

LEMMA 3. Let $a''(u) \leq -\rho < 0$. Then

$$\left| \int_{\alpha}^{\beta} \exp 2\pi i a(u) du \right| \leq C\rho^{-1/2}.$$

The following two propositions are (using Abel's transformation) immediate corollaries of Lemmas 1 and 2.

LEMMA 4. If (i) $a(u) \rightarrow \infty$, (ii) $a'(u)$ decreases monotonically to zero, (iii) $b_n \rightarrow 0$, (iv) $\sum |\Delta b_n| < \infty$, $\Delta b_n = b_n - b_{n+1}$, then the series

$$(5) \quad \sum b_n \exp 2\pi i [a(n) + n\theta]$$

converges uniformly on every arc $\delta \leq 0 \leq 1 - \delta$.

* We take these lemmas in the form stated by Hille [10, 11]. In [10] several bibliographical references are given.

† See footnote on p. 587.

LEMMA 5. If (iii) and (iv) in Lemma 4 are replaced respectively by (iii') $b_n/a'(n) \rightarrow 0$, (iv') $\sum |\Delta b_n|/a'(n) < \infty$, then the series (5) converges for every value of θ although not necessarily uniformly (in fact it converges uniformly over every interval $0 \leq \theta \leq 1 - \delta$).

4. We shall now prove

THEOREM 3. If $0 < \alpha < 1$, $\beta > 0$, the function

$$(6) \quad f^{\alpha, \beta}(\theta) = \sum_{n=1}^{\infty} n^{-\beta} \exp 2\pi i(n^\alpha + n\theta)$$

which is continuous in every interval $\delta \leq \theta \leq 1 - \delta$, satisfies the following inequalities:

$$f^{\alpha, \beta}(\theta) = \begin{cases} O(\theta^{-1+\beta/(1-\alpha)}), & \alpha + \beta < 1 \\ O(\log |\theta|), & \alpha + \beta = 1 \\ O(1), & \alpha + \beta > 1 \end{cases} \text{ as } \theta \rightarrow 0^+;$$

$$f^{\alpha, \beta}(\theta) = \begin{cases} O(|\theta|^{-(1-\alpha/2-\beta)/(1-\alpha)}), & \frac{1}{2}\alpha + \beta < 1 \\ O(1), & \frac{1}{2}\alpha + \beta \geq 1 \end{cases} \text{ as } \theta \rightarrow 0^-.$$

Let

$$S_n^{\alpha, \beta}(\theta) = \sum_{\nu=1}^n \nu^{-\beta} \exp 2\pi i(\nu^\alpha + \nu\theta).$$

Put $a(u) = u^\alpha + u\theta$ and assume $|\theta| \leq \frac{1}{2}$. From Lemma 2 it follows that

$$(7) \quad S_n^{\alpha, 0}(\theta) = \int_1^n \exp 2\pi i a(u) du + O(1),$$

and, by Lemma 3,*

$$\left| \int_1^n \exp 2\pi i a(u) du \right| \leq A n^{1-\alpha/2}.$$

Hence

$$(8) \quad |S_n^{\alpha, 0}(\theta)| \leq A n^{1-\alpha/2}, \quad |\theta| \leq \frac{1}{2}.$$

For subsequent discussion we need more precise estimates of $S_n^{\alpha, 0}(\theta)$. To obtain them the cases $0 < \theta \leq \frac{1}{2}$, $-\frac{1}{2} \leq \theta < 0$ have to be treated separately. In the first case we have, by Lemma 2,

$$\left| \int_1^n \exp 2\pi i a(u) du \right| \leq \frac{A}{\alpha n^{\alpha-1} + \theta} \leq \frac{A}{\theta},$$

* We use A as a general notation of a constant which does not depend on θ .

whence

$$\left. \begin{aligned} (9) \quad & |S_n^{\alpha,0}(\theta)| \leq An^{1-\alpha} \\ (10) \quad & |S_n^{\alpha,0}(\theta)| \leq \frac{A}{\theta} \end{aligned} \right\} 0 < \theta \leq \frac{1}{2}.$$

In the second case we put $t = -\theta$,

$$(11) \quad a(u) = u^\alpha - tu, \quad a'(u) = \alpha u^{\alpha-1} - t, \quad 0 < t \leq \frac{1}{2},$$

$$(12) \quad N_1 = \left[\left(\frac{2t}{\alpha} \right)^{-1/(1-\alpha)} \right], \quad N_2 = \left[\left(\frac{t}{2\alpha} \right)^{-1/(1-\alpha)} \right].$$

Now if $n \leq N_1$, again by Lemma 2,

$$\left| \int_1^n \exp 2\pi i a(u) du \right| \leq \frac{A}{\alpha n^{\alpha-1} - t} \leq An^{1-\alpha},$$

and again

$$(13) \quad |S_n^{\alpha,0}(\theta)| \leq An^{1-\alpha}, \quad n \leq N_1.$$

By (8),

$$(14) \quad |S_n^{\alpha,0}(\theta)| \leq An^{1-\alpha/2} \leq A |\theta|^{-1-\alpha/[2(1-\alpha)]}, \quad N_1 < n \leq N_2.$$

Finally, when $n > N_2$, we write

$$\int_1^n \exp 2\pi i (u^\alpha - tu) du = \int_1^{N_2} \dots + \int_{N_2}^n \dots$$

The first integral on the right gives the same contribution as (14) while the second integral can be estimated by Lemma 2 (with an obvious modification). Thus we get

$$\left| \int_{N_2}^n \exp 2\pi i (u^\alpha - tu) du \right| \leq \frac{A}{t - \alpha N_2^{\alpha-1}} \leq \frac{A}{t},$$

and

$$(15) \quad |S_n^{\alpha,0}(\theta)| \leq |\theta|^{-1-\alpha/[2(1-\alpha)]}, \quad n > N_2.$$

Now, by Abel's partial summation,

$$S_n^{\alpha,\beta}(\theta) = \sum_{\nu=1}^{n-1} S_\nu^{\alpha,0}(\theta) \Delta \nu^{-\beta} + n^{-\beta} S_n^{\alpha,0}(\theta).$$

For any fixed value of $\theta \neq 0$, $|\theta| \leq \frac{1}{2}$, the second term of the right-hand member tends to 0 as $n \rightarrow \infty$. Hence

$$(16) \quad f^{\alpha, \beta}(\theta) = \lim_{n \rightarrow \infty} S_n^{\alpha, \beta}(\theta) = \sum_{p=1}^{\infty} \Delta p^{-\beta} S_p^{\alpha, 0}(\theta), \quad 0 < |\theta| \leq \frac{1}{2}.$$

We write

$$f^{\alpha, \beta}(\theta) = \sum_{p=1}^{N_1} \Delta p^{-\beta} S_p^{\alpha, 0}(\theta) + \sum_{p=N_1+1}^{\infty} \Delta p^{-\beta} S_p^{\alpha, 0}(\theta) \equiv P + Q.$$

By (9) and (13)

$$|P| \leq A \sum_{p=1}^{N_1} p^{-1-\beta} p^{1-\alpha} = A \sum_{p=1}^{N_1} p^{-\alpha-\beta}.$$

Hence, for $0 < |\theta| \leq \frac{1}{2}$,

$$P = \begin{cases} O(N_1^{1-\alpha-\beta}) = O(|\theta|^{-1+\beta/(1-\alpha)}), & \text{if } \alpha + \beta < 1, \\ O(\log N_1) = O(\log |\theta|), & \text{if } \alpha + \beta = 1, \\ O(1), & \text{if } \alpha + \beta > 1. \end{cases}$$

At this juncture we have again to distinguish between the cases $0 < \theta \leq \frac{1}{2}$ and $-\frac{1}{2} \leq \theta < 0$. In the first case we apply (10) which gives

$$Q = O\left(\frac{1}{\theta} \sum_{p=N_1+1}^{\infty} p^{-1-\beta}\right) = O(\theta^{-1} N_1^{-\beta}) = O(\theta^{-1+\beta/(1-\alpha)}).$$

Being combined with the estimates above for P this furnishes the proof of the first part of Theorem 3.

When $-\frac{1}{2} \leq \theta < 0$ we write

$$Q = \sum_{p=N_1+1}^{\infty} \dots = \sum_{p=N_1+1}^{N_2} \dots + \sum_{p=N_2+1}^{\infty} \dots \equiv R + S,$$

and apply (14) and (15). This yields

$$|R| \leq A \sum_{p=N_1+1}^{N_2} p^{-1-\beta} p^{1-\alpha/2} = A \sum_{p=N_1+1}^{N_2} p^{-\beta-\alpha/2}.$$

It follows that

$$R = \begin{cases} O(N_2^{1-\beta-\alpha/2}) = O(|\theta|^{-(1-\beta-\alpha/2)/(1-\alpha)}), & \text{if } \beta + \frac{\alpha}{2} < 1, \\ O\left(\log \frac{N_2}{N_1}\right) = O(1), & \text{if } \beta + \frac{\alpha}{2} = 1, \\ o(1), & \text{if } \beta + \frac{\alpha}{2} > 1. \end{cases}$$

Finally, S is readily estimated by using (15) which gives

$$\begin{aligned} S &= O\left(\sum_{p=N+1}^{\infty} p^{-1-\beta} |\theta|^{-1-\alpha/[2(1-\alpha)]}\right) \\ &= O(|\theta|^{-1-\alpha/[2(1-\alpha)]} N_2^{-\beta}) = O(|\theta|^{-(1-\beta-\alpha/2)/(1-\alpha)}). \end{aligned}$$

On combining these results we obtain a proof of the second part of Theorem 3.

Theorem 3 shows that the behavior of $f^{\alpha,\beta}(\theta)$ in the neighborhood of $\theta=0$ is different for $\theta \rightarrow 0^+$ and for $\theta \rightarrow 0^-$. In the interval $0 < \theta \leq \frac{1}{2}$ the function $f^{\alpha,\beta}(\theta)$ is always integrable. In the interval $-\frac{1}{2} \leq \theta < 0$ we are sure of integrability only if $\beta > \alpha/2$. If $\beta = \alpha/2$ we get only $f^{\alpha,\beta}(\theta) = O(|\theta|^{-1})$ and the function is probably not integrable.* It will be integrable if we introduce additional logarithms, as is shown in the next

THEOREM 4. *The sum of the series*

$$(17) \quad \sum_{n=2}^{\infty} n^{-\alpha/2} (\log n)^{-\gamma} \exp 2\pi i(n^{\alpha} + n\theta), \quad \gamma > 1,$$

is $O(|\theta|^{-1} \log^{-\gamma}(1/|\theta|))$, and, consequently, the series is the Fourier series of its sum.

Although this theorem is important for our purposes, the proof need not be gone into, as it is essentially the same as that of Theorem 3.†

5. We now prove

THEOREM 5. *If $1 \leq \alpha/2 + \beta \leq 2$, $0 < \alpha < 1$, $\beta > 0$, the function $f^{\alpha,\beta}(\theta)$ satisfies a Lipschitz condition of order $\alpha/2 + \beta - 1$.*

The case $\alpha/2 + \beta = 1$ is contained in Theorem 3 and the other extreme case is a corollary of it. If $1 < \alpha/2 + \beta < 2$ it follows from Lemma 5 that the series is everywhere convergent. Using (16) we have, with $N = [1/|h|]$,

$$\begin{aligned} |f^{\alpha,\beta}(\theta + h) - f^{\alpha,\beta}(\theta)| &\leq \sum_{p=1}^{\infty} \Delta p^{-\beta} |S_p^{\alpha,0}(\theta + h) - S_p^{\alpha,0}(\theta)| \\ &= \sum_{p=1}^N \dots + \sum_{p=N+1}^{\infty} \dots \equiv P + Q. \end{aligned}$$

From (8) it follows that

* It is certainly not integrable if $\beta < \alpha/2$, for otherwise the series $\sum n^{-1-\beta} \exp 2\pi i(n^{\alpha} + n\theta)$ would be the Fourier series of a function of bounded variation (indeed absolutely continuous) satisfying a Lipschitz condition of order $\alpha/2 + \beta$ (see Theorem 5) and, by Theorem 1, its exponent of convergence would be $\leq 2/(\alpha/2 + \beta + 2)$, which is easily seen to be impossible. It is, however, obvious that for any α , $\beta > 0$, the series (6) are Fourier-Riemann series.

† The same argument gives a more general result concerning the functions we obtain by introducing logarithms into the denominator of the series (6).

$$|Q| \leq \sum_{p=N+1}^{\infty} \Delta p^{-\beta} (|S_p^{\alpha,0}(\theta+h)| + |S_p^{\alpha,0}(\theta)|) = O\left(\sum_{p=N+1}^{\infty} p^{-\beta-1} p^{1-\alpha/2}\right) \\ = O(|h|^{\alpha/2+\beta-1}).$$

On the other hand we have

$$|P| \leq |h| \sum_{p=1}^N \Delta p^{-\beta} \max |(S_p^{\alpha,0})'|.$$

By the well known theorem of S. Bernstein, if $T_n(\theta)$ is any trigonometric polynomial of order n , and if $|T_n(\theta)| \leq M$, then $|T_n'(\theta)| \leq Mn$. In view of (8) this yields at once

$$\left| \frac{d}{d\theta} S_p^{\alpha,0}(\theta) \right| \leq A p^{2-\alpha/2},$$

whence

$$|P| = O\left(|h| \sum_{p=1}^N p^{-1-\beta} p^{2-\alpha/2}\right) = O(|h| N^{2-\alpha/2-\beta}) = O(|h|^{\alpha/2+\beta-1}).$$

Theorem 5 is thus proved.

THEOREM 6. If $1 \leq \alpha/2 + \beta \leq 2$, $0 < \alpha < 1$, $\beta > 0$, $\gamma > 0$, then the sum of the series

$$(18) \quad f^{\alpha,\beta,\gamma}(\theta) = \sum_{n=2}^{\infty} n^{-\beta} (\log n)^{-\gamma} \exp 2\pi i(n\alpha + n\theta)$$

satisfies a Lipschitz condition of order $\frac{1}{2}\alpha + \beta - 1$.

The proof is the same as in the case of Theorem 5.*

THEOREM 7. The series

$$\sum_{n=2}^{\infty} n^{-1-\alpha/2} (\log n)^{-\gamma} \exp 2\pi i(n\alpha + n\theta), \quad 0 < \alpha < 1, \gamma > 1,$$

is the Fourier series of a function of bounded variation satisfying a Lipschitz condition of order α , and if γ is sufficiently near to 1, its coefficients c_n have the property that $\sum |c_n|^k$ diverges for $k = 2/(\alpha + 2)$.

This follows from Theorems 4 and 6. Theorem 2 now follows from Theorem 7.

* We may also deduce Theorem 6 from Theorem 5 if we take into account that (18) is a "Faltung" of (6) and $\sum (\log n)^{-\gamma} \cos 2\pi n\theta$ which is a Fourier-Lebesgue series for every $\gamma > 0$. It is easy to see that the modulus of continuity of (6) will be preserved.

III. ON A THEOREM OF FEJÉR AND RIESZ

1. The following result has been obtained by Fejér and Riesz [4].

THEOREM 1. Every analytic function $f(z)$ regular for $|z| \leq 1$ satisfies the inequality

$$(1_p) \quad \int_D |f(z)|^p |dz| \leq \frac{1}{2} \int_C |f(z)|^p |dz|,$$

where C denotes the circle $|z| = 1$ and D is its arbitrary diameter.

It is well known that it suffices to prove the inequality (1_p) for any special value of p ; the general result then follows by a familiar argument.* Fejér and Riesz started with the case $p=2$. An alternative proof of (1_p) which is given below begins with $p=1$. This proof is based on the following

LEMMA. Let $u(z)$ and $v(z)$ be conjugate, not necessarily real,† harmonic functions such that $v(0)=0$ and that $f(z)=u(z)+iv(z)$ is regular for $|z| \leq 1$. Then, with the same notations as before,

$$(2) \quad \int_D |v(z)| |dz| \leq \int_D \left| \frac{v(z)}{z} \right| |dz| \leq \frac{1}{2} \int_C |u(z)| |dz|.$$

On setting

$$Q_r(t) = \sum_{n=1}^{\infty} r^n \sin nt = \frac{r \sin t}{1 - 2r \cos t + r^2}, \quad z = re^{it},$$

we have

$$v(re^{i\theta}) = \frac{1}{\pi} \int_0^{2\pi} u(e^{it}) Q_r(t - \theta) dt.$$

Without loss of generality we may assume that the diameter D is the segment $(-1, 1)$ of the X -axis. Then

$$\int_D \left| \frac{v(z)}{z} \right| |dz| = \int_0^1 [|v(r)| + |v(-r)|] r^{-1} dr \leq M \int_0^{2\pi} |u(e^{it})| dt,$$

* It is well known that the condition of $f(z)$ being regular on C is not necessary and may be replaced by less stringent conditions. In the proof of Theorem 2 below we shall use the inequality (1_p) under the assumption that $\Re(f)$ is continuous for $|z| \leq 1$.

† A complex harmonic function $v(z)=v_1(z)+iv_2(z)$ is said to be conjugate to $u_1(z)+iu_2(z)$ if $v_1(z)$ is conjugate to $u_1(z)$, and $v_2(z)$ to $u_2(z)$.

where M is the upper bound, with respect to α , of

$$\frac{1}{\pi} \int_0^1 [|Q_r(\alpha)| + |Q_r(\alpha + \pi)|] r^{-1} dr.$$

Suppose, as we may, that $0 < \alpha < \pi$. On setting $\sin \alpha = h$, $\cos \alpha = k$, we see that the last integral is equal to

$$\frac{h}{\pi} \left[\int_0^1 \frac{dr}{(r-k)^2 + h^2} + \int_0^1 \frac{dr}{(r+k)^2 + h^2} \right] = \frac{1}{2}.$$

Our lemma is thus established.†

Now we notice that if $g(z)$ is analytic, the function $-ig(z)$ is conjugate to $g(z)$. Consequently, applying our lemma to the functions $u(z) = zf(z)$ and $v(z) = -izf(z)$, we get the inequality (1₁) and, hence, the whole Theorem 1.

2. We now prove

THEOREM 2. *Let $f(z) = u(z) + iv(z)$ be regular for $|z| \leq 1$, where u and v are real and $v(0) = 0$. There exists a constant A_p depending only on p , and uniformly bounded in every interval $1 \leq p \leq p_0$, such that*

$$(3_p) \quad \int_D |v(z)|^p |dz| \leq A_p^p \int_C |u(z)|^p |dz|, \quad p \geq 1.$$

A preliminary remark is worth making. It has been proved by M. Riesz [22] that for any $p > 1$ we have

$$\int_0^{2\pi} |v(e^{i\theta})|^p d\theta \leq M_p \int_0^{2\pi} |u(e^{i\theta})|^p d\theta,$$

where M_p depends only on p , and so

$$\begin{aligned} \int_D |v(z)|^p |dz| &\leq \int_D |f(z)|^p |dz| \leq \frac{1}{2} \int_C |f(z)|^p |dz| \\ (4) \quad &\leq 2^{p-1} \int_C [|u(z)|^p + |v(z)|^p] |dz| \\ &\leq 2^{p-1} (M_p + 1) \int_C |u(z)|^p |dz|. \end{aligned}$$

† The constant $\frac{1}{2}$ in (2) cannot be improved for, otherwise, we could improve the inequality (1_p), which is known to be impossible (Fejér and Riesz, loc. cit.). Another example is given by the pair of conjugate functions $u(Rz)$ and $v(Rz)$, where

$$u(z) = P_r(\theta) = \frac{1}{2} \frac{1-r^2}{1-2r \cos \theta + r^2}, \quad v(z) = Q_r(\theta),$$

and R is sufficiently near to 1.

However we cannot put $A_p^p = 2^{p-1}(M_p + 1)$, since M_p is known to be unbounded in the neighborhood of $p = 1$, so that Theorem 2 is not a consequence of (1_p) and of M. Riesz's theorems on conjugate functions, although every single inequality (3_p) , for $p > 1$, is such a consequence.

Assume again for simplicity that D is the interval $(-1, 1)$ of the X -axis. To any continuous function $u(e^{i\theta})$ defined on $|z| = 1$, there corresponds a function $v(z) = T\{u\}$, conjugate to the Poisson integral of $u(e^{i\theta})$, defined for $-1 < r < 1$. The functional $v = T\{u\}$ is additive and the inequality (3_p) is certainly true for $p = 1$ and $p = p_0$. By a theorem of M. Riesz [21], the upper bound (with respect to all continuous functions) of the ratio

$$\left(\int_{-1}^1 |v(r)|^p dr \right)^{1/p} / \left(\int_0^{2\pi} |u(e^{i\theta})|^p d\theta \right)^{1/p}$$

is a convex function of $1/p$, $p \geq 1$. Hence, if A_p denotes the smallest possible value for which (3_p) is true and if $1 \leq p \leq p_0$, the number A_p does not exceed $\max(A_1, A_{p_0})$.

THEOREM 3. *Under the conditions of Theorem 2 we also have*

$$(4_p) \quad \int_D \left| \frac{v(z)}{z} \right|^p |dz| \leq A_p^p \int_C |u(z)|^p |dz|,$$

where A_p is a constant analogous to, but not necessarily the same as, the constant A_p of Theorem 2.

In fact, if $a_0 = f(0)$, we find, arguing as above, that

$$\begin{aligned} \int_D \left| \frac{v(z)}{z} \right|^p |dz| &\leq \int_D \left| \frac{f(z) - a_0}{z} \right|^p |dz| \\ &\leq 2^{p-1}(M_p + 1) \int_C |u(z) - a_0|^p |dz| \\ &\leq A_p^{*p} \int_C |u(z)|^p |dz|, \end{aligned}$$

since

$$|a_0|^p = |f(0)|^p \leq \left(\frac{1}{2\pi} \int_0^{2\pi} |u(e^{i\theta})| d\theta \right)^p \leq \frac{1}{2\pi} \int_0^{2\pi} |u(e^{i\theta})|^p d\theta.$$

The rest of the proof is the same.

3. Additional remarks. (i) The function $u(z) = P_r(\theta)$ shows that Theorem 2 is false if in the left-hand member of the inequality (3) we replace v by u , but, of course, the new inequality is true if $1 + \epsilon \leq p \leq p_0$, for every $\epsilon > 0$.

(ii) Let $u(z)$ be real and harmonic for $|z| < 1$. Applying the Lemma to the

conjugate functions $\partial u/\partial \theta$ and $-r\partial u/\partial r$ we obtain the following result: If a function $u(e^{i\theta})$ is of bounded variation, the corresponding harmonic function defined by Poisson's integral is of (uniformly) bounded variation on any radius (cf. Prasad [19] where a more general result is proved).

(iii) Theorem 2 is probably false for any $0 < p < 1$. It is certainly false e.g. for $p = \frac{1}{2}$, as the example of conjugate functions $dP_r(\theta)/d\theta$ and $dQ_r(\theta)/d\theta$ shows (see footnote on page 600).

IV. ON A THEOREM ON CONJUGATE FUNCTIONS

1. The following is one of the several definitions of an integral given by Denjoy [2].

A function $f(x)$ defined for $a \leq x \leq b$ and continued outside (a, b) by the condition of periodicity is said to be integrable B on (a, b) if, for an arbitrary subdivision $a = a_0 < a_1 < a_2 < \dots < a_n = b$ and arbitrary set of values ξ_i , $a_{i-1} \leq \xi_i \leq a_i$, the expression

$$(1) \quad J(f; t) = \sum_{i=1}^n f(\xi_i + t)(a_i - a_{i-1})$$

tends in measure to a limit J , when $\max(a_i - a_{i-1}) \rightarrow 0$.[†] J is then called the value of the integral of f over (a, b) .

It is not difficult to grasp the meaning of the above definition. Instead of one (periodic) function $f(x)$, we consider the whole family $f_i(x)$ derived from $f(x)$ by translating the argument x by t and construct for each of them the Riemannian approximating sums. Even if the function $f(x)$ (and, consequently, any $f_i(x)$) is not integrable R , it may happen that "on the whole" the sums $J(f; t)$ are near to a number J , and the nearer, the smaller $\max(a_i - a_{i-1})$ is. Thus, the integral B is what may be called "Riemann's integral in measure."

This definition has found a rather unexpected application in the theory of trigonometric series by the following theorem of Kolmogoroff [14]:

THEOREM A. *If $f(x)$ is integrable L and*

$$(2) \quad f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

the (generalized) sum $\tilde{f}(x)$ of the conjugate series

$$(3) \quad \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx)$$

is integrable B , and, moreover, (3) is the Fourier (-Denjoy) series of $\tilde{f}(x)$.

[†] In other words, for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon)$, such that, if only $\max(a_i - a_{i-1}) < \delta$, the measure of the set of values of t for which $|J - J(f; t)| > \epsilon$ is less than ϵ .

Kolmogoroff's proof is based on an inequality concerning the measure of the set of points for which $|\tilde{f}(x)| \geq R$. As the proofs of this inequality, so far published [13, 25], are not simple, an alternative proof of Theorem A would be, perhaps, of some interest. The proof given below uses a theorem (Theorem C) also due to Kolmogoroff, which may be considered now as fairly simple (cf. Hardy [5]).

2. We begin by proving the following theorem of Denjoy [2, 1] (which is not necessary for the proof of Theorem A).

THEOREM B. *If $f(x)$ is integrable L in (a, b) it is also integrable B and both definitions give the same value of the integral.[†]*

Let

$$J = (L) \int_a^b f(x) dx, \quad J^* = (L) \int_a^b |f(x)| dx.$$

Integrating (1) we get

$$(4) \quad \int_a^b |J(f; t)| dt \leq \sum_{i=1}^n (a_i - a_{i-1}) \int_a^b |f(\xi_i + t)| dt = (b-a)J^*.$$

Suppose that $J^* < \epsilon^2/(3(b-a))$. Then the left-hand member in (4) does not exceed $\epsilon^2/3$ and the measure of the set of values of t for which $|J(f; t)| > \epsilon/3$ does not exceed ϵ . In the general case we put $f = f_1 + f_2$ and introduce the integrals J_1, J_1^*, J_2, J_2^* , analogous to J, J^* . We may suppose that f_1 is continuous and that $J_2^* \leq \epsilon^2/(3(b-a))$. Then $|J(f_2; t)| \leq \epsilon/3$ except in a set of measure $\leq \epsilon$. On the other hand, if $\max (a_i - a_{i-1})$ is sufficiently small, we have for every t the inequality $|J(f_1; t) - J_1| < \epsilon/3$ and so (assuming as we may, that $\epsilon < b-a$),

$$\begin{aligned} |J(f; t) - J| &\leq |J(f_1; t) - J_1| + |J(f_2; t)| + |J_2| \\ &\leq \epsilon/3 + \epsilon^2/(3(b-a)) + \epsilon^2/(3(b-a)) < \epsilon \end{aligned}$$

except in a set of measure $\leq \epsilon$.

3. The theorem which we will use in the proof of Theorem A and which we take for granted is as follows.

THEOREM C. *If $f(x)$ is integrable L over $(0, 2\pi)$, and $\tilde{f}(x)$ is conjugate to f , then*

$$(5) \quad \left(\int_0^{2\pi} |\tilde{f}(x)|^{1-\epsilon} dx \right)^{1/(1-\epsilon)} \leq A_\epsilon \int_0^{2\pi} |f(x)| dx,$$

where A_ϵ is a constant depending only on $\epsilon > 0$.

[†] The proof given in the text is due to Dr. S. Saks.

Now it is obvious that if we replace in (1) f by \tilde{f} , we obtain the function $J(\tilde{f}; t)$ conjugate to $J(f; t)$. Hence, from (5), with $\epsilon = \frac{1}{2}$ we get

$$(6) \quad \int_0^{2\pi} |J(\tilde{f}; t)|^{1/2} dt \leq A^{1/2} \left(\int_0^{2\pi} |f(t)| dt \right)^{1/2}.$$

Suppose first that the right-hand member of (6) does not exceed $\epsilon^{3/2}$. Then the set of values of t for which $J(\tilde{f}; t)$ exceeds ϵ is less than ϵ . In the general case we put again $f = f_1 + f_2$, where f_1 has a continuous derivative (so that \tilde{f}_1 is continuous) and the integral of $|f_2|$ is small. In the equality $J(\tilde{f}; t) = J(\tilde{f}_1; t) + J(\tilde{f}_2; t)$ the term $J(\tilde{f}_1; t)$ is small for every t , provided that $\max(a_i - a_{i-1})$ is small (the Fourier series of \tilde{f}_1 has no constant term) and $J(\tilde{f}_2; t)$ is small, except in a set of small measure. This shows that \tilde{f} is integrable B and the value of the integral is 0, as was to be expected.

4. To prove the second part of Theorem A, we have to show that the products $f(x) \cos kx$ and $f(x) \sin kx$ are integrable B and that the corresponding integrals are $-\pi b_k, \pi a_k, k = 1, 2, \dots$. We may suppose that $a_0 = a_1 = b_1 = \dots = a_k = b_k = 0$. Then it is not difficult to verify that the conjugate functions of $f(x) \cos kx, f(x) \sin kx$, are $\tilde{f}(x) \cos kx, \tilde{f}(x) \sin kx$ respectively. Hence the products $\tilde{f}(x) \cos kx, \tilde{f}(x) \sin kx$ are integrable B and their integrals over $(0, 2\pi)$ vanish.

V. ON AN EXTREME CASE IN THE THEORY OF FRACTIONAL INTEGRALS

1. Hardy and Littlewood have proved [7] that if $f(x)$ belongs to $L^p (p > 1)$ in an interval (a, b) , where $-\infty < a < b \leq \infty$, then the function $f_\alpha(x)$, the fractional integral of order α of $f(x)$, belongs to L^q , provided that

$$(1) \quad 1/p - 1/q = \alpha, \quad 0 < \alpha < 1/p, \quad p > 1.$$

As may be shown by very simple examples [7], this theorem is no longer true when $p = 1$. The main purpose of this note is to find a substitute theorem for this case and to give some indications concerning the case $\alpha = 1/p$. Since these theorems have some applications in the theory of Fourier series, Weyl's definition of fractional integral [27] will be more convenient for us and we shall use it throughout, instead of the familiar Riemann-Liouville definition. According to Weyl's definition

$$f_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-t)^{\alpha-1} f(t) dt, \quad 0 < \alpha < 1,$$

where the integrable function f has the period 2π and the constant coefficient of its Fourier series vanishes. The latter condition will be tacitly assumed throughout this paper, wherever it is necessary.

The arguments will be based on the theorem just mentioned, which it will be necessary for our purposes to state in its complete form.

THEOREM 1 (Hardy-Littlewood). *If $f(x) \in L^p$, $p > 1$, in the interval $(0, 2\pi)$ and if the relations (1) are satisfied, then*

$$(2) \quad \mathfrak{M}_\alpha(f_\alpha) < M \mathfrak{M}_p(f)$$

with

$$(3) \quad M = \mu q^{\alpha'/p'},$$

where μ is an absolute constant.†

2. We begin by proving the following

THEOREM 1. *Suppose that $f \in L^r$, $r > 1$, and that $\mathfrak{M}_r(f) \leq 1$. Then there exist two constants $\lambda > 0$ and Λ independent of f , such that*

$$(4) \quad \int_0^{2\pi} \exp \lambda |f_{1/r}(x)|^{r'} dx < \Lambda.$$

This result shows that the function $f_{1/r}(x)$, which by the theorem of Hardy and Littlewood is integrable in any power, is integrable exponentially.

Put in (2)

$$\alpha = 1/r, \quad p = \frac{rk}{r+k-1} < r,$$

$$p' = \frac{rk}{(r-1)(k-1)}, \quad q = r'k > p, \quad k = 2, 3, \dots$$

Then, since $f \in L^p$, and since $\mathfrak{M}_p(f)$ is an increasing function of p we deduce from (2), (3) that

$$(5) \quad \begin{aligned} \mathfrak{M}_{r'k}(f_{1/r}) &\leq D_k \mathfrak{M}_p(f) \\ &\leq D_k \mathfrak{M}_r(f) \leq D_k, \end{aligned}$$

where

$$D_k = \mu q^{\alpha'/p'} = \mu (r'k)^{(k-1)/(r'k-1)} < \mu (r'k)^{1/r'}.$$

Raising the inequality (5) to the power $r'k$, multiplying it by $\lambda^k/k!$, and summing from $k=2$ on, we get, by Stirling's formula,

† We use the familiar notation

$$\mathfrak{M}_r(\phi) = \left(\frac{1}{2\pi} \int_0^{2\pi} |\phi|^r dx \right)^{1/r}, \quad r' = \frac{r}{r-1}.$$

The numerical value of μ in (3) is irrelevant for our purposes. When the Riemann-Liouville definition is used (in the interval $(0, \infty)$) we may put e.g. $\mu = \max 1/\Gamma(1+\alpha)$. For the definition adopted in this paper the value of μ ten times as large will certainly be sufficient.

$$(6) \quad \sum_{k=2}^{\infty} \frac{\lambda^k}{k!} \int_0^{2\pi} |f_{1/r}|^{r'k} dx \leq 2\pi \sum_{k=2}^{\infty} \frac{(\mu^{r'} r' \lambda)^k k^k}{k!} \leq C \sum_{k=2}^{\infty} (\mu^{r'} r' \lambda e)^k$$

$$< \frac{C}{1 - \mu^{r'} r' \lambda e} = B,$$

where C is an absolute constant and λ is assumed to be so small that $\mu^{r'} r' \lambda e < 1$. Let $\psi(x) = e^x - 1 - x$. Noticing that for $x \geq 0$, $e^x \leq 2\psi(x) + C$ (see footnote on page 587), and taking into account only the extreme terms of the inequality (6), we see that, with a modified value of B , we may replace in the first of them the lower limit of summation by 0, and this is just the inequality (4).

3. If we put (which we have no right to do) in the second relation (1) $p=1$, we should obtain $q=1/(1-\alpha)$. But the theorem is false for $p=1$; to state the correct form of this extreme case we introduce the class $L^{1,k}$ of functions f such that $|f|(\log^+ |f|)^k$ is integrable. We have then

THEOREM 2. *If $f \in L^{1,1-\alpha}$, $0 < \alpha < 1$, then $f_\alpha \in L^\beta$, $\beta = 1/(1-\alpha)$, and*

$$(7) \quad \mathfrak{M}_\beta(f_\alpha) \leq M \int_0^{2\pi} |f| (\log^+ |f|)^{1-\alpha} dx + N,$$

where the constants M and N do not depend on f .†

Given any integrable function $\phi(x)$, $0 \leq x \leq 2\pi$, we shall denote by $\sigma_n[\phi]$ the first arithmetical means of the Fourier series of ϕ . It is well known that the two inequalities

$$\mathfrak{M}_r(\phi) \leq A, \quad \mathfrak{M}(\sigma_n[\phi]) \leq A,$$

where A is a constant and $r \geq 1$, are equivalent. Therefore, if we wish to prove that $f_\alpha \in L^\beta$ it is sufficient to show the existence of a number A such that

$$(8) \quad \left| \int_0^{2\pi} \sigma_n[f_\alpha] g(-x) dx \right| \leq A$$

for every (periodic) g with $\mathfrak{M}_{\beta'}(g) \leq 1$. It is well known that

$$(9) \quad f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad (c_0 = 0) \quad \text{implies}$$

$$f_\alpha(x) \sim \sum_{n=-\infty}^{\infty} c_n |n|^{-\alpha} \exp\left(-\frac{\pi i \alpha}{2} \operatorname{sg} n\right) e^{inx}.$$

From this it is easily seen that the left-hand member of (8) is equal to

† It may be added that Theorems 1 and 2 are valid in the case of the Riemann-Liouville definition, at least if we suppose that the interval of integration is finite. The use of arithmetical means in the proof below of Theorem 2 is not essential and could easily be avoided.

$$\left| \int_0^{2\pi} f(-x) \sigma_n[g_\alpha] dx \right|.$$

Using W. H. Young's inequality† we see that this expression does not exceed

$$(10) \quad \int_0^{2\pi} \Phi(|f|) dx + \int_0^{2\pi} \Psi(|\sigma_n[g]|) dx,$$

where Φ and Ψ are conjugate. We take

$$(11) \quad \Psi(x) = \exp(\lambda x^\beta) - 1,$$

λ being the same constant as occurs in Theorem 1. Since Ψ is convex, we have by the inequality of Jensen and the inequality (4)

$$(12) \quad \begin{aligned} \Psi(|\sigma_n[g_\alpha(x)]|) &\leq \Psi\left(\frac{1}{\pi} \int_0^{2\pi} |g_\alpha(x+t)| K_n(t) dt\right) \\ &\leq \frac{1}{\pi} \int_0^{2\pi} \Psi(|g_\alpha(x+t)|) K_n(t) dt, \\ \int_0^{2\pi} \Psi(|\sigma_n[g_\alpha]|) dx &\leq \int_0^{2\pi} \Psi(|g_\alpha|) dx < \int_0^{2\pi} \exp \lambda |g_\alpha|^\beta dx < \Lambda, \end{aligned}$$

where $K_n(t)$ denotes the Fejér kernel. It follows from (11) that, for y large, the conjugate function $\Phi(y)$ is asymptotically equal to $\lambda^{-1/\beta} y (\log y)^{1/\beta}$, and so the first term in (10) has a finite value M . Consequently (8) is true with $A = \Lambda + M$ and Theorem 2 is established.

4. We now prove

THEOREM 3. *If $0 < \alpha \leq 1$ and $f \in L^{1,\alpha}$, the (complex) Fourier coefficients c_n of f satisfy the inequality*

$$(13) \quad \left(\sum_{n=1}^{\infty} n^{-1} |c_n|^{1/\alpha} \right)^\alpha < A_\alpha \int_0^{2\pi} |f| (\log^+ |f|)^\alpha dx + B_\alpha,$$

with A_α and B_α depending only on α . For $\alpha > 1$ the theorem is false.

† Let $\phi(x)$, $x \geq 0$, be a continuous increasing function, with $\phi(0) = 0$, and let $\psi(y)$ be the function inverse to $\phi(x)$. If

$$\Phi(x) = \int_0^x \phi(u) du, \quad \Psi(y) = \int_0^y \psi(v) dv,$$

then, for every $a \geq 0$, $b \geq 0$, we have

$$(*) \quad ab \leq \Phi(a) + \Psi(b).$$

The sign \leq in (*) degenerates into $=$ if, and only if, $b = \phi(a)$. The functions Φ and Ψ are called conjugate, of course, in the sense different from that used in the theory of Fourier series. For a very simple proof of Young's inequality (*) see Oppenheim [16].

We assume for simplicity that f is real and so $c_{-n} = \bar{c}_n$. Similarly, although we suppose in the proof that $c_0 = 0$, the inequality (13) remains valid without this assumption. From (9) and Theorem 2 it follows that

$$(14) \quad f_{1-\alpha}(x) \sim \sum_{n=-\infty}^{\infty} c_n |n|^{\alpha-1} \epsilon_n e^{inx} \in L^{1/\alpha}, \quad |\epsilon_n| = 1.$$

Now we use the following theorem of Hardy and Littlewood [6]:

If $\phi \sim \sum \gamma_n e^{inx} \in L^p$, $1 < p \leq 2$, then

$$(15) \quad \sum_{n=1}^{\infty} |\gamma_n|^p n^{p-2} < A_p \int_0^{2\pi} |\phi|^p dx,$$

with A_p independent of ϕ .

Applying this theorem, with $p = 1/\alpha$, $\phi = f_{1-\alpha}$, to the series (14) and taking into account (7), we obtain (13) for $\frac{1}{2} \leq \alpha < 1$. We shall not consider here the case $\alpha = 1^+$, and, for $0 < \alpha < \frac{1}{2}$, since the inequality (13) can be strengthened[†], we shall be contented with proving the convergence of the series (13).

LEMMA. Let $f(x)$, $g(x)$ be non-negative in $(0, 2\pi)$ and let $\phi(u)$, $\psi(u)$, $u \geq 0$, be two non-negative and non-decreasing convex functions. Put

$$(16) \quad \chi(u) = \phi(\psi(u)), \quad h(x) = \int_0^{2\pi} f(t)g(x+t)dt.$$

If

$$(17) \quad \int_0^{2\pi} f(x)dx = 1, \quad \int_0^{2\pi} \psi(g(x))dx \leq 1,$$

then

$$\int_0^{2\pi} \chi(h(x))dx \leq \int_0^{2\pi} \phi(f(x))dx.$$

Let $1/\kappa$, $\kappa \geq 1$, be the value of the second integral in (17). Using twice Jensen's inequality, we have

$$\begin{aligned} \chi(h(x)) &= \phi\left[\psi\left(\int_0^{2\pi} f(t)g(x+t)dt\right)\right] \leq \phi\left[\int_0^{2\pi} f(t)\psi(g(x+t))dt\right] \\ &\leq \phi\left[\int_0^{2\pi} \kappa f(t)\psi(g(x+t))dt\right] \leq \int_0^{2\pi} \phi(f(t))\kappa\psi(g(x+t))dt, \\ \int_0^{2\pi} \chi(h(x))dx &\leq \int_0^{2\pi} \phi(f(t))dt \int_0^{2\pi} \kappa\psi(g(t))dt = \int_0^{2\pi} \phi(f(t))dt. \end{aligned}$$

[†] See the next Note VI.

COROLLARY. If $f \in L^{1,\alpha}$, $g \in L^{1,\beta}$, $\alpha \geq 0$, $\beta \geq 0$, then $h \in L^{1,\alpha+\beta}$.

Let $\phi(u) = u(\log^+ u)^\alpha$, $\psi(u) = u(\log^+ u)^\beta$. We may plainly assume that conditions (17) are satisfied. Then it is sufficient to notice that, for $u \geq u_0$, we have

$$\chi(u) = u(\log u)^\beta \{ \log (u \log^\beta u) \}^\alpha \geq u(\log u)^{\alpha+\beta}.$$

Suppose now that $\frac{1}{2} \leq \alpha < \frac{1}{2}$ and set $g=f$ in the integral (16). It is well known that then

$$h(x) \sim 2\pi \sum_{n=1}^{\infty} |c_n|^2 e^{inx}.$$

Since $h \in L^{1,2\alpha}$ and $\frac{1}{2} \leq 2\alpha < 1$, we obtain, by applying Theorem 3 in the case already established, the convergence of the series

$$(18) \quad \sum_{n=1}^{\infty} n^{-1} |c_n|^{2 \cdot 1/(2\alpha)} = \sum_{n=1}^{\infty} n^{-1} |c_n|^{1/\alpha}.$$

We proceed similarly when $\frac{1}{3} \leq \alpha < \frac{1}{2}$, and so on.

In order to show that the condition $0 < \alpha \leq 1$ cannot be removed, consider the function

$$f(x) = \sum_{n=2}^{\infty} (\log n)^{-\alpha} (\log_2 n)^{-\beta} \cos nx.$$

It may be shown that in the neighborhood of $x=0^+$,

$$f(x) = O[x^{-1}(\log 1/x)^{-(\alpha+1)}(\log_2 1/x)^{-\beta}],$$

and, consequently, $f \in L^{1,\alpha}$ if only $\beta > 1$. If, moreover, $1 < \beta < \alpha$, the series (18) diverges. To get the needed estimate observe that, by Abel's transformation,

$$f(x) \equiv \sum_{n=2}^{\infty} a_n \cos nx = \sum_{n=2}^{\infty} \Delta^2 a_n \frac{\sin^2(n+1) \frac{x}{2}}{4 \sin^2 \frac{x}{2}}.$$

Now we break up the last sum into two, the first being extended over the range $2 \leq n \leq 1/x$. In the first sum the coefficient of $\Delta^2 a_n$ is $O(n^2)$, in the second it is $O(x^{-2})$. It simplifies slightly the proof if we use the fact that $\Delta^2 a_n \geq 0$ for $n \geq n_0$.

VI. SOME THEOREMS ON FOURIER COEFFICIENTS

1. Given a sequence of (complex) numbers $c_1, c_2, \dots, c_n, \dots$, we shall denote by $c_1^*, c_2^*, \dots, c_n^*, \dots$ the sequence $|c_1|, |c_2|, \dots, |c_n|, \dots$ rearranged in descending order of magnitude.

Hardy and Littlewood [9] have established the following theorems.†

THEOREM A. Suppose that

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad c_{-n} = \bar{c}_n,$$

and that $|f| \log^+ |f| \in L$. Then $\sum_1^\infty n^{-1} c_n^*$ is convergent and $\sum \exp(-k/|c_n|)$ is convergent for every $k > 0$.

THEOREM B. Suppose that $\sum_1^\infty |c_n| (\log(1/|c_n|))^{-1}$ is convergent. Then $\sum c_n e^{inx}$ is the Fourier series of a function f , such that $\exp(k|f|)$ is integrable for every $k > 0$.

Our object here is to generalize these theorems in two directions. First, we consider slightly more general types of integrability and, secondly, the results are extended to general, uniformly bounded, orthogonal systems.

Let ϕ_2, ϕ_3, \dots ‡ be a system of functions, orthogonal and normal in a finite interval (a, b) and uniformly bounded,

$$(1) \quad |\phi_n(x)| \leq M.$$

These conditions will be assumed in the following discussion.

THEOREM 1. If $|f| (\log^+ f)^\alpha \in L$ in (a, b) , $\alpha > 0$, then

- (i) the series $\sum \exp(-k/|c_n|^{1/\alpha})$ converges for every $k > 0$.
 (ii) If, moreover, $\alpha \leq 1$, we have $\sum n^{-1} c_n^{*1/\alpha} < \infty$.

THEOREM 2. If the series $\sum |c_n| (\log(1/|c_n|))^{-\alpha}$, $\alpha > 0$, converges, the series

$$(2) \quad \sum_{n=2}^{\infty} c_n \phi_n(x)$$

is the Fourier series of a function f such that $\exp(k|f|^{1/\alpha})$ is integrable for every $k > 0$.

† The results are stated without proofs. A result less strong than Theorem A, viz., the convergence of the series $\sum n^{-1} |c_n|$, is proved in Zygmund [29, Theorem 3]. Since the argument used there can be applied, with slight modifications, to general uniformly bounded orthogonal systems, it yields also the result of Hardy and Littlewood. The latter result is, in turn, contained in the following theorem: If $f(x) \sim c_0 + c_1 e^{ix} + \dots + c_n e^{inx} + \dots$, then the series $\sum n^{-1} c_n^*$ converges.

‡ It is slightly more convenient to denote the system by ϕ_2, ϕ_3, \dots , and not by ϕ_1, ϕ_2, \dots . Correspondingly, c_2^*, c_3^*, \dots denotes the sequence $|c_2|, |c_3|, \dots$ rearranged in descending order.

2. The proof will be based on a series of lemmas.

LEMMA 1. Let $y = \phi(x)$, $x \geq 0$, be a non-negative, continuous, strictly increasing function with $\phi(0) = 0$. Let $x = \psi(y)$ be the function inverse to $\phi(x)$. Then, for every $a, b \geq 0$, the inequality

$$(3) \quad ab \leq \Phi(a) + \Psi(b), \quad \Phi(x) = \int_0^x \phi(u) du, \quad \Psi(y) = \int_0^y \psi(v) dv$$

holds. The sign of equality occurs in (3) if, and only if, $b = \phi(a)$. (Cf. footnote † on page 607.)

LEMMA 2. Let

$$f(x) \sim \sum_{n=2}^{\infty} c_n \phi_n(x),$$

where

$$|c_n| \leq n^{-1}(\log n)^{\alpha-1}, \quad \alpha > 0, \quad n = 2, 3, \dots$$

Then, for $\lambda > 0$ sufficiently small, we have

$$(4) \quad \int_a^b \exp(\lambda |f|^{1/\alpha}) dx \leq A. \dagger$$

The function $(\log x)^{\alpha-1}$ decreases for $x > e^{\alpha-1}$. Let n_0 be an integer $> e^{\alpha-1}$. Without loss of generality we may assume that $c_n = 0$ for $n \leq n_0$. If $\mu \geq 2$, we have, by the F. Riesz theorem (cf. M. Riesz [21]),

$$\begin{aligned} \int_a^b |f|^\mu dx &\leq M^{\mu-2} \left[\sum_{n=n_0+1}^{\infty} n^{-\mu'} (\log n)^{(\alpha-1)\mu'} \right]^{\mu-1} \\ &\leq M^{\mu-2} \left(\int_2^{\infty} x^{-\mu'} (\log x)^{(\alpha-1)\mu'} dx \right)^{\mu-1}. \end{aligned}$$

The last factor does not exceed $A^{\mu-1}$ where $A = A_\alpha$ is a constant independent of μ , if only $\mu \geq \mu_0 \geq 2$.

Put $\mu = \beta k$, where $\beta = 1/\alpha$. Let k_0 denote an integer, such that $\beta k \geq \mu_0$ for $k \geq k_0$. Then

$$\frac{\lambda^k}{k!} \int_a^b |f|^{\beta k} dx \leq M^{-2} A^{-1} \lambda^k (AM)^{\beta k} (\beta k)^k / k!, \quad k \geq k_0,$$

and, by Stirling's formula, we obtain

† We designate by A any constant (not necessarily the same in all the formulas) which does not depend on f .

$$(5) \quad \int_a^b \sum_{k=k_0}^{\infty} (k!)^{-1} (\lambda |f|^{\beta})^k < A,$$

if only $\lambda e A^{\beta} M^{\beta} \beta < 1$. Since

$$\sum_{k=0}^{k_0-1} (k!)^{-1} (\lambda u)^{\beta k} < \sum_{k=k_0}^{\infty} (k!)^{-1} (\lambda u)^{\beta k}$$

for $u \geq u_0$, the inequality (4) follows from (5).

LEMMA 3. If $|f|(\log^+ |f|)^{\alpha} \in L$, $\alpha > 0$, and if c_n , $n=2, 3, \dots$, are the Fourier coefficients of f with respect to $\{\phi_n\}$, then

$$(6) \quad \sum_{n=2}^{\infty} n^{-1} (\log n)^{\alpha-1} c_n^* \leq A \int_a^b |f| (\log^+ |f|)^{\alpha} dx + A.$$

Since the order of the functions ϕ_n is irrelevant, we may suppose that $c_n^* = |c_n|$. Put $\epsilon_n = \overline{\text{sg } c_n}$ and consider the partial sums s_N of the series

$$(7) \quad \sum_{n=2}^{\infty} \epsilon_n n^{-1} (\log n)^{\alpha-1} \phi_n(x).$$

Using Young's inequality we obtain

$$(8) \quad \sum_{n=2}^N n^{-1} (\log n)^{\alpha-1} c_n^* = \int_a^b f(x) s_N(x) dx \leq \int_a^b \Phi(|f|) dx + \int_a^b \Psi(|s_N|) dx.$$

Put

$$(9) \quad \begin{aligned} \Psi(x) &= x \exp(\lambda_0 x^{\beta}) - x, & \beta &= 1/\alpha, \text{ and hence} \\ \Phi(x) &\sim (\lambda_0)^{-\alpha} x (\log x)^{\alpha} \text{ as } x \rightarrow \infty, \end{aligned}$$

where λ_0 is any positive constant less than the constant λ occurring in (4). Since $\Psi(x) \leq \exp(\lambda x^{\beta})$, $x \geq x_0$, we get, from (8) and (4),

$$(10) \quad \sum_{n=2}^{\infty} n^{-1} (\log n)^{\alpha-1} c_n^* \leq \int_a^b \Phi(|f|) dx + A.$$

Since $\Phi(x) < 2\lambda_0^{-\alpha} x (\log x)^{\alpha}$, $x \geq x_0$, (6) follows from (10).

3. Now it is not difficult to prove Theorem 1. Let B_0 denote the right-hand side of (6). Then

$$(11) \quad c_n^* \sum_{\nu=2}^n \nu^{-1} (\log \nu)^{\alpha-1} \leq \sum_{\nu=2}^n c_{\nu}^* \nu^{-1} (\log \nu)^{\alpha-1} < B_0.$$

Since the coefficient of c_n^* in the first term is $\geq \rho (\log n)^{\alpha}$, ρ being a constant

independent of n , the following three inequalities are consequences of (11):

$$(12) \quad c_n^* \leq B_0 \rho^{-1} (\log n)^{-\alpha}, \quad \log n \leq (B_0 / (\rho c_n^*))^\beta, \quad n \leq \exp (B_0 / (\rho c_n^*))^\beta.$$

From the second of them and from (6) we get

$$\left(\sum_{n=2}^{\infty} n^{-1} c_n^{*1/\alpha} \right)^\alpha \leq \rho^{\alpha-1} B_0 = \rho^{\alpha-1} \left\{ A \int_a^b |f| (\log^+ |f|)^{\alpha} dx + A \right\},$$

which is the second part of Theorem 1.† To prove the first part, we notice that the function $x^{-1}(\log x)^{\alpha-1}$ decreases for $x \geq n_0 \geq e^{\alpha-1}$, and so, from the third inequality (12) and (6), we obtain

$$n^{-1}(\log n)^{\alpha-1} \geq \{B_0 / (\rho c_n^*)\}^{\beta(\alpha-1)} \exp(-B_1 / c_n^{*\beta}), \quad B_1 = (B_0 / \rho)^\beta, \quad n \geq n_0,$$

$$\sum_{n=n_0}^{\infty} (c_n^*)^{2-1/\alpha} \exp(-B_1 (c_n^*)^{-1/\alpha}) \leq \rho^{(\alpha-1)/\alpha} B_0^{1/\alpha}.$$

This gives statement (i) of Theorem 1, for some $k > 0$. To prove it for every $k > 0$ it suffices to notice (rejecting a large number of terms from the convergent series (10)) that $c_n = o((\log n)^{-\alpha})$, and to repeat the previous argument.

4. We now pass to Theorem 2.

LEMMA 4. Let $c_n > 0$, $b_n \geq b_{n+1} > 0$, $\alpha > 0$, and

$$(13) \quad \sum_{n=3}^{\infty} c_n \left(\log \frac{1}{c_n} \right)^{-\alpha} \leq C_\alpha < \infty, \quad \sum_{n=3}^{\infty} n^{-1} (\log n)^{\alpha-1} b_n \leq B_\alpha < \infty.$$

There exists a number $\sigma > 0$ depending only on α , and such that

$$(14) \quad S = \sum_{n=3}^{\infty} c_n b_n \leq (\sigma C_\alpha + \frac{1}{2}) B_\alpha.$$

From the second inequality it follows that $b_n \leq B_\alpha \rho^{-1} (\log n)^{-\alpha}$. Break up the sum S into two, $S = S_1 + S_2$, where S_1 contains the indices n for which $b_n \leq \sigma B_\alpha (\log (1/c_n))^{-\alpha}$, σ being defined by the equality $(\rho\sigma)^\beta = 3$. It is obvious that $S_1 \leq \sigma C_\alpha B_\alpha$. If n occurs in S_2 , we have

$$B_\alpha \rho^{-1} (\log n)^{-\alpha} \geq b_n \geq B_\alpha \sigma (\log 1/c_n)^{-\alpha},$$

and hence $c_n \leq n^{-3}$. Therefore

$$S_2 \leq \sum_{n=3}^{\infty} n^{-3} b_n \leq \sum_{n=3}^{\infty} n^{-2} (\log n)^{-1} b_n \leq \frac{1}{2} \sum_{n=3}^{\infty} n^{-1} (\log n)^{-1} b_n$$

$$\leq \frac{1}{2} \sum_{n=3}^{\infty} n^{-1} (\log n)^{\alpha-1} b_n = \frac{B_\alpha}{2},$$

† The condition $\alpha \leq 1$ is essential. See V, Theorem 3.

and the lemma follows.

Suppose now that in the series (2) we have not only $c_3=0$, but that also a number of subsequent coefficients vanish, $c_4=\dots=c_{n_0}=0$, n_0 being so large that

$$C_0 = \sum_{n=n_0+1}^{\infty} |c_n| (\log 1/|c_n|)^{-\alpha} \leq 1/(2\sigma).$$

It follows that the coefficient of B_α in (14) does not exceed 1 if C_α is replaced there by C_0 . Let s_N , $N > n_0$, denote the partial sums of the series (2), and let g be any function with Fourier coefficients b_n and $\Phi(|g|)$ integrable. Then

$$\left| \int_a^b s_N g dx \right| = \left| \sum_{n=n_0+1}^N c_n b_n \right| \leq \sum_{n=n_0+1}^{\infty} |c_n| |b_n|.$$

On rearranging the terms in the last sum according to the decreasing magnitude of $|b_n|$ and applying Lemma 4, and the inequality (10) (with a slightly different notation where c_n^* , f have been replaced by b_n^* , g), we get

$$(15) \quad \left| \int_a^b s_N g dx \right| \leq \sum_{n=2}^{\infty} n^{-1} (\log n)^{\alpha-1} b_n^* \leq \int_a^b \Phi(|g|) dx + D.$$

On the other hand, if g is chosen conveniently (see Lemma 1) the left-hand member in (15) is equal to

$$\int_a^b \Psi(|s_N|) dx + \int_a^b \Phi(|g|) dx,$$

the last integral being finite. Comparing this with the right-hand side of (15), we get

$$(16) \quad \int_a^b |\Psi(|s_N|)| dx \leq D.$$

By the theorem of Riesz-Fischer, the series (2) is the Fourier series of a function $f \in L^2$ and a subsequence of $\{s_N\}$ converges almost everywhere to f . By Fatou's well known lemma, the inequality (16) implies

$$\int_a^b \Psi(|f|) dx \leq D.$$

It follows that $\exp(k|f|^\theta)$ is integrable for some $k > 0$. Rejecting the restriction concerning the first coefficients of the series, we may assert the integrability of $\exp(k|f - s_{n_0}|^\theta)$, where n_0 is sufficiently large. Since s_{n_0} is bounded, $\exp(k|f|^\theta)$ is again integrable for some $k > 0$.

To prove that it is integrable for every $k > 0$, it suffices to observe that for any $\lambda > 0$ and $c'_n = \lambda c_n$ the series $\sum |c'_n| (\log (1/|c'_n|))^{-\alpha}$ converges and so $\exp (k\lambda |f|^\beta)$ is integrable for every $\lambda > 0$.

5. We now prove

THEOREM 3. *If $\sum n^{r-1} |c_n|^r$, $r > 1$, converges, the series (2) is the Fourier series of a function f such that $\exp (k|f|^{r'})$ is integrable for all values of $k > 0$.*

We shall only sketch the proof, which is analogous to, and even a little simpler than, that of Theorem 2. Using Hölder's inequality we see that the series $\sum c_n b_n$ converges, even absolutely, for any $\{b_n\}$, such that $\sum n^{-1} |b_n|^{r'} < \infty$. In particular, it converges if b_n are the Fourier coefficients of a function g such that $|g| (\log^+ |g|)^{1/r'}$ is integrable (see (iii) of Theorem 1). Since, roughly speaking, $\exp x^{r'}$ and $x (\log x)^{1/r'}$ are conjugate in the sense of Young, the integrability of $\exp (k|f|^{r'})$ for some $k > 0$, and hence for every $k > 0$, follows.

Remark. In the case of trigonometric series and $r \geq 2$, Theorem 3 is a corollary of Theorem 1, Note V (using the inequality (15) of that note).

VII. ON A THEOREM OF PALEY AND WIENER

In a recent paper Paley and Wiener [17] proved the following theorem: If $f(x)$ is defined over $(-\pi, \pi)$ as an odd function and is non-decreasing and integrable over $(-\pi, \pi)$, then its conjugate function $\tilde{f}(x)$ is also integrable. Here we propose to give a simpler proof of this theorem, or rather of an equivalent

THEOREM. *If $f(x)$ is odd in $(-\pi, \pi)$, non-increasing and integrable over $(0, \pi)$, then its conjugate function $\tilde{f}(x)$ is also integrable.*

The theorem is trivial if $f(x)$ is bounded on $(0, \pi)$. On the other hand there is no loss of generality if we assume that $f(x)$ is not bounded only in the neighborhood of $x=0$ and that $f(x) \geq 0$, $0 < x \leq \pi$. Our proof is based on the following obvious

LEMMA. *If $f(x)$ is integrable over $(0, \pi)$ then the functions*

$$\phi(x) = x^{-2} \int_0^x t |f(t)| dt,$$

$$\psi(x) = \int_x^\pi t^{-1} |f(t)| dt$$

are also integrable.

Now, assuming $0 < x < \pi$, we have

$$\begin{aligned}
 -\pi \tilde{f}(x) &= \int_{-\pi}^{\pi} f(t) \cot \frac{t-x}{2} dt \\
 &= \int_{-\pi/2}^{\pi/2} \cdots + \int_{-3\pi/2}^{-\pi/2} \cdots + \int_{\pi/2}^{3\pi/2} \cdots + \left(\int_{-\pi}^{-3\pi/2} \cdots + \int_{3\pi/2}^{\pi} \cdots \right) \\
 &\equiv J_1(x) + J_2(x) + J_3(x) + J_4(x).
 \end{aligned}$$

Here

$$\begin{aligned}
 |J_1(x)| &\leq \int_0^{\pi/2} f(t) O(tx^{-2}) dt = O(\phi(x)) \in L, \\
 |J_2(x)| &\leq f\left(\frac{x}{2}\right) \int_{\pi/2}^{3\pi/2} O\left(\frac{1}{t}\right) dt = O\left(f\left(\frac{x}{2}\right)\right) \in L, \\
 |J_4(x)| &= O\left(\int_{3\pi/2}^{\pi} \frac{f(t) dt}{t}\right) = O\left(\psi\left(\frac{3x}{2}\right)\right) \in L, \\
 \int_0^{\pi/2} |J_3(x)| dx &= O\left\{ \int_0^{\pi/2} dx \int_0^{\pi/2} |f(x+t) - f(x-t)| \frac{dt}{t} \right\} \\
 &= O\left\{ \int_0^{\pi/4} \frac{dt}{t} \int_{2t}^{\pi/2} |f(x+t) - f(x-t)| dx \right\} \\
 &= O\left\{ \int_0^{\pi/4} \frac{dt}{t} \left[\int_t^{\pi/2-t} f(x) dx - \int_{3t}^{\pi/2+t} f(x) dx \right] \right\} \\
 &= O\left\{ \int_0^{\pi/4} \frac{dt}{t} \left[\int_t^{3t} f(x) dx - \int_{\pi/2-t}^{\pi/2+t} f(x) dx \right] \right\} \\
 &= O\left\{ \int_0^{\pi/4} \frac{dt}{t} [2tf(t) + O(t)] \right\} < \infty.
 \end{aligned}$$

This proves that $f(x)$ is integrable over $(0, \pi)$.

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THE RIEMANN MULTIPLE-SPACE AND ALGEBROID FUNCTIONS*

BY

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1. **Introduction.** The present paper considers the extension of the Riemann surface‡ to the case of several complex variables. The resulting configuration will be called Riemann multiple-space§ (R. M. S.), and the first object is to give its construction, or definition. It is then shown that the R. M. S. is a generalized manifold.|| The property of being a generalized manifold is shown to be a topologically invariant property of a complex, and a simple characterization of a GM_n is given. The locus of non-spherical points¶ of the R. M. S. is proved to be a sub-complex of dimension not greater than $2n-4$,** where n is the number of independent variables; an example due to Osgood†† shows that it can actually attain that dimensionality.

2. **Properties of the generalized manifold.** We prove certain properties which are needed in what follows.

LEMMA 1. *A generalized n -manifold is a simple n -circuit.‡‡*

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† Some of the results of the present paper were announced in preliminary form in the abstract bearing the same title presented by B. O. Koopman (at that time National Research Council Fellow) in the Bulletin of the American Mathematical Society, vol. 33 (1927), p. 406.

‡ For a treatment of the case of one independent variable, see H. Weyl, *Die Idee der Riemannschen Fläche*, 2d edition, Leipzig, 1923.

§ Terms often used are Riemann hypersurface or Riemann space; but it seems undesirable to use these, inasmuch as their use in the present connection involves a contradiction with other standard mathematical usage.

|| O. Veblen, *Analysis Situs*, chapter III, pp. 95-96 in second edition; Colloquium Series, vol. 5, part 2, New York, 1931. A generalized manifold of n dimensions (GM_n) is defined as the set of points on an n -circuit such that the cells of higher dimensions incident with any given i -cell have the incidence relations of a GM_{n-i-1} . The only GM_0 is a pair of 0-cells.

Terminology will be as defined in Veblen, or as in Lefschetz's *Topology*, Colloquium Series, vol. 12, New York, 1930. (Lefschetz I.)

An n -circuit is an n -complex which (1) is the closure of its n -cells; (2) has an even number of n -cells incident with each of its $(n-1)$ -cells; (3) contains no proper sub-complex satisfying (1) and (2).

¶ A point of a k -complex will be called a *spherical point* if it has a neighborhood on the complex which is homeomorphic to a k -cell.

** Note that this result does not of itself imply that the R. M. S. is a generalized manifold, nor does the latter imply the former.

†† W. F. Osgood, *Lehrbuch der Funktionentheorie*, vol. 2, first part, chapter 2, §21. (Osgood II.)

‡‡ A simple n -circuit is an n -circuit each of whose $(n-1)$ -cells is incident with exactly two of its n -cells.

This lemma is stated for convenience in reference. It follows from the facts that the n -manifold is an n -circuit and that a GM_0 is a pair of 0-cells.

LEMMA 2. A definition of GM_n , $n > 0$, equivalent to the original one, is the following. A GM_n is a connected n -complex K_n such that the cells of higher dimensions incident with any given i -cell have the incidence relations of a GM_{n-i-1} .

Using induction, we assume the lemma proved for dimensions less than n . As our proof will require no assumption for the case $n=1$, it remains only to prove the lemma for general n under the assumption of the induction.

The new definition differs from the original one only in the replacement of " n -circuit" by "connected n -complex." As any n -circuit is connected, we need merely show that under the new definition a GM_n is an n -circuit. As the cells of higher dimensions incident with any i -cell have the incidence relations of an $(n-i-1)$ -complex, it follows that every point of K_n is on the closure of at least one n -cell. As the incidence relations between the cells incident with any $(n-1)$ -cell E_{n-1} are those of a GM_0 , it follows that E_{n-1} is incident with just two n -cells. Hence K_n is an n -cycle each of whose $(n-1)$ -cells is incident with just two n -cells.

If K_n were not an n -circuit there would be two sub-complexes M_n^1 and M_n^2 , each an n -cycle, containing all the n -cells of K_n but having no common n -cells. As M_n is connected, M_n^1 and M_n^2 would have at least one common cell, say an i -cell E_i . As the lemma is assumed true for dimensions less than n , the cells E^j of higher dimensions, of K_n , incident with E_i , would have the incidence relations of an $(n-i-1)$ -circuit c_{n-i-1} . Since M_n^1 and M_n^2 are n -cycles, those of the cells of the set E^j belonging to M_n^j would have the incidence relations of an $(n-i-1)$ -cycle c_{n-i-1}^j , $j=1, 2$, which could be considered as a sub-complex of c_{n-i-1} . But that is impossible, as an $(n-i-1)$ -circuit cannot have two sub-complexes each of which is an $(n-i-1)$ -cycle, and distinct. Hence K_n must be an n -circuit, and the proof is complete.

LEMMA 3. A complex K_n is a generalized manifold if and only if it is connected, is the closure of its n -cells, and satisfies the following condition: If K_n^s is the set of spherical points of K_n , K_n is locally connected* by curves which can be taken in K_n^s whenever their end points are in K_n^s .

To prove the necessity, let P be any point of K_n , and E_i the cell of K_n on which it lies. Let N be any neighborhood of P on K_n , and $N' \subset N$ the neighborhood consisting of all the cells of K_n' on whose closures P lies, where K_n' is the complex obtained by subdividing K_n regularly enough times so

* Local connectedness in the ordinary sense is meant; in the terminology of Lefschetz I this means local 0-connectedness.

that such an N' exists. Let P_1 and P_2 be any two points in N' . Now we consider E_i and all the cells of higher dimensions that are incident with it. The latter have the incidence relations of a generalized $(n-i-1)$ -manifold. As this manifold is, according to Lemma 1, a simple circuit, we can name a sequence of $(n-i-1)$ - and $(n-i-2)$ -cells such that the corresponding sequence of n - and $(n-1)$ -cells of K_n has the following properties: (1) each cell is incident with the adjacent ones in the sequence; (2) P_1 is on the first cell or on its boundary; (3) P_2 is on the last cell or on its boundary. The rest of the proof is obvious.

To prove the sufficiency, suppose the condition satisfied and let i be an integer such that for every E_r , $r < i$, the incident cells E_r of higher dimensions have the incidence relations of a GM_{n-r-1} . We shall prove that the property holds also when $r = i$. Let E_i be any i -cell. The cells E^i of higher dimensions incident with E_i have the incidence relations of a complex k_{n-i-1} , since K_n is the closure of its n -cells. Let E_{i+j} be any $(i+j)$ -cell incident with E_i . By hypothesis, the cells E^{i+j} incident with E_{i+j} have the incidence relations of a $GM_{n-i-j-1}$. Now in considering k_{n-i-1} , E_{i+j} corresponds to a $(j-1)$ -cell e_{j-1}^{i+j} of k_{n-i-1} . As the incidence relations of the cells of E^{i+j} are the same as the incidence relations of the corresponding cells of k_{n-i-1} incident with e_{j-1}^{i+j} , it follows that the latter relations are likewise those of a $GM_{n-i-j-1}$. Thus e_{j-1}^{i+j} satisfies the condition imposed on a $(j-1)$ -cell and its incident cells of higher dimensions on k_{n-i-1} in order that k_{n-i-1} should be a GM_{n-i-1} . Since a similar statement can be made for any $(i+j)$ -cell incident with E_i , $j > 0$, it follows from Lemma 2 that each connected part of k_{n-i-1} is a GM_{n-i-1} .*

Now if k_{n-i-1} were not connected, we could let P^1 and P^2 be points on n -cells of K_n corresponding to $(n-i-1)$ -cells of two unconnected parts of k_{n-i-1} , sufficiently near to some point P of E_i to satisfy the condition of Lemma 3 for some neighborhood N of P containing no points on the boundary of E_i . Then a curve C would exist joining P^1 to P^2 on K_n and in N . Then C would contain a point Q on E_i , as no cell of the group corresponding to the first part of k_{n-i-1} could be incident with any cell of the group corresponding to the second part. Since Q would have an n -cell neighborhood, by the invariance of the combinatorial manifold† it follows that k_{n-i-1} would be a combinatorial $(n-i-1)$ -sphere. As the latter is connected, we would then have a contradiction to the hypothesis that k_{n-i-1} is not connected. Consequently it is connected, and therefore a GM_{n-i-1} .

* Cf. Veblen, loc. cit., pp. 96-97.

† E. R. van Kampen. For references, see Lefschetz I. The linked complex of E_i has the construction of a regular subdivision of k_{n-i-1} . We have not used linked complexes in the proofs, as they would have necessitated longer proofs.

It now follows by induction that for every r -cell E_r , $r=0, 1, \dots, n-1$, the incident cells of higher dimensions have the incidence relations of a GM_{n-r-1} . Since K_n is connected, it follows from Lemma 2 that K_n is a GM_n , and the proof of Lemma 3 is complete.

The following combinatorial characterization of the GM_n is an easily proved consequence of Lemma 3.

COROLLARY. *A necessary and sufficient condition that a complex K be a GM_n is that it have the following properties: firstly, it is connected; secondly, every cell is an n -cell or on the boundary of an n -cell; thirdly, given any i -cell E_i , $i < n$, and any two n -cells E_n^1 and E_n^2 incident with E_i , there exists a sequence of cells of K having the following properties: (1) E_n^1 is the first cell of the sequence and E_n^2 is the last; (2) the cells of the sequence are alternately n -cells and $(n-1)$ -cells; (3) each cell of the sequence is incident with the adjacent cells of the sequence; (4) all the cells of the sequence are incident with E_i .*

This result might be described more briefly in the following terms. *A generalized n -manifold is a connected n -complex which is locally a simple n -circuit.**

3. The Riemann multiple-space. Let a region R be given in the $(2n+2)$ -space of the complex variables w, z_1, \dots, z_n , together with a function $F(w, z_1, \dots, z_n) = F(w, z)$ with the following properties: (1) F is single-valued and analytic at all points of R ; (2) $F=0$ for some points in R ; (3) if we continue analytically from a point P at which $F=0$, over a path which may go outside R , and return to the point P , then if the continued function vanishes at P it must be identical with the original function $F(w, z)$ at P ; (4) F is irreducible, that is, it is not a product of two functions each satisfying the preceding conditions and having the same locus of points when equated to zero.†

Given a point P on the locus $F=0$, let F be factored into a product of irreducible analytic factors F^i , each vanishing at P . It will be proved below that no two of the F^i can be equivalent at P .‡

The Riemann multiple-space for the locus $F=0$ in R is defined as the following Hausdorff space. A point P on $F=0$ together with one of the irreducible functions F^i at P , (P, F^i) , constitute a point of the space. If F^i and F^j are equivalent at P , (P, F^i) and (P, F^j) are the same point of the R. M. S. A neighborhood of (P, F^i) consists of the set of points (Q, F^i) for which Q is in a neighborhood of P on $F=0$ in (w, z) -space, and $F^i = F^i \Phi$ at and near Q , where Φ is analytic at Q .

* Because of this fact, it may seem that we could have dispensed with the entire section on the GM_n . However this is not the case, as the results are used in the later proofs.

† In order to treat a function such as $w - \log z$, near a point at which $w - \log z = 0$ we use the branch which vanishes at the point.

‡ F^1 and F^2 are equivalent at P if $F^1 = F^2 \Omega$ at and near P , where Ω is analytic and not zero at P .

From this definition it is evident that the topological properties of the R. M. S. are independent of any change in coordinates. In order to carry through our later proofs we make a change of coordinates if necessary, so that for no (z^0) is $F(w, z^0)$ zero for all w neighboring any value determining a point in R . That this can be done follows easily from a theorem of the authors dealing with a somewhat similar situation in the case of reals.*

Given a point (w^0, z^0) , with $F(w^0, z^0) = 0$, we apply the Weierstrass Preparation Theorem†, giving us, near (w^0, z^0) ,

$$(3.1) \quad F(w, z) \equiv \prod_k [F_k(w, z)] \Omega(w, z).$$

Here Ω is analytic and not zero at (w^0, z^0) , the product is finite, and F_k is an irreducible algebroid polynomial, in general not singular, with vertex at (z^0) . Thus F_k has the general form

$$F_k(w, z) = w^N + \psi_1(z)w^{N-1} + \cdots + \psi_N(z),$$

where the ψ 's are analytic at (z^0) , and all the roots of F_k coincide in the value w^0 when $(z) = (z^0)$.

From the properties of algebroid polynomials‡ it follows that these F_k 's can be taken as those mentioned in the definition of R. M. S. No two of them are equivalent, since in that case they would be identical and from the hypotheses on F and R it would follow that $F(w, z)$ would be reducible, contrary to hypothesis.

We observe that to each point of the locus $F=0$ correspond one or more (but a finite number of) points of the R. M. S. The points of the R. M. S. shall at times be considered in association with the corresponding points in (w, z) -space on the locus $F=0$, and at other times, as is ordinarily the case when $n=1$, in association with the corresponding points in (z) -space.

According to Theorem 6.II of KB, if any closed sub-set of R is given, a complex K_{2n+2} can be found containing the sub-set, such that the locus $F=0$ in K_{2n+2} is a sub-complex of even dimension, with analytic cells. In this case the dimension is $2n$, and we denote the sub-complex by K_{2n} . We denote by K_{2n-2} the complex of all cells of K_{2n} of dimensions less than $2n-1$.

* On the covering of analytic loci by complexes, these Transactions, vol. 34 (1932), pp. 231-251; Theorem 5.I. We shall refer to this paper as KB. On p. 233 of this paper the words "In irreducible-C factorization" should be inserted at the beginning of the last sentence in Theorem 2.V, and also in Corollary 2.VI. In the last line on p. 233 the words "at the same points as" should be replaced by "identically if and only if the same is true for."

See also S. Lefschetz and J. H. C. Whitehead, *Analytical complexes*, these Transactions, vol. 35 (1933), pp. 510-517; §4.

† Osgood II, chapter 2, §2.

‡ Osgood II, chapter 2, §§5, 7.

We shall now state and prove a simple set of rules for determining the R. M. S., and shall later use these rules in establishing certain properties of the R. M. S.

LEMMA 4. *The R. M. S. can be determined from K_{2n} as the locus L_{2n} which we now describe. We keep all of the $2n$ - and $(2n-1)$ -cells of K_{2n} . At each point, say T , of any cell E_p , $p < 2n-1$, we consider all the incident $(2n-1)$ - and $2n$ -cells, and using them alone apply the test described in the concluding sentence of Lemma 3 to the neighborhood of T , finding that the incident $2n$ - and $(2n-1)$ -cells are thus grouped into a finite number of sets, for each of which the condition of Lemma 3 is satisfied. For each of these sets we assign a point to L_{2n} , corresponding to T . Then L_{2n} consists of $(K_{2n} - K_{2n-2})$ and these new points, with neighborhood on L_{2n} determined as on K_{2n} except at points corresponding to points on K_{2n-2} , where it is determined in an obvious manner by use of the incident $2n$ - and $(2n-1)$ -cells appearing in the tests mentioned above.*

In applying this procedure at the boundary of K_{2n+2} , we must consider K_{2n} enlarged by the addition of part of the locus $F=0$ outside of K_{2n+2} .

Before proving the lemma we observe that we shall show later that L_{2n} is a complex and that, as we should expect from the above lemma, corresponding to each cell of K_{2n-2} is a finite positive number of cells of L_{2n} ; but corresponding to each cell of $K_{2n} - K_{2n-2}$ is just one cell of L_{2n} .

We begin by observing that each point of $K_{2n} - K_{2n-2}$ yields just one point of the R. M. S.: the locus in (z) -space where values of w coincide is defined by equating discriminants to zero, hence is at most $(2n-2)$ -dimensional. If any point of a $2n$ -cell or of a $(2n-1)$ -cell projected onto that locus, every point of the cell would project onto the locus, as all points of a cell have similar neighborhoods, and the cells project in one-to-one manner onto cells of the (z) -space. (The cells are obtained from cells in the (z) -space by two successive steps of the kind described on page 249 of KB, where at each step we obtain a cell of the first class.) Since the locus in question in (z) -space is at most $(2n-2)$ -dimensional, we would then have a contradiction to the invariance of dimensionality.* Consequently, at each point of $K_{2n} - K_{2n-2}$, w is a single-valued, and hence analytic, function of the z 's, and according to the definition of R. M. S. each such point therefore yields just one point of the R. M. S.

Now consider the points of the R. M. S. corresponding to a given point P of K_{2n-2} . For each of the irreducible functions F^i , vanishing at P , into which F factors, we obtain a point on the R. M. S. Let P^0 be the projection of P on (z) -space. Analytic continuation in a neighborhood of P^0 , avoiding

* Brouwer. See Lefschetz I for references.

points where the discriminant of F vanishes, never leads from one function F^i to a distinct function F^j ,* and furthermore such continuation can be made a test for distinguishing the functions F^i . In so testing, the paths can be made to avoid the projection, K_{2n-2}^1 , on (z) -space of K_{2n-2} without affecting the results, since any path avoiding points where the discriminant of F vanishes can be deformed into a path of the kind wanted in such a way that none of the intermediate positions of the path pass through any point of the part of K_{2n-2}^1 for which the discriminant vanishes. This is because (z) -space is $2n$ -dimensional. Consequently, for each point of the R. M. S. corresponding to P the part of the R. M. S. corresponding to $(K_{2n} - K_{2n-2})$ hangs together near the point in the way described in Lemma 3, and must therefore be one of the sets designated in Lemma 4. This proves that the process of Lemma 4 determines all of the points of the R. M. S. corresponding to P , and each on the boundary of the proper cells of $(K_{2n} - K_{2n-2})$.

Since each such set of cells of $(K_{2n} - K_{2n-2})$ must determine one of the functions F^i , it follows that no unwanted points are determined by the process of Lemma 4.

Consequently we have exactly the R. M. S. determined, and the proof of Lemma 4 is complete.

LEMMA 5. *The R. M. S. (locus L_{2n}) is a complex.*

We begin with the cells of $K_{2n} - K_{2n-2}$, which can be taken as part of a representation of L_{2n} , as we have already seen. Now consider points of L_{2n} arising from points of K_{2n-2} , in the light of Lemma 4. All points of a given cell of K_{2n-2} have similar neighborhoods on $K_{2n} - K_{2n-2}$, in fact, neighborhoods which are composed of parts of the same cells. From that fact and Lemma 4 it follows that corresponding to each cell of K_{2n-2} we have a finite number of cells of points of L_{2n} , each incident with certain of the cells of higher dimension of $K_{2n} - K_{2n-2}$. Now L_{2n} is closed, as follows upon consideration of Lemma 4, and of the fact that if a given cell of a complex is incident with certain cells of higher dimensions, then any cell on its boundary is incident with these cells of higher dimensions. Consequently L_{2n} is a complex, as we wished to prove.

THEOREM 1. *The Riemann multiple-space (L_{2n}) is a set of generalized manifolds (mod boundary of K_{2n+2}).*

By this we mean that it is a complex consisting of a number of parts each

* Osgood II, chapter 2, §§10, 11. We do not find there a general treatment of R. M. S., as the points for which the discriminant vanishes are not treated.

of which satisfies the definition of generalized manifold except at the boundary of K_{2n+2} .

According to Lemmas 5, 4 and 3, L_{2n} satisfies the condition for a set of generalized manifolds, except at the boundary of K_{2n+2} . Consequently Theorem 1 is valid.

4. Non-spherical points. Any point of L_{2n} which does not have a neighborhood on the R. M. S. homeomorphic to a $2n$ -cell shall be called a *non-spherical point*. We shall prove that the non-spherical points form a sub-complex of dimension not greater than $2n-4$.

THEOREM 2. *The R. M. S. L_{2n} can be formed from K_{2n} (locus $F=0$), by the process described by Veblen,* used in his proof that every n -circuit is a singular generalized n -manifold.*

With the $2n$ - and $(2n-1)$ -cells we get the correct result, since each $(2n-1)$ -cell not on the boundary of K_{2n+2} is incident with just two $2n$ -cells. Now we use induction, supposing that we have got the correct result with all cells down to those of dimension $p+1$, and next consider those of dimension p . Under each of the two methods, that given by Veblen and that given in Lemma 4 (under the test of Lemma 3), we replace a given p -cell E_p of K_{2n-2} by a finite number of p -cells, each incident with certain groups of cells of higher dimensions. In the first case, we have one p -cell for each group of incident cells of higher dimensions which remain connected near E_p when E_p is removed, and in the second case we have a similar test, but consider only the incident cells of dimensions $2n$ and $2n-1$. But from the corollary to Lemma 3 we see that, since we know that we have a generalized manifold insofar as cells of dimensions greater than p are tested, we obtain the same result by each of the two methods. Consequently Theorem 2 is valid.

THEOREM 3. *The non-spherical points of the R. M. S. (L_{2n}) form a sub-complex of dimension at most $2n-4$.*

As the set of spherical points is evidently an open set on L_{2n} , the set of non-spherical points must be closed. As it must consist of a certain number of cells, it is therefore a complex. It remains to prove that this complex is of dimension at most $2n-4$.

It is shown in KB† that near any point P on the locus $F=0$, above a point where the discriminant is zero, but where the discriminant of the discriminant is not zero (upon second application of the Weierstrass Preparation Theorem), the locus of similarly described points near P is obtained by equating

* Loc. cit.

† §4, pp. 236-242.

w and z_n each to an analytic function of (z_1, \dots, z_{n-1}) . We denote by J_{2n-2} a $(2n-2)$ -cell neighborhood of P consisting of such similar points. Let J_{2n-2}^1 denote the projection on (z) -space of J_{2n-2} , with equation $z_n = \psi(z_1, \dots, z_{n-1})$. We now cover a neighborhood of the points of J_{2n-2}^1 in (z) -space by a set of analytic cells of dimensions $2n-1$ and $2n$, as follows. Let E_{2n-2} be a flat cell, part of the locus $y_n = 0$ in the $2n$ -space of the complex variables (y_1, \dots, y_n) . Cover a neighborhood of E_{2n-2} in (y) -space by E_{2n-2} and a set of flat $(2n-1)$ -cells and $2n$ -cells, each incident with E_{2n-2} , alternating in order, arranged in cyclic order. Next make the transformation (with non-vanishing Jacobian) $y_i = z_i$, $i=1, \dots, n-1$, and $y_n = z_n - \psi(z_1, \dots, z_{n-1})$. This transformation gives us the set of cells covering a neighborhood of J_{2n-2}^1 , that we wanted. Above any of the $2n$ - or $(2n-1)$ -cells of this neighborhood, near J_{2n-2} , w equals a finite number of distinct-valued analytic functions of $(z) = (z_1, \dots, z_n)$, since these cells contain no points for which the discriminant vanishes. Corresponding to a circuit of the $2n$ - and $(2n-1)$ -cells incident with E_{2n-2} in (y) -space we will now have a circuit around J_{2n-2} (we can consider a curve going around it), and if we go around enough times (a finite number) we must come back to the original value of w , hence back to the original point of the R. M. S. at which we started the curve. Hence the point P has a neighborhood on L_{2n} consisting of J_{2n-2} and a set of incident $2n$ - and $(2n-1)$ -cells (not cells of L_{2n}) arranged in cyclic order, and alternating. Of course, P might be on a $(2n-3)$ -cell of L_{2n} , or even on one of lower dimension, but that does not affect our work. Consequently P is a spherical point.

Therefore the non-spherical points of L_{2n} must project onto points of (z) -space for which the discriminant of the discriminant is zero. The locus of such points is at most $(2n-4)$ -dimensional, and hence the locus of non-spherical points cannot contain any cell of dimension higher than $2n-4$. For such a cell would project onto (z) -space in a cell of the same dimension $\geq 2n-3$, which would contradict the result just obtained. Thus the proof is complete.

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ON A CERTAIN CORRESPONDENCE BETWEEN SURFACES IN HYPERSPACE*

BY
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1. INTRODUCTION

Consider a surface S and a point x on S . Let the parametric vector equation of S be

$$(1) \quad x = x(u, v).$$

The ambient space of the osculating planes at the point x to all of the curves through x is a certain space $S(2, 0)$ called the *two-osculating space of S at x* . This space is determined by the six points

$$(2) \quad x, x_u, x_v, x_{uu}, x_{uv}, x_{vv}.$$

It is the purpose of this paper to find all surfaces \bar{S} in one-to-one point correspondence with S , such that the two-osculating space $\bar{S}(2, 0)$ of \bar{S} coincides with the two-osculating space $S(2, 0)$ of S at corresponding points. We shall find that the surface \bar{S} is not arbitrary, but that the functions \bar{x} satisfy certain third-order partial differential equations studied by Lane† and by Bompiani.‡ A similar statement holds for the surface \bar{S} .

Let the surfaces S and \bar{S} be in one-to-one point correspondence so that the corresponding points have the same curvilinear coordinates.

In order that $\bar{S}(2, 0)$ at \bar{x} coincide with $S(2, 0)$ at x , it is necessary and sufficient that the functions

$$(3) \quad \bar{x}, \bar{x}_u, \bar{x}_v, \bar{x}_{uu}, \bar{x}_{uv}, \bar{x}_{vv}$$

be expressible as linear, homogeneous functions of the functions (2). The parametric vector equation of \bar{S} will therefore be of the form

$$(4) \quad \bar{x} = \bar{x}(u, v) = Ax_{uu} + Bx_{uv} + Cx_{vv} + \alpha x_u + \beta x_v + \gamma x.$$

We shall call the case in which $S(2, 0)$ is a space of five dimensions and in which the coefficients A, B, C of (4) satisfy the inequality

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† E. P. Lane, *Integral surfaces of pairs of partial differential equations of the third order*, these Transactions, vol. 32 (1930), pp. 782-793. Hereafter referred to as Lane, *Surfaces*.

‡ E. Bompiani, *Determinazione delle superficie integrali d'un sistema di equazioni a derivate parziali lineari ed omogenee*, Rendiconti del Reale Istituto Lombardo di Scienze e Lettere, vol. 52 (1919), pp. 820-830. Hereafter referred to as Bompiani, *Surfaces*.

$$(5) \quad B^2 - 4AC \neq 0$$

the non-parabolic case, and the case in which $S(2, 0)$ is a space of five dimensions and in which

$$(6) \quad B^2 - 4AC = 0$$

the parabolic case. By proper choice of ϕ , ψ , and λ in the transformation

$$\bar{u} = \phi(u, v), \quad \bar{v} = \psi(u, v), \quad \bar{x} = \lambda \bar{x}',$$

in the non-parabolic case, we may write (4) in the form

$$(7) \quad \bar{x} = x_{uv} + \alpha x_u + \beta x_v + \gamma x;$$

and in the parabolic case in the form

$$(8) \quad \bar{x} = x_{uu} + \alpha x_u + \beta x_v + \gamma x.$$

We shall denote by $S(3, 0)$ the ambient space of the three-dimensional spaces osculating all of the curves on S through x . The space $S(3, 0)$ is determined by the six points (2) and the points

$$(9) \quad x_{uuu}, x_{uuv}, x_{uvv}, x_{vvv}.$$

2. THE NON-PARABOLIC CASE

If we differentiate \bar{x} defined by (7) with respect to u and v we obtain the following expressions:

$$(10) \quad \begin{aligned} \bar{x}_u &= x_{uuv} + \alpha x_{uu} + \beta x_{uv} + (\alpha_u + \gamma)x_u + \beta_u x_v + \gamma_u x, \\ \bar{x}_v &= x_{uvv} + \alpha x_{uv} + \beta x_{vv} + \alpha_v x_u + (\beta_v + \gamma)x_v + \gamma_v x. \end{aligned}$$

The points \bar{x}_u, \bar{x}_v are in the space $S(2, 0)$ if, and only if, the functions x defining the surface S satisfy a system of differential equations of the form

$$(11) \quad \begin{aligned} x_{uuv} &= ax_{uu} + hx_{uv} + bx_{vv} + lx_u + mx_v + dx, \\ x_{uvv} &= a'x_{uu} + h'x_{uv} + b'x_{vv} + l'x_u + m'x_v + d'x. \end{aligned}$$

It follows therefore that in the non-parabolic case $S(3, 0)$ is of dimensions no higher than seven.

Subcase a. Suppose that $S(3, 0)$ is a space of seven dimensions. It follows that the functions x satisfy the equations (11) and no other third-order differential equations. Under these conditions some of the integrability conditions* of system (11) are

$$a' = b = 0, ah' + l' - a^2 - a_v = 0, b'h + m - b'^2 - b'_u = 0.$$

Equations (10) may be written in the form

* Bompiani, *Surfaces*, p. 632.

$$(12) \quad \begin{aligned} \bar{x}_u &= (a + \alpha)x_{uu} + (h + \beta)x_{uv} + (l + \alpha_u + \gamma)x_u + (m + \beta_u)x_v + (d + \gamma_u)x, \\ \bar{x}_v &= (h' + \alpha)x_{uv} + (b' + \beta)x_{vv} + (l' + \alpha_v)x_u + (m' + \beta_v + \gamma)x_v + (d' + \gamma_v)x. \end{aligned}$$

From (12) we see that the points \bar{x}_{uu} , \bar{x}_{uv} , \bar{x}_{vv} lie in $S(2, 0)$ if, and only if,

$$(13) \quad \alpha + a = 0, \quad \beta + b' = 0.$$

Therefore the point \bar{x} defined by the expression

$$(14) \quad \bar{x} = x_{uv} - ax_u - b'x_v + \gamma x$$

generates a surface \bar{S} whose two-osculating space $\bar{S}(2, 0)$ at \bar{x} coincides with the two-osculating space $S(2, 0)$ at x .

From (12) and (14) we find that the expressions for \bar{x}_u and \bar{x}_v may be written in the form

$$(15) \quad \begin{aligned} \bar{x}_u &= [a(h - b') + l - a_u + \gamma]x_u + [d + \gamma_u + \gamma(h - b')]x + (h - b')\bar{x}, \\ \bar{x}_v &= [b'(h' - a) + m' - b'_v + \gamma]x_v + [d' + \gamma_v + \gamma(h' - a)]x + (h' - a)\bar{x}. \end{aligned}$$

Therefore the lines g joining corresponding points x and \bar{x} of S and \bar{S} form a congruence G , and the surfaces S and \bar{S} sustain C nets* in relation C ; the developables of G intersect S and \bar{S} in these C nets. Conversely if two nets are in relation C their sustaining surfaces have coincident two-osculating spaces at corresponding points.

Subcase b. Suppose that $S(3, 0)$ is of six dimensions. By proper choice of the notation, the functions x satisfy a system of differential equations of the form

$$(16) \quad \begin{aligned} x_{uuv} &= ax_{uu} + hx_{uv} + bx_{vv} + lx_u + mx_v + dx, \\ x_{uvv} &= a'x_{uu} + h'x_{uv} + b'x_{vv} + l'x_u + m'x_v + d'x \\ x_{uuu} &= Ax_{vvv} + a''x_{uu} + h''x_{uv} + b''x_{vv} + l''x_u + m''x_v + d''x, \end{aligned}$$

but no other third-order differential equations.

From (7) we find that

$$(17) \quad \begin{aligned} \bar{x}_u &= (a + \alpha)x_{uu} + (h + \beta)x_{uv} + bx_{vv} + (l + \alpha_u + \gamma)x_u + (m + \beta_u)x_v \\ &\quad + (d + \gamma_u)x, \\ \bar{x}_v &= a'x_{uu} + (h' + \alpha)x_{uv} + (\beta + b')x_{vv} + (l' + \alpha_v)x_u + (m' + \beta_v + \gamma)x_v \\ &\quad + (d' + \gamma_v)x. \end{aligned}$$

It follows from (17) and (16) that the points \bar{x}_{uu} , \bar{x}_{uv} , \bar{x}_{vv} lie in $S(2, 0)$ if, and only if,

$$(18) \quad \beta + b' = 0, \quad b = 0, \quad A(a + \alpha) = 0.$$

* V. G. Grove, *The transformation C of nets in hyperspace*, these Transactions, vol. 33 (1931), pp. 733-741.

If we use (18) we may write equations (17) in the form

$$\begin{aligned}
 \bar{x}_u &= (a + \alpha)x_{uu} + [l + \alpha_u + \gamma - \alpha(h - b')]x_u + [m - b'_u + b'(h - b')]x_v \\
 &\quad + [d + \gamma_u - \gamma(h - b')]x + (h - b')\bar{x}, \\
 (19) \quad \bar{x}_v &= a'x_{uv} + [l' + \alpha_v - \alpha(h' + \alpha)]x_u + [m' - b'_v + \gamma + b'(h' + \alpha)]x_v \\
 &\quad + [d' + \gamma_v - \gamma(h' + \alpha)]x + (h' + \alpha)\bar{x}.
 \end{aligned}$$

Some of the integrability conditions of system (16) with $b=0$ are

$$\begin{aligned}
 (20) \quad Aa' &= 0, \quad a^2 + a'h + a_v = a'a'' + ah' + a'b' + a'_u + l', \\
 &\quad b'(h - b') + m - b'_u = a'b''.
 \end{aligned}$$

A. Suppose first that $A \neq 0$, $a' = 0$. Under conditions (18) equations (19) may be written in the form

$$\begin{aligned}
 (21) \quad \bar{x}_u &= [l - a_u + \gamma + a(h - b')]x_u + [d + \gamma_u - \gamma(h - b')]x + (h - b')\bar{x}, \\
 \bar{x}_v &= [m' - b'_v + \gamma + b'(h' - a)]x_v + [d' + \gamma_v - \gamma(h' - a)]x + (h' - a)\bar{x}.
 \end{aligned}$$

It follows therefore that if $A \neq 0$, $a' = 0$, the surfaces S and \bar{S} sustain C nets, and the lines g joining corresponding points x and \bar{x} form a congruence G , the developables of G intersecting these surfaces in their C nets.

B. Suppose that $A = 0$. Under this condition another integrability condition of system (16) is $b'' = 0$. Equations (19) may now be written in the form

$$\begin{aligned}
 (22) \quad \bar{x}_u &= (a + \alpha)x_{uu} + [l + \alpha_u + \gamma - \alpha(h - b')]x_u + [d + \gamma_u - \gamma(h - b')]x \\
 &\quad + (h - b')\bar{x}, \\
 \bar{x}_v &= a'x_{uv} + [l' + \alpha_v - \alpha(h' + \alpha)]x_u + [m' - b'_v + \gamma + b'(h' + \alpha)]x_v \\
 &\quad + [d' + \gamma_v - \gamma(h' + \alpha)]x + (h' + \alpha)\bar{x}.
 \end{aligned}$$

It follows that the tangent to $v = \text{const.}$ on \bar{S} intersects the osculating plane to $v = \text{const.}$ on S . The tangent planes to S and \bar{S} at x and \bar{x} respectively intersect in a point; they will intersect in a line if, and only if, $a' = a + \alpha = 0$, that is, if, and only if, the parametric nets on S and \bar{S} are in relation C . In this latter case the lines joining corresponding points x and \bar{x} form a congruence.

3. THE PARABOLIC CASE

Let us consider the parabolic case. If we differentiate \bar{x} defined by (8) with respect to u and v , we obtain

$$\begin{aligned}
 (23) \quad \bar{x}_u &= x_{uuu} + \alpha x_{uu} + \beta x_{uv} + (\alpha_u + \gamma)x_u + \beta_u x_v + \gamma_u x, \\
 \bar{x}_v &= x_{uvv} + \alpha x_{uv} + \beta x_{vv} + \alpha_v x_u + (\beta_v + \gamma)x_v + \gamma_v x.
 \end{aligned}$$

It follows therefore that if the points \bar{x}_u, \bar{x}_v lie in $S(2, 0)$ the functions x must satisfy a system of differential equations of the form

$$(24) \quad \begin{aligned} x_{uuu} &= ax_{uu} + hx_{uv} + bx_{vv} + lx_u + mx_v + dx, \\ x_{uuv} &= a'x_{uu} + h'x_{uv} + b'x_{vv} + l'x_u + m'x_v + d'x. \end{aligned}$$

It follows therefore that $S(3, 0)$ is of dimensions no higher than seven.

Subcase a. Suppose that $S(3, 0)$ is of seven dimensions, that is, that the functions x do not satisfy a third third-order differential equation.

The system (24) has the following integrability conditions*:

$$(25) \quad \begin{aligned} b &= 0, \quad h = b', \quad a_v = a'_u + a'h' + l', \\ h_v + ah' + l &= h'_u + a'h + h'^2 + m', \\ ab' + m &= b'_u + b'h', \quad l_v + al' = l'_u + a'l + h'l' + d', \\ m_v + am' + d &= m'_u + a'm + h'm', \\ d_v + ad' &= d'_u + a'd + d'h'. \end{aligned}$$

It follows from (23) and (24) that the functions \bar{x}_u and \bar{x}_v are defined by the expressions

$$(26) \quad \begin{aligned} \bar{x}_u &= (a + \alpha)x_{uu} + (h + \beta)x_{uv} + (l + \alpha_u + \gamma)x_u + (m + \beta_u)x_v + (d + \gamma_u)x, \\ \bar{x}_v &= a'x_{uu} + (h' + \alpha)x_{uv} + (b' + \beta)x_{vv} + (l' + \alpha_v)x_u + (m' + \beta_v + \gamma)x_v \\ &\quad + (d' + \gamma_v)x. \end{aligned}$$

From (26) we find that the points \bar{x}_{uu} , \bar{x}_{uv} , \bar{x}_{vv} lie in the space $S(2, 0)$ if and only if

$$(27) \quad \alpha + h' = 0, \quad \beta + b' = 0.$$

Therefore the surface \bar{S} generated by the point \bar{x} defined by the expression

$$(28) \quad \bar{x} = x_{uu} - h'x_u - b'x_v + \gamma x$$

is such that the two-osculating space $\bar{S}(2, 0)$ at \bar{x} coincides with the space $S(2, 0)$ at x for every choice of γ .

If we make use of equation (28) we may write equation (26) in the form

$$(29) \quad \begin{aligned} \bar{x}_u &= \mu x_u + f x + A \bar{x}, \\ \bar{x}_v &= r x_u + \mu x_v + g x + B \bar{x}, \end{aligned}$$

wherein

$$(30) \quad \begin{aligned} \mu &= h'(a - h') + l - h'_u + \gamma = a'b' + m' - b'_v + \gamma, \\ f &= d + \gamma_u - \gamma(a - h'), \quad A = a - h', \quad B = a', \\ g &= d' + \gamma_v - a'\gamma, \quad r = a'h' + l' - h'_v. \end{aligned}$$

* Lane, *Surfaces*, p. 792.

We may readily verify that as $x(\bar{x})$ moves along the curve $v = \text{const.}$ on $S(\bar{S})$ the point

$$y = \bar{x} - \mu x, \quad r \neq 0,$$

describes a curve whose tangent at y is the line g joining x to \bar{x} . Moreover there exists no other curve on $S(\bar{S})$ along which $x(\bar{x})$ may move so that the line g will generate a developable surface. We may readily verify that *the lines g generate a congruence G composed of the tangents to a one-parameter family of asymptotic curves on the surface generated by the point y* . However the point y defined by the expression

$$y = \bar{x} - \mu x, \quad r = 0,$$

is a fixed point, and the lines g form a bundle of lines through this fixed point.

Subcase b. Suppose that the space $S(3, 0)$ is of six dimensions.

A. The points x_{uv} , x_{vv} , as may be seen from (23), will lie in the space $S(2, 0)$ if

$$(31) \quad \beta = -h, \quad \alpha = -h',$$

and if x satisfies the equations (24) and a differential equation of the form

$$(32) \quad x_{vvv} = a''x_{uu} + h''x_{uv} + b''x_{vv} + l''x_u + m''x_v + d''x.$$

Some of the integrability conditions of the system composed of equations (24) and (32) are

$$b = 0, \quad h = b', \quad m + b'(a - h') - b'_u = 0.$$

We may readily verify that *the point \bar{x} defined by*

$$(33) \quad \bar{x} = x_{uu} - h'x_u - b'x_v + \gamma x$$

generates a surface \bar{S} whose two-osculating space $\bar{S}(2, 0)$ at \bar{x} coincides with the two-osculating space $S(2, 0)$ of S at x . Moreover the tangent planes to S at x and \bar{S} at \bar{x} intersect in a line h . The projectivity determined on h by the pencils of tangent lines to S and \bar{S} at x and \bar{x} is parabolic. The lines g joining x to \bar{x} form a congruence of tangents to a one-parameter family of asymptotic curves on a surface.

B. The space $\bar{S}(2, 0)$ of \bar{S} at \bar{x} will also coincide with the space $S(2, 0)$ at x if

$$\beta + b' = 0,$$

and if x satisfies equations (24) and a differential equation of the form

$$(34) \quad x_{uu} = a''x_{uu} + h''x_{uv} + b''x_{vv} + l''x_u + m''x_v + d''x.$$

Two of the integrability conditions of such a system are

$$b = 0, \quad b' = 0.$$

It follows therefore that any point defined by the expression

$$\bar{x} = x_{uu} + \alpha x_u + \gamma x$$

(α and γ arbitrary) in the osculating plane to $v = \text{const.}$ on S at x generates a surface \bar{S} whose two-osculating space $\bar{S}(2, 0)$ at \bar{x} coincides with the space $S(2, 0)$ at x . The tangent planes to S and \bar{S} at x and \bar{x} intersect in a point.

Suppose that in the expression (4) $A = B = C = 0$. By a transformation of the curvilinear coordinates we may write (4) in the form

$$(35) \quad \bar{x} = x_u + \gamma x.$$

By repeated differentiations we find that $\bar{S}(2, 0)$ coincides with $S(2, 0)$ if, and only if, the functions x satisfy a system of differential equations composed of equations of the form (24) and (34). It follows that the space $S(3, 0)$ of S at x is of six dimensions. Conversely if the functions satisfy such a system, a point \bar{x} defined by (35) generates a surface of the required type.

4. THE CONJUGATE CASE

Suppose now that S sustains a conjugate net. By proper choice of the parameters we may take this net to be the parametric net. The functions x therefore satisfy an equation of the Laplace type

$$(36) \quad x_{uv} = ax_u + bx_v + cx.$$

It follows from (36) that $S(2, 0)$ is a space of four dimensions and that $S(3, 0)$ is a space of not more than six dimensions.

Let the point \bar{x} be defined by the expression

$$(37) \quad \bar{x} = Ax_{uu} + Cx_{vv} + \alpha x_u + \beta x_v + \gamma x,$$

wherein not both A and C are zero.

A. Suppose first that $\bar{S}(3, 0)$ is of six dimensions. We find readily that there exist no surfaces \bar{S} distinct from S such that the spaces $\bar{S}(2, 0)$ and $S(2, 0)$ coincide.

B. Suppose that $S(3, 0)$ is of five dimensions. We find from (37) that

$$(38) \quad \begin{aligned} \bar{x}_u = & Ax_{uuu} + (A + \alpha)x_{uu} + (bC + C_u)x_{vv} + [C(a_v + a^2) + a\beta + \alpha_u + \gamma]x_u \\ & + [C(c + ab + b_v) + b\beta + \beta_u]x_v + [C(c_v + ac) + \beta c + \gamma_u]x. \end{aligned}$$

A symmetrical expression obtains for \bar{x}_v . It follows that if $A \neq 0$, the func-

tions x satisfy an equation of the form

$$(39) \quad x_{uuu} = a'x_{uu} + b'x_{vv} + l'x_u + m'x_v + d'x.$$

In order that \bar{x}_v lie in the space $S(2, 0)$, and that $S(3, 0)$ be a space of five dimensions the coefficient C must be zero.

Some of the integrability conditions of the system composed of equations (36) and (39) are

$$b' = 0, \quad m' = 0, \quad a'_v - a_u = c + ab + a_u.$$

Hence the curves $v = \text{const.}$ on S are plane curves. With the expression for \bar{x}_v and $C = 0$, we find that \bar{x}_{vv} lies in $S(2, 0)$ if, and only if, $\beta = 0$. Hence \bar{x} lies in the plane of the curve $v = \text{const.}$

If we set $A = 1$, we find that the points \bar{x}_u, \bar{x}_v are defined by the expressions

$$(40) \quad \begin{aligned} \bar{x}_u &= [l' + \gamma + \alpha_u - \alpha(a' + \alpha)]x_u \\ &\quad + [d' + \gamma_u - \gamma(a' + \alpha)]x + (a' + \alpha)\bar{x}, \\ \bar{x}_v &= (c + ab + a_u + \alpha_v)x_u + (b^2 + b_u + \alpha b + \gamma)x_v \\ &\quad + (c_u + bc + \alpha c + \gamma_v - a\gamma)x + a\bar{x}. \end{aligned}$$

The tangent planes to S and \bar{S} at x and \bar{x} intersect in a line. Hence if $S(3, 0)$ is a space of five dimensions, and if S sustains a conjugate net, the point \bar{x} defined by (37) will describe a surface \bar{S} whose two-osculating space $S(2, 0)$ at \bar{x} coincides with $S(2, 0)$ at x if and only if each curve of one of the component families of curves of the conjugate net is a plane curve, and the point \bar{x} is a point in the plane of the curve. The lines g joining x and \bar{x} form a congruence.

Suppose that \bar{x} lies in the tangent plane of S at x , that is, suppose that in (37) $A = C = 0$. We readily verify that if $S(3, 0)$ is of six dimensions the space $\bar{S}(2, 0)$ at \bar{x} cannot coincide with the space $S(2, 0)$ at x for distinct surfaces S and \bar{S} . If $S(3, 0)$ is a space of five dimensions, the point \bar{x} must lie in the tangent to one of the curves of the conjugate net, and that family of curves is a family of plane curves.

5. THE ASYMPTOTIC CASE

Suppose that S sustains a one-parameter family of asymptotic curves. Let the notation be so chosen that the curves $v = \text{const.}$ are the asymptotics. It follows that the functions x defining S satisfy the differential equation

$$(41) \quad x_{uu} = ax_u + bx_v + cx.$$

It follows that the space $S(3, 0)$ is a space of six dimensions at most.

Let \bar{x} be defined by an expression of the form

$$(42) \quad \bar{x} = Bx_{uv} + Cx_{vv} + \alpha x_u + \beta x_v + \gamma x,$$

wherein not both B and C are zero.

A. We may readily verify that if $S(3, 0)$ is a space of six dimensions, there exists no surface \bar{S} distinct from S with the desired property.

B. Suppose therefore that $S(3, 0)$ is a space of five dimensions. It follows from (42) that the points \bar{x}_u and \bar{x}_v are in $S(2, 0)$ if, and only if, $C=0$, and the functions x satisfy a differential equation of the form

$$(43) \quad x_{uvv} = h'x_{uv} + b'x_{vv} + l'x_u + m'x_v + d'x.$$

Two of the integrability conditions of the system composed of equations (41) and (43) are

$$(44) \quad b = 0, \quad c - b'_u + b'(a - b') = 0.$$

It follows therefore that the surface S is ruled.

If in (42) we set $C=0$, $B=1$, we find that

$$(45) \quad \begin{aligned} \bar{x}_u &= (a + \beta)x_{uv} + (a_v + a\alpha + \alpha_u + \gamma)x_u \\ &\quad + (c + \beta_u)x_v + (c_v + \alpha c + \gamma_u)x, \\ \bar{x}_v &= (a' + \alpha)x_{uv} + (\beta + b')x_{vv} + (l' + \alpha_v)x_u \\ &\quad + (m' + \beta_v + \gamma)x_v + (d' + \gamma_v)x. \end{aligned}$$

The points \bar{x}_{uu} , \bar{x}_{uv} , \bar{x}_{vv} lie in $S(2, 0)$ if, and only if, $\beta = -b'$. Equation (45) may be written in the form

$$(46) \quad \begin{aligned} \bar{x}_u &= [a_v + a\alpha + \alpha_u + \gamma - \alpha(a - b')]x_u + [c_v + \alpha c + \gamma_u - \gamma(a - b')]x \\ &\quad + (a - b')\bar{x}, \\ \bar{x}_v &= [l' + \alpha_v - \alpha(a' + \alpha)]x_u + [m' - b'_v + \gamma + b'(a' + \alpha)]x_v \\ &\quad + [d' + \gamma_v - \gamma(a' + \alpha)]x + (a' + \alpha)\bar{x}. \end{aligned}$$

The point \bar{x} defined by the expression

$$\bar{x} = x_{uv} + \alpha x_u - b'x_v + \gamma x$$

for arbitrary values of α and γ generates a surface \bar{S} whose two-osculating space $\bar{S}(2, 0)$ at \bar{x} coincides with the two-osculating space $S(2, 0)$ of S at x .

The point r defined by the expression $r = x_u - b'x$ is readily characterized as the only point, on the generator through x of the ruled surface, describing a surface for which the osculating plane to the curve $u = \text{const.}$ at r lies in the space of three dimensions tangent to the ruled surface along the generator through x . We find that

$$r_0 + \alpha r = \bar{x} - (\alpha b' + \gamma + b'_0)x.$$

It follows that the lines g joining x to \bar{x} form a congruence. The line g passes through x and intersects the tangent line to the curve $u = \text{const.}$ on the surface generated by the point r .

Suppose that \bar{x} lies in the tangent plane to S at x . We readily verify that $\bar{S}(2, 0)$ at \bar{x} will coincide with $S(2, 0)$ at x if and only if \bar{x} lies in the tangent line of the asymptotic curve on S through x , and if the functions x defining the surface satisfy a differential equation of the form (43).

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THE SOLUTIONS OF THE MATHIEU EQUATION WITH A COMPLEX VARIABLE AND AT LEAST ONE PARAMETER LARGE*

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Introduction. The Mathieu differential equation

$$(1) \quad \frac{d^2 u}{dz^2} + \{\Delta - \Omega \cos 2z\}u = 0,$$

also commonly known as the equation of the elliptic cylinder functions, is too well known to require any introduction. Its solutions govern problems of the greatest diversity in astronomy and theoretical physics, and have accordingly been the subjects of a vast number of investigations.†

The differential equation as such depends upon two independent parameters, designated in the form written above by Δ and Ω . In the present discussion these are to be taken real but are to be numerically unrestricted except that at least one is to be large. The variable will be permitted to range over the complex plane.

Since the coefficient of the differential equation is an even simply periodic analytic function of z , it is known from Floquet's theory of such equations that the solutions are in general of the structure

$$u(z) = c_1 e^{\mu z} \phi(z) + c_2 e^{-\mu z} \phi(-z),$$

in which the function $\phi(z)$ is periodic. The *characteristic exponent*, μ , is a constant as to z but depends in an intricate way upon the parameters Δ and Ω . If it is real, the equation obviously possesses a solution which for large real values of the variable becomes exponentially infinite, i.e., a so called *unstable* solution. In the alternative case the exponent is pure imaginary and the solutions remain bounded along the axis of reals, i.e., are of the so called *stable* type. The intermediate case in which $\mu = 0$ is of especial im-

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† Cf. for the literature and for partial enumerations of applications of the equation: Strutt, M. J. O., *Lamé-Mathieusche und verwandte Funktionen in Physik und Technik*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 1, No. 3, Berlin, 1932; Whittaker and Watson, *A Course in Modern Analysis*, 3d edition, 1920, Cambridge University Press; Humbert, P., *Fonctions de Lamé et Fonctions de Mathieu*, Mémorial des Sciences Mathématiques, X, Paris, 1926; Van der Pol, B., and Strutt, M. J. O., *On the stability of the solutions of Mathieu's equation*, The Philosophical Magazine, vol. 5 (1928), p. 18.

portance, for the equation then admits one solution known as a *Mathieu function* which is periodic. The second solution, a *Mathieu function of the second kind*, is then not periodic and is of a functional structure distinct from that indicated above.

With either of the parameters Δ and Ω fixed, the relation $\mu=0$ restricts the remaining one to a denumerably infinite set of values called the *characteristic values*. Broadly speaking the determination of these values and of the corresponding Mathieu functions is the matter of prime importance in the applications of the equation which belong more immediately to the domain of physics, while the determination of the characteristic exponent in terms of a fixed set of parameters is generally the peculiar requirement of the applications to astronomy.

When the values of the parameters are small the solution of the differential equation is generally and appropriately essayed through the means of convergent series expansions. When at least one of the parameters is large, on the other hand, the methods of asymptotic representation are adapted and have been generally applied. Though the literature covering investigations of this latter type is large it can hardly be said that the results recorded are by any means complete. Restrictions upon the range of the parameters are generally made and frequently only the forms of the Mathieu functions, i.e., of the solutions with the period 2π , are considered. Again, when forms asymptotic with respect to one parameter are obtained their dependence upon the remaining secondary parameter may not be considered, the results being established, therefore, only for a fixed configuration of the parameters relative to each other. Finally the investigations have almost exclusively been restricted to the case of a real variable. The most recent report on the status of the theory* says on this point: "While we believe that the theory of the Hill and Mathieu differential equations with *real* variables and parameters has to a certain extent been rounded out, it is to be emphasized that no such assertion can be made concerning these equations with *complex* variables and parameters. . . . Only when the problems bearing upon this point have been adequately treated may it be hoped to round out the theory of the Lamé equation as has been done in the case of the equation of Mathieu. Such an investigation would not only throw new light upon many differential equations of mathematical physics, but would make possible the application of certain of the functions obtained to problems of practical importance."

The present investigation is devoted to a general consideration of the asymptotic solutions of the Mathieu equation over the complex plane and for all real configurations of the parameter values in which at least one is

* Strutt, loc. cit. (Vorwort).

numerically large. The analytic forms which represent the solutions asymptotically are found to differ in essentially different parameter configurations, while in its dependence upon the variable such a representation even for a specific solution and with one and the same configuration of parameters requires the employment of a variety of analytic forms. In general a special form is required for the description in the neighborhood of any point in which the coefficient of the equation vanishes, while outside such neighborhoods several forms again are made necessary by the incidence of the Stokes' phenomenon.

The limitation of the discussion to real parameter values was imposed to keep the extent of the investigation within its present bounds. The method in no way requires such a restriction.* In the matter of the method the present paper is based upon earlier papers of the author† which gave a general derivation of the asymptotic solutions of differential equations of the type

$$\frac{d^2u}{dz^2} + \{ \rho^2 \chi_0^2(z) + \rho \chi_1(z) + \chi_2(z, \rho) \} u = 0,$$

in which ρ is a large complex parameter and the coefficient $\chi_0^2(z)$ vanishes at some point of the domain considered. Aside from the considerations peculiar to the Mathieu equation, however, the presence of two independent parameters makes of the present discussion something more than a specialization of the general theory cited. With one parameter assigned to a primary role it must be shown that the hypotheses of the theory cited are met *uniformly* with respect to the secondary parameter which has remained free. This is essential to assure the uniform validity of the conclusions, i.e., that the degree of approximation afforded by the asymptotic representation is maintained during a variation of the parameters within the bounds of a given configuration.

By way of arrangement there have been grouped in chapter 1 such general considerations as are to be subsequently available. Of the following chapters each is given to the deductions peculiar to a specific configuration of parameters. Throughout the paper the forms of two fundamental pairs of solutions are deduced. This is desirable because of the fact that the members of any one pair of solutions may and do become asymptotically indistinguishable in certain regions of the complex plane. Aside from the general asymptotic

* An analogous application of the method to a study of the Bessel functions with both the variable and the parameter complex was made by the author in the papers cited below.

† These Transactions, as follows: *On the asymptotic solutions of ordinary differential equations, etc.*, vol. 33 (1931), p. 23; *On the asymptotic solutions of differential equations, etc.*, vol. 34 (1932), p. 447; *The asymptotic solutions of certain linear ordinary differential equations of the second order*, vol. 36, p. 90. These papers will be referred to in the text by the designations L_1 , L_2 and L_3 .

otic forms the special forms which apply to real values of the variable are noted, and the forms of the solutions of the associated Mathieu equation,

$$(2) \quad \frac{d^2 v}{dz^2} + \{\Omega \cosh 2z - \Delta\}v = 0,$$

are deduced. The asymptotic equations for the characteristic values are given, and the characteristic exponent is asymptotically determined.

CHAPTER 1

GENERAL CONSIDERATIONS

1.1. The parameter configurations. The effect of replacing the variable z by $z + \pi/2$ in the equation (1) is merely to alter the sign of the cosine function, i.e., to replace the parameter Ω by its negative. There is, therefore, no loss of generality in assuming, as will henceforth be done, that Ω ranges only over the positive values and zero. The parameter Δ , on the other hand, is to range unrestrictedly over all real values.

For any positive Ω , however small it may be, the term $\Omega \cos 2z$ becomes dominant over Δ when z reaches a domain sufficiently remote from the axis of reals. In any such domain therefore the character of the differential equation is essentially altered if Ω is replaced by zero, and it may accordingly be expected that formulas which are to be valid *uniformly* for $\Omega \geq 0$ may be obtained only for regions of the z plane in which $|\vartheta(z)|$ is bounded. This fact suggests the grouping into separate configurations of those sets of parameter values in which Ω is relatively small. They are indicated as II and IX in Figure 1 below, the precise specifications to be later determined.

When $\Omega > 0$, the function $\{\Delta - \Omega \cos 2z\}$ vanishes at an infinite set of points in the complex plane. As z moves at a suitable distance about any such point the asymptotic forms which represent a given solution of the differential equation must be altered, i.e., replaced by others, at certain specifiable intervals. This so called Stokes' phenomenon depends quantitatively upon the order of the zero which is encircled, and since this order changes from the first to the second when Ω and $|\Delta|$ become equal, it may be expected that results obtained on the assumption that the parameters are sufficiently different in numerical value may not remain *uniformly* valid when these values are allowed to approach equality. This fact serves as the motivation for considering as distinct configurations those indicated in Figure 1 by the designations IV and VII, in which the parameters numerically approximate each other. They will be precisely defined at appropriate points in the discussion which follows. The division of the half-plane of the coordinates (Δ, Ω) into configurations is, therefore, such as is indicated in the

figure, the hypothesis that at least one parameter be large having the effect of excluding from consideration a neighborhood of the point O .

1.2. **The hypotheses of the general theory.** The differential equation (1) may be transformed in a variety of ways into an equation of the general form

$$(3) \quad \frac{d^2 u}{ds^2} + \{\rho^2 \chi_0^2(s, \sigma) + \rho \chi_1(s, \sigma)\} u = 0,$$

in which ρ , the primary parameter, and σ , the secondary parameter, are expressible in terms of Δ and Ω . The particular substitutions and hence the particular equations which result are to depend upon the parameter configuration which obtains, and will therefore be made at appropriate points as the discussion proceeds.

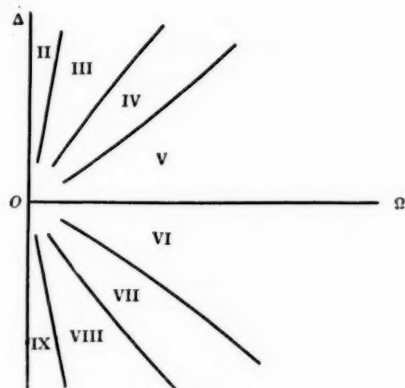


FIG. 1

Equations of the type (3) in which, however, the parameter σ is absent (i.e., fixed) are familiar, the asymptotic forms of their solutions having been deduced* under hypotheses which for the present purposes may be enumerated in the following way:

(i) *The range of the complex variable s is to be a region R_s in which the functions*

$$(s - s_0)^{-\nu} \chi_0^2(s) \text{ and } \chi_1(s)$$

are analytic, s_0 being some point of R_s and ν being some real non-negative constant. Except in some fixed neighborhood of s_0 the several functions

$$(4) \quad \{\chi_0(s)\}^{-1}, \left\{ \int_{s_0}^s \chi_0 ds \right\}^{-1}, \{\chi_1(s)/\chi_0^2(s)\}, \left\{ \int (\chi_1/\chi_0) ds \right\} \left\{ \int_{s_0}^s \chi_0 ds \right\}^{-1}$$

are to be bounded.

* Papers L_2 and L_3 cited above. In the formulas of paper L_3 the variables λ , χ_1 , ϕ , and ξ must be replaced by $i\rho$, $i\chi_1$, 2ϕ and $2i\xi$ respectively in order that they may appear as given here.

It is convenient to have at hand the following definitions:

$$(5) \quad k = \frac{-i\chi_1(s_0)}{4\chi_0'(s_0)}, \quad \eta(s) = \frac{i\chi_1(s)}{\chi_0(s)} + \frac{2k\chi_0(s)}{\int_{s_0}^s \chi_0 ds},$$

$$\phi(s) = \chi_0(s) - \frac{i\eta(s)}{2\rho}, \quad \Phi(s) = \int_{s_0}^s \phi(s) ds, \quad \xi = \rho\Phi.$$

It follows then, as may be shown, from the hypothesis (i) that the functions

$$(6) \quad \omega(\phi) \equiv \frac{3}{4} \left(\frac{\phi'}{\phi} \right)^2 - \frac{1}{2} \left(\frac{\phi''}{\phi} \right) - \frac{\nu(\nu+4)}{4(\nu+2)^2} \left(\frac{\phi}{\Phi} \right)^2,$$

$$\omega_1 \equiv \frac{\eta^2}{4} + \frac{k\chi_0^2 \int_{s_0}^s \eta ds}{\Phi \int_{s_0}^s \chi_0 ds} - \frac{k(4\eta\chi_0 - i\eta^2\rho^{-1})}{2\Phi},$$

$$\Psi \equiv \frac{\Phi^{\nu/(2\nu+4)}}{\phi^{1/2}}$$

are continuous in the region R , inclusive of the point $s=s_0$. A second and third hypothesis* made are the following:

(ii) *The differential equation (3) is to be in normal form, i.e., such that either $\chi_1 \equiv 0$ or else $\nu = 2$ and*

$$\{3\chi_0'\chi_1' - 2\chi_0''\chi_1\}_{s=s_0} = 0.$$

(iii) *Either the region R , is to be bounded, or else there are to exist constants M and H such that the relations*

$$\int \left| \frac{\omega(\phi)}{\phi} ds \right| < M, \quad \int \left| \frac{\omega_1(s)}{\phi} ds \right| < M$$

are satisfied for all arcs of integration in R , on which $|s-s_0| > H$ and on which $\vartheta(\xi)$ varies monotonically with $|\xi|$.

When the secondary parameter σ is not fixed but is permitted to vary, the formulas to be taken from the theory cited will be valid *uniformly* only if the hypotheses stated are satisfied uniformly with respect to σ . Specifically the functions (4) must be uniformly bounded in R , the functions (6) must be uniformly bounded in any fixed finite part of R , and the hypothesis (iii) must be fulfilled with constants M and H which are independent of σ .

* The hypothesis (iv) of papers L_2 and L_3 is not repeated here. It is obviously satisfied in every case of the present discussion.

1.3. **The solutions.** When the equation (3) satisfies the several hypotheses and the primary parameter ρ is sufficiently large, the relation defining the variable ξ determines a map of the region R , upon a corresponding region R_ξ in the complex ξ plane. This map is conformal except possibly at the point corresponding to s_0 where, if $\nu \neq 0$, the region R_ξ has a branch point whose order depends upon ν .

The relations

$$(7) \quad \Xi^{(l)}: (l-1)\pi + \epsilon \leq \arg \xi \leq (l+1)\pi - \epsilon,$$

with l an integral index and ϵ an arbitrarily small but fixed positive constant, define in the domain R_ξ the (overlapping) sub-regions $\Xi^{(l)}$. These correspond to respective sub-regions of R , which will likewise be denoted by $\Xi^{(l)}$.

For any index h the differential equation (3) possesses a fundamental pair of solutions $u_{h,1}(s), u_{h,2}(s)$, which are characterized by the fact that they are of peculiarly simple asymptotic forms as compared with the general solution for values of s which are in the corresponding sub-region $\Xi^{(h)}$ and which are not too near the point s_0 . When s passes the bounds of the sub-region $\Xi^{(h)}$ this simplicity is lost and devolves upon a new set of solutions which are in turn associated in the manner indicated with the new sub-region in which s is then to be found. If $\nu \neq 0$ the forms referred to give valid representations of the respective solutions only so long as $|\xi| \geq N$, where N is a constant whose magnitude is determined by the degree of approximation which the asymptotic representation is required to afford. The excepted region $|\xi| \leq N$ corresponds in R , to a neighborhood of the point s_0 , and in this region a distinct representation must in general be employed.

The solutions $u_{h,j}(s), j=1, 2$, with a particular index h are thus because of their simplicity especially adapted for use in any deduction in which the associated region $\Xi^{(h)}$ plays a peculiar role. In terms of them, however, any other solutions may be simply expressed. In particular, it will be noted that if the point z_a corresponds to s_a under the correspondence of the variables which relates the equations (1) and (3), then the principal solutions $u(z), U(z)$, of the equation (1) relative to z_a , i.e., those determined by the values

$$(8a) \quad u(z_a) = 0, \quad \frac{du(z_a)}{dz} = 1, \quad U(z_a) = 1, \quad \frac{dU(z_a)}{dz} = 0,$$

are given by the formulas

$$(8b) \quad u = \left(\frac{dz}{ds} \right)_{s=s_a} \left\{ \frac{u_{h,2}(s_a)u_{h,1}(s) - u_{h,1}(s_a)u_{h,2}(s)}{W} \right\},$$

$$U = - \left\{ \frac{u'_{h,2}(s_a)u_{h,1}(s) - u'_{h,1}(s_a)u_{h,2}(s)}{W} \right\},$$

in which h may be any index, the primes denote differentiation with respect to s , and W designates the Wronskian

$$W = u'_{h,1}(s)u_{h,2}(s) - u'_{h,2}(s)u_{h,1}(s),$$

which is a constant.

The principal solutions relative to the origin ($z_a=0$) will be designated throughout the discussion by $u_o(z)$ and $u_e(z)$. Inasmuch as the coefficient of the differential equation is an even function, they will be respectively odd and even functions of z as is to be indicated by the subscripts chosen. The principal solutions relative to the point $z_a=\pi/2$ will be denoted by $u_a(z)$ and $u_g(z)$.

1.4. The asymptotic solutions when $\nu=1$. The special case of most frequent occurrence in the discussion which follows is that in which $\nu=1$, i.e., in which the zero of the coefficient $\chi^2(s)$ is a simple one. It is convenient, therefore, to note at this point for general reference the specific formulas which then apply in the relations of the preceding section, in so far as they are later to be used. Thus, for $h=-1, 0, 1, 2$ the solutions $u_{h,j}(s)$ are described by the following formulas:

When $|\xi| \geq N$ and s is in $\Xi^{(l)}$,

$$(9a) \quad u_{h,j}(s) = \rho^{-1/6} \phi^{-1/2} \{ A_{j,1}^{h,l} e^{i\xi} + A_{j,2}^{h,l} e^{-i\xi} \}, \quad j = 1, 2,$$

with coefficients to be obtained from the following table:

| (h, l) | $(-1, -1)$ | $(-1, 0)$ | $(-1, 1)$ | $(0, -1)$ | $(0, 0)$ | $(0, 1)$ | $(1, -1)$ | $(1, 0)$ | $(1, 1)$ | $(2, -1)$ | $(2, 0)$ | $(2, 1)$ | $(2, 2)$ |
|-----------------|------------|-----------|-----------|-----------|----------|----------|-----------|----------|----------|-----------|----------|----------|----------|
| $A_{1,1}^{h,l}$ | [1] | [1] | 0 | [1] | [1] | [1] | [1] | [1] | [1] | 0 | 0 | [1] | [1] |
| $A_{1,2}^{h,l}$ | 0 | [i] | [i] | [-i] | 0 | 0 | [-i] | 0 | 0 | [-i] | [-i] | [-i] | 0 |
| $A_{2,1}^{h,l}$ | 0 | 0 | [i] | 0 | 0 | [i] | [-i] | [-i] | 0 | [-i] | [-i] | 0 | 0 |
| $A_{2,2}^{h,l}$ | [1] | [1] | [1] | [1] | [1] | [1] | 0 | [1] | [1] | 0 | [1] | [1] | [1] |

and, when $|\xi| \leq N$,

$$(10a) \quad u_{h,j}(s) = (2\pi/3)^{1/2} \Psi e^{(3/2-j)\pi i/2} [\gamma_{1,j}^{(h)} \xi^{1/3} J_{-1/3}(\xi) + \gamma_{2,j}^{(h)} \xi^{1/3} J_{1/3}(\xi)],$$

with the coefficients

| h | -1 | 0 | 1 | 2 |
|----------------------|----------------|----------------|----------------|-----------------|
| $\gamma_{1,1}^{(h)}$ | 1 | $e^{-\pi i/3}$ | $e^{-\pi i/3}$ | $e^{-2\pi i/3}$ |
| $\gamma_{2,1}^{(h)}$ | 1 | $e^{\pi i/3}$ | $e^{\pi i/3}$ | $e^{2\pi i/3}$ |
| $\gamma_{1,2}^{(h)}$ | $e^{\pi i/3}$ | $e^{\pi i/3}$ | 1 | 1 |
| $\gamma_{2,2}^{(h)}$ | $e^{-\pi i/3}$ | $e^{-\pi i/3}$ | 1 | 1 |

The symbols J in these formulas designate Bessel functions in the familiar manner, and the symbol $[\]$ will be used throughout the discussion in the sense that $[Q]$ designates a quantity which differs from Q by terms of the order of ρ^{-1} and of the order of N^{-1} uniformly in σ .

From formulas thus given the evaluations

$$u_{h,1}(s_a) = \frac{[1]e^{i\xi_a}}{\rho^{1/6}\phi_a^{1/2}}, \quad u_{h,2}(s_a) = \frac{[1]e^{-i\xi_a}}{\rho^{1/6}\phi_a^{1/2}},$$

when $|\xi_a| \geq N$ and ξ_a is in $\Xi^{(h)}$, and $W = [2i]\rho^{2/3}$, will be immediately noted. Direct substitution in the relations (8b) leads, therefore, to the following formulas:

When $|\xi| \geq N$, z is in $\Xi^{(l)}$ and z_a is in $\Xi^{(h)}$,

$$\begin{aligned} u &= \frac{[1]}{2i} \left(\frac{dz}{ds} \right)_{s=z_a} \left(\frac{1}{\rho\phi_a\phi} \right)^{1/2} \left\{ e^{-i\xi_a} (A_{1,1}^{h,l} e^{i\xi} + A_{1,2}^{h,l} e^{-i\xi}) \right. \\ &\quad \left. - e^{i\xi_a} (A_{2,1}^{h,l} e^{i\xi} + A_{2,2}^{h,l} e^{-i\xi}) \right\}, \\ (11a) \quad U &= \frac{[1]}{2} \left(\frac{\phi_a}{\phi} \right)^{1/2} \left\{ e^{-i\xi_a} (A_{1,1}^{h,l} e^{i\xi} + A_{1,1}^{h,l} e^{-i\xi}) + e^{i\xi_a} (A_{2,1}^{h,l} e^{i\xi} + A_{2,2}^{h,l} e^{-i\xi}) \right\}, \end{aligned}$$

and when $|\xi| \leq N$ and z_a is in $\Xi^{(h)}$,

$$\begin{aligned} u &= \left(\frac{dz}{ds} \right)_{s=z_a} \left(\frac{\pi}{6\rho\phi_a\phi} \right)^{1/2} \xi^{1/6} \left\{ e^{-i\xi_a - i\pi/4} [\gamma_{1,1}^{(h)} \xi^{1/3} J_{-1/3}(\xi) + \gamma_{2,1}^{(h)} \xi^{1/3} J_{1/3}(\xi)] \right. \\ &\quad \left. + e^{i\xi_a + \pi/4} [\gamma_{1,2}^{(h)} \xi^{1/3} J_{-1/3}(\xi) + \gamma_{2,2}^{(h)} \xi^{1/3} J_{1/3}(\xi)] \right\}, \\ (11b) \quad U &= \left(\frac{\pi\phi_a}{6\phi} \right)^{1/2} \xi^{1/6} \left\{ e^{i\xi_a + \pi/4} [\gamma_{1,1}^{(h)} \xi^{1/3} J_{-1/3}(\xi) + \gamma_{2,1}^{(h)} \xi^{1/3} J_{1/3}(\xi)] \right. \\ &\quad \left. + e^{i\xi_a - \pi/4} [\gamma_{1,2}^{(h)} \xi^{1/3} J_{-1/3}(\xi) + \gamma_{2,2}^{(h)} \xi^{1/3} J_{1/3}(\xi)] \right\}. \end{aligned}$$

From these forms certain terms, depending upon the indices, may under certain conditions be omitted as asymptotically negligible in comparison with others. The precise evaluations will be deferred to the points where applications of the formulas are to be made.

1.5. The "associated" Mathieu equation. The *associated* Mathieu equation (2) is obtainable from the equation (1) by substituting in the latter iz in place of the variable z . Its solutions may, therefore, be derived from those discussed above by this simple change of variable. In particular it may be observed that the principal solutions relative to the origin, to be denoted by $v_o(z)$ and $v_e(z)$, are respectively odd and even functions of z , and that they are given by the formulas

$$(12) \quad \begin{aligned} v_o(z) &\equiv -iu_o(iz), \\ v_e(z) &\equiv u_e(iz). \end{aligned}$$

1.6. The solutions for general values of z . The hypotheses stated in §1.2 under which the forms of the solutions of the equation (1) are obtainable through the medium of the equation (3) restrict the variable to a region R_z in which the coefficient $(\Delta - \Omega \cos 2z)$ has at most one zero. It will be found in the subsequent discussion that this region over which the forms are directly deducible is in each case either the strip

$$(13) \quad 0 \leq x \leq \pi/2, \text{ where } z = x + iy,$$

or some closely related domain. It remains, therefore, to consider the extension of the asymptotic representations over the remaining parts of the z plane. A method by which this may be done is to be outlined as follows.

Since the coefficient of the differential equation is an even periodic function with the period π , the function $u(n\pi - z)$ is a solution whenever $u(z)$ is such and n is an integer. Hence each member of the several relations

$$(14) \quad \begin{aligned} (a) \quad u_o(z) &\equiv -u_o(\pi - z) + 2u_o(\pi/2)u_\beta(\pi - z), \\ (b) \quad u_e(z) &\equiv -u_e(\pi - z) + 2u_e(\pi/2)u_\beta(\pi - z), \\ (c) \quad u_o(z) &\equiv u_o(\pi - z) - 2u'_o(\pi/2)u_\alpha(\pi - z), \\ (d) \quad u_e(z) &\equiv u_e(\pi - z) - 2u'_e(\pi/2)u_\alpha(\pi - z) \end{aligned}$$

is a solution of the differential equation. The identities are established, therefore, by the fact that in each relation both members and likewise their derivatives take the same values at the point $z = \pi/2$. A similar comparison of values at the point $z = 2^p\pi$, whatever the integer p , establishes the further relations

$$(15) \quad \begin{aligned} (a) \quad u_o(z) &\equiv -u_o(2^{p+1}\pi - z) + 2u_o(2^p\pi)u_e(z - 2^p\pi), \\ (b) \quad u_e(z) &\equiv u_e(2^{p+1}\pi - z) + 2u'_e(2^p\pi)u_o(z - 2^p\pi). \end{aligned}$$

Let it be supposed now that the forms of the solutions have been deduced and so are known for all values of the variable which lie in the strip (13). It is to be shown then by the method of induction that they are deducible over the strip S_p where p is any integer and S_p is defined by the relation

$$(16) \quad S_p: \quad 0 \leq x \leq 2^p\pi.$$

To begin with, let z lie in the region S_0 . Then either z or $\pi - z$ lies in the strip (13). In the former case the representations of $u_o(z)$ and $u_e(z)$ are known by hypothesis, whereas in the latter they are given by the identities (14) in which the forms of the right-hand members are known. Proceeding, let the

representations be considered known in the region S_p with any specific p , and let z lie in the strip S_{p+1} . Then either z lies in S_p and the forms are already known, or else both the values $(2^{p+1}\pi - z)$ and $(z - 2^p\pi)$ lie in S_p and the forms of the right-hand members of the relations (15) are known. In the latter event the identities furnish the representations sought in the part of S_{p+1} not included in S_p .

Finally the odd and even functional characters of the solutions $u_o(z)$, $u_e(z)$ may be drawn upon to extend their representations into the left-hand half-plane, and with the forms of these solutions at hand the representations of $u_a(z)$ and $u_b(z)$ may be drawn from the identities (14).

1.7. **The characteristic values.** With any given value of Ω there are known to be associated specific *characteristic values* of Δ for which the differential equation (1) admits a periodic solution with the period 2π . These periodic solutions are enumerable, and are each either an odd or an even function of z .^{*} With a scheme of enumeration which will become clear as the subsequent quantitative discussion proceeds, the characteristic values for which the odd solution $u_o(z)$ has the period 2π will be denoted by $S_n(\Omega)$, while those for which the period occurs in the even solution $u_e(z)$ will be designated by $C_n(\Omega)$. The equations of which these values are the roots are called *characteristic equations*.

Consider the characteristic equations for the values $S_n(\Omega)$. From the identity (15a) it is seen at once that a necessary and sufficient condition that 2π be a period of $u_o(z)$ is that $u_o(\pi) = 0$, an equation which in virtue of the relation (14c), with $z = \pi$, may be written

$$u'_o(\pi/2)u_a(0) = 0.$$

If the root in question is one for which the factor $u'_o(\pi/2)$ vanishes, it follows from the identity (14c) that $u_o(z)$ admits no smaller period than 2π . On the other hand, if the root is one for which $u_a(0)$ is zero, then the solutions $u_o(z)$ and $u_a(z)$ are linearly dependent. It follows that $u_o(z)$ vanishes at $z = \pi/2$, and hence from the relation (14a) that $u_o(z)$ admits the period π . With the enumeration to be chosen the characteristic equations for odd periodic solutions are accordingly the following:

$$(17) \quad \begin{aligned} (a) \quad & u_o(\pi/2) = 0, \quad \text{roots } S_{2n}(\Omega), \\ & u_o(z) \text{ periodic with the primitive period } \pi; \\ (b) \quad & u'_o(\pi/2) = 0, \quad \text{roots } S_{2n+1}(\Omega), \\ & u_o(z) \text{ periodic with the primitive period } 2\pi. \end{aligned}$$

^{*} Cf. Whittaker and Watson, loc. cit., §19.2.

The characteristic equations for even solutions may be similarly deduced. Thus from the identity (15b), with $p=0$, the condition that 2π be a period of $u_e(z)$ is seen to be $u_e'(\pi)=0$. From the derived relation (14b), taken at $z=\pi$, the condition is found to be

$$u_e(\pi/2)u_\beta'(0) = 0.$$

If for the root in question $u_e(\pi/2)$ is zero, the identity (14b) shows that a smaller period than 2π is precluded. In the alternative the factor $u_\beta'(0)$ is zero, $u_e(z)$ and $u_\beta(z)$ are dependent and hence $u_e'(z)$ vanishes at $z=\pi/2$. It follows from the relation (14d) then that $u_e(z)$ admits the period π . In this instance, therefore, the characteristic equations are

$$\begin{aligned} (a) \quad & u_e'(\pi/2) = 0, \text{ roots } C_{2n}(\Omega), \\ & u_e(z) \text{ periodic with the primitive period } \pi; \\ (18) \quad (b) \quad & u_e(\pi/2) = 0, \text{ roots } C_{2n+1}(\Omega), \\ & u_e(z) \text{ periodic with the primitive period } 2\pi. \end{aligned}$$

1.8. The Mathieu functions. When Δ is a characteristic value $S_n(\Omega)$ or $C_n(\Omega)$, the corresponding periodic solution $u_o(z)$ or $u_e(z)$ is after suitable normalization known as a Mathieu function, and is respectively designated by $se_n(z, \Omega)$ or $ce_n(z, \Omega)$. Two modes of normalization have been commonly employed. The first* uses the stipulation that the coefficients of $\sin nz$ and $\cos nz$ in the respective Fourier expansions of $se_n(z, \Omega)$ and $ce_n(z, \Omega)$ be unity, i.e.,

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^{\pi} se_n(x, \Omega) \sin nx \, dx &= 1, \\ \frac{1}{\pi} \int_{-\pi}^{\pi} ce_n(x, \Omega) \cos nx \, dx &= 1 + \delta_{0,n}, \quad \delta_{0,n} = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases} \end{aligned}$$

Since the integrands in these relations are even functions, the intervals of integration may, of course, be reduced to $(0, \pi)$. It may, however, be further observed that in virtue of the equations (17), (18), and (14),

$$\begin{aligned} (19) \quad se_n(z, \Omega) &\equiv (-1)^{n+1} se_n(\pi - z, \Omega), \\ ce_n(z, \Omega) &\equiv (-1)^n ce_n(\pi - z, \Omega), \end{aligned}$$

i.e., the Mathieu functions are each either even or odd in the variable $z-\pi/2$. The ranges of integration above may, therefore, be reduced further to $(0, \pi/2)$, the formulas which result being

* Cf. Whittaker and Watson, loc. cit.

$$(20) \quad \begin{aligned} \operatorname{se}_n(z, \Omega) &= \left\{ \frac{\pi u_0(z)}{4 \int_0^{\pi/2} u_0(x) \sin nx \, dx} \right\}_{\Delta=S_n(\Omega)}, \\ \operatorname{ce}_n(z, \Omega) &= \left\{ \frac{\pi u_e(z)}{(4 - 2\delta_{0,n}) \int_0^{\pi/2} u_e(x) \cos nx \, dx} \right\}_{\Delta=C_n(\Omega)}. \end{aligned}$$

A second mode of normalization* is based on the requirements

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{se}_n^2(x, \Omega) \, dx = 1, \quad \frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{ce}_n^2(x, \Omega) \, dx = 1 + \delta_{0,n}.$$

In this case the formulas obtained are

$$(21) \quad \begin{aligned} \operatorname{se}_n(z, \Omega) &= \left\{ \frac{\pi^{1/2} u_0(z)}{2 \left(\int_0^{\pi/2} u_0^2(x) \, dx \right)^{1/2}} \right\}_{\Delta=S_n(\Omega)}, \\ \operatorname{ce}_n(z, \Omega) &= \left\{ \frac{\pi^{1/2} u_e(z)}{2^{1-\delta_{0,n/2}} \left(\int_0^{\pi/2} u_e^2(x) \, dx \right)^{1/2}} \right\}_{\Delta=C_n(\Omega)}. \end{aligned}$$

1.9. Other periodic solutions. The characteristic equations for values of Δ which yield periodic solutions with periods other than π or 2π may be deduced by considerations similar to those of §1.7. The identities

$$(22) \quad \begin{aligned} (a) \quad u(z) &\equiv -u(2p\pi - z) + 2u(p\pi)u_o(p\pi - z), \\ (b) \quad u(z) &\equiv u(2p\pi - z) - 2u'(p\pi)u_o(p\pi - z), \\ (c) \quad u(z) &\equiv -u((2p-1)\pi - z) + 2u((p-\tfrac{1}{2})\pi)u_o(p\pi - z), \\ (d) \quad u(z) &\equiv u((2p-1)\pi - z) - 2u'((p-\tfrac{1}{2})\pi)u_o(p\pi - z). \end{aligned}$$

are easily verified when p is any integer and $u(z)$ is an arbitrary solution of the differential equation. With the use of them it can be shown, as is outlined below, that periodic solutions with the periods indicated occur for values of Δ which are roots of the respective equations

$$(23) \quad u_o(n\pi/2) = 0, \quad \text{odd solutions with period } n\pi,$$

$$u'_o(n\pi/2) = 0, \quad \text{odd solutions with period } 2n\pi,$$

$$(24) \quad u'_e(n\pi/2) = 0, \quad \text{even solutions with period } n\pi,$$

$$u_e(n\pi/2) = 0, \quad \text{even solutions with period } 2n\pi.$$

* Cf. Strutt, loc. cit.

Moreover, if n is the smallest integer for which an equation is satisfied, the period indicated is primitive.

Consider the equations (23). Their sufficiency for the indicated periodicities may be verified by observing that they imply through the pertinent identities (22) respectively that $u_o(z+n\pi) = \pm u_o(z)$. Conversely, if $2n\pi$ is a period of $u_o(z)$, then $u_o(n\pi) = 0$ by the identity (22a), and this leads when n is even through the relation (22b) to the one or the other of the equations (23). If n is odd the result follows from the identities (22c) and (22d), together with the fact that at least one of the solutions $u_a(z)$ and $u_b(z)$ must differ from zero at the point $z = (n+1)\pi/2$.

The necessity and sufficiency of the equations (24) for solutions of their associated types is proved similarly, though in some instances the identities (22) must be differentiated prior to their application.

1.10. The characteristic exponent. When the parameters Δ and Ω are both fixed the differential equation in general admits no periodic solution. In this case it is known from Floquet's theory of differential equations with simply periodic coefficients that there are two solutions of the forms

$$e^{\mu z}\phi(z), \text{ and } e^{-\mu z}\phi(-z),$$

in which $\phi(z)$ is a periodic function with the period π , while μ , the so called *characteristic exponent*, is a constant which depends upon Δ and Ω . The equation for μ is*

$$e^{2\pi\mu} - 2\Theta e^{\pi\mu} + 1 = 0,$$

whence

$$(25a) \quad \mu = \frac{1}{\pi} \cosh^{-1} \Theta = \frac{i}{\pi} \cos^{-1} \Theta,$$

with $\Theta = u_e(\pi)$. The alternative evaluation

$$(25b) \quad \Theta = 2u_e(\pi/2)u_b(0) - 1$$

may be obtained from the relation (14b).

It is evident that μ is either real or pure imaginary according as $\Theta > 1$ or $\Theta < 1$. In the former case the solutions noted above become infinite near the one or the other extremity of the axis of reals and are called unstable; in the latter case they remain bounded for real values of z and are called stable.

1.11. Certain elliptic integrals. It will be found now and again in the discussion which follows, that the comparison and identification of certain

* Cf. Horn, J., *Gewöhnliche Differentialgleichungen*, Leipzig, 1905, p. 242.

superficially dissimilar formulas will depend upon the approximate or asymptotic evaluation of certain elliptic integrals of the type

$$(26) \quad G(\tau, h^2) = \int_0^{\pi/2} \frac{1 - \tau \sin^2 \zeta}{\{1 - h^2 \sin^2 \zeta\}^{1/2}} d\zeta.$$

The value of h will in every case be either near zero or near 1, and τ will be either 1 or h^2 .

In terms of the standard complete elliptic integrals

$$K = \int_0^{\pi/2} \frac{d\zeta}{\{1 - h^2 \sin^2 \zeta\}^{1/2}}, \quad E = \int_0^{\pi/2} \{1 - h^2 \sin^2 \zeta\} d\zeta,$$

it is evident that

$$G(\tau, h^2) = K + \frac{\tau}{h^2}(E - K).$$

Hence on substituting for these integrals their expansions in powers of h , it is found that when h^2 is nearly zero

$$(26a) \quad \begin{aligned} G(1, h^2) &= \frac{\pi}{4} \left\{ 1 + \frac{h^2}{8} + h^4 O(1) \right\}, \\ G(h^2, h^2) &= \frac{\pi}{2} \left\{ 1 - \frac{h^2}{4} + h^4 O(1) \right\}. \end{aligned}$$

On the other hand, when h^2 is nearly 1 the Landen Transformation*

$$h \sin \zeta = \sin (2t - \zeta)$$

yields the form

$$G(\tau, h^2) = \frac{-\tau}{h} + \frac{2}{1+h} \int_0^{t_1} \frac{\left(1 - \frac{\tau}{h}\right) + \frac{2\tau}{h} \cos^2 t}{\cos t \{1 + \epsilon^2 \tan^2 t\}^{1/2}} dt,$$

in which

$$(27) \quad t_1 = \sin^{-1} \left\{ \frac{1+h}{2} \right\}^{1/2}, \quad \epsilon = \frac{1-h}{1+h}.$$

The quantity $\epsilon^2 \tan^2 t$ is uniformly small of the order of ϵ . Hence the radical may be replaced by its binomial expansion, whereupon the integration leads to the formula

* Cf. Hancock, H., *Elliptic Integrals*, New York, 1917, p. 84.

$$G(\tau, h^2) = \frac{-\tau}{h} + \left(\frac{2}{1+h}\right) \left(\frac{2\tau \sin t_1}{h} - \frac{\epsilon^2(h-\tau) \sin t_1}{4h \cos^2 t_1} \right) \\ + \left\{ \left(\frac{h-\tau}{2h} \right) + \frac{\epsilon^2}{8} \left(1 - \frac{5\tau}{h} \right) \right\} \log \frac{1 + \sin t_1}{1 - \sin t_1} + o(\epsilon^2).$$

For the special values of τ this reduces to

$$(26b) \quad G(1, h^2) = \frac{-1}{h} + \frac{2}{h} \left(\frac{2}{1+h} \right)^{1/2} + \frac{1-h}{h(1+h)} \log \frac{1-h}{8} + O(\epsilon^2 \log \epsilon), \\ G(h^2, h^2) = -h + 2h \left(\frac{2}{1+h} \right)^{1/2} - \frac{1-h}{1+h} \log \frac{1-h}{8} + O(\epsilon^2 \log \epsilon).$$

CHAPTER 2

THE CONFIGURATION II

2.1. The differential equation. When the relative values of the parameters Δ and Ω are such that the point (Ω, Δ) in Figure 1 lies in the region II at a sufficient distance from O , i.e., more specifically when Δ is large and positive, and with a constant M_1 (to be specified below) the relation

$$(2.1) \quad 0 \leq \Omega \leq \frac{1}{M_1} \Delta$$

is fulfilled, the substitutions

$$(2.2) \quad \rho = \Delta^{1/2}, \quad \sigma^2 = \Omega/\Delta, \quad s = z^*$$

give to the equation (1) the form (3) with

$$(2.3) \quad \chi_0 \equiv \phi, \quad \chi_1 \equiv 0, \\ \phi^2 \equiv 1 - \sigma^2 \cos 2s.$$

Let the variable z be restricted to any finite region of the complex plane. Then a number M_1 may be determined such that for all admitted values of z

$$(2.4a) \quad |y| \leq \frac{1}{2} \cosh^{-1} \frac{M_1}{2}, \quad z = x + iy.$$

The constant M_1 of the relation (2.1), which determines the parameter values to be included in the present configuration, is to be one with which the condition (2.4a) is fulfilled. The primary parameter ρ is to be thought of as

* The distinction between s and z , which in the present instance is non-existent, is drawn for the purpose of making the formulas subsequently useful in a case when these variables are not the same.

bounded below but not above, and the secondary parameter σ is evidently restricted to the range

$$(2.5) \quad 0 \leq \sigma^2 \leq \frac{1}{M_1}.$$

The relation (2.4a), together with

$$(2.4b) \quad 0 \leq x \leq \frac{\pi}{2},$$

defines a strip of the z plane which is to be designated as R_s . The corresponding domain of the variable s is

$$(2.6) \quad R_s: \quad 0 \leq s' \leq \pi/2, \quad |s''| \leq \frac{1}{2} \cosh^{-1}(M_1/2), \quad s = s' + is''.$$

This region includes the origin and it is readily verified that with $s_0=0$ the hypothesis (i) of §1.2 is fulfilled uniformly in σ with $\nu=0$. The hypotheses (ii) and (iii) are likewise fulfilled, since $\chi_1=0$ and R_s is bounded. From the formulas (5) it is seen that in the present instance $\eta(s) \equiv \omega_1(s) \equiv k=0$, in consequence of which

$$\omega(\phi) \equiv 1 + \frac{1}{4\phi^2} - \frac{5(1-\sigma^4)}{4\phi^4}, \quad \Psi \equiv \phi^{-1/2}.$$

These functions are bounded uniformly in σ and hence the requirements enumerated in §1.2 are completely fulfilled.

2.2. The solutions. Since the case in hand is one in which $\nu=0$, there exist solutions of the differential equation which maintain a single asymptotic form over the entire region R_s . Such solutions with their respective forms are

$$(2.7) \quad \begin{aligned} u_{0,1}(s) &= \phi^{-1/2} e^{i\xi} [1], \\ u_{0,2}(s) &= \phi^{-1/2} e^{-i\xi} [1]. \end{aligned}$$

Their Wronskian has the value $W = [2i]\rho$. The principal solutions relative to the point $z=0$ are accordingly computed directly from the formula (8b), with $h=0$, $s_a=0$, to be

$$(2.8) \quad \begin{aligned} u_o(z) &= \frac{1}{2i} \left\{ \frac{1}{\rho^2 \phi_1 \phi} \right\}^{1/2} \{ e^{i\xi} [1] - e^{-i\xi} [1] \}, \\ u_e(z) &= \frac{1}{2} \left\{ \frac{\phi_1}{\phi} \right\}^{1/2} \{ e^{i\xi} [1] + e^{-i\xi} [1] \}, \end{aligned}$$

with

$$(2.8a) \quad \begin{aligned} \rho\phi &= \{\Delta - \Omega \cos 2z\}^{1/2}, \quad \rho\phi_1 = \{\Delta - \Omega\}^{1/2}, \\ \xi &= \int_0^z \{\Delta - \Omega \cos 2z\}^{1/2} dz. \end{aligned}$$

Inasmuch as

$$e^{i\xi}[1] - e^{-i\xi}[1] = [2i] \sin [\xi],$$

with analogous formulas involving the other trigonometric functions, it is seen in particular that for real values of the variable

$$\begin{aligned} u_a(x) &= \frac{[1]}{\{(\Delta - \Omega)(\Delta - \Omega \cos 2x)\}^{1/4}} \sin \left[\int_0^x \{\Delta - \Omega \cos 2x\}^{1/2} dx \right], \\ (2.8b) \quad u_b(x) &= \left\{ \frac{\Delta - \Omega}{\Delta - \Omega \cos 2x} \right\}^{1/4} [1] \cos \left[\int_0^x \{\Delta - \Omega \cos 2x\}^{1/2} dx \right]. \end{aligned}$$

The principal solutions relative to $z = \pi/2$ are similarly found to be given by the formulas

$$\begin{aligned} (2.9) \quad u_a(z) &= \frac{1}{2i} \left\{ \frac{1}{\rho^2 \phi_2 \phi} \right\}^{1/2} \{ e^{i(\xi - \xi_2)} [1] - e^{-i(\xi - \xi_2)} [1] \}, \\ u_b(z) &= \frac{1}{2} \left\{ \frac{\phi_2}{\phi} \right\}^{1/2} \{ e^{i(\xi - \xi_2)} [1] + e^{-i(\xi - \xi_2)} [1] \}, \end{aligned}$$

with

$$(2.9a) \quad \rho \phi_2 = \{\Delta + \Omega\}^{1/2}, \quad \xi - \xi_2 = \int_{\pi/2}^z \{\Delta - \Omega \cos 2z\}^{1/2} dz.$$

When z is real they are

$$\begin{aligned} (2.9b) \quad u_a(x) &= \frac{-[1]}{\{(\Delta + \Omega)(\Delta - \Omega \cos 2x)\}^{1/4}} \sin \left[\int_x^{\pi/2} \{\Delta - \Omega \cos 2x\}^{1/2} dx \right], \\ u_b(x) &= \left\{ \frac{\Delta + \Omega}{\Delta - \Omega \cos 2x} \right\}^{1/4} [1] \cos \left[\int_x^{\pi/2} \{\Delta - \Omega \cos 2x\}^{1/2} dx \right]. \end{aligned}$$

In the special case that $\sigma=0$ (i.e., $\Omega=0$) the differential equation (1) is directly integrable, and it is verified immediately that the formulas above are correct when the symbols $[\quad]$ are omitted. It may be concluded, therefore, in the discussion of this chapter that the quantities $[1]$ reduce to 1 when $\sigma^2=0$.

2.3. The solutions of the associated Mathieu equation. The principal solutions of the associated Mathieu equation (2) relative to the origin may be derived from the functions (2.8) by the substitutions (12) as was noted in §1.5. Their forms so obtained are

$$\begin{aligned} (2.10) \quad v_a(z) &= \frac{[1]}{\{(\Delta - \Omega)(\Delta - \Omega \cosh 2z)\}^{1/4}} \sinh \left[\int_0^z \{\Delta - \Omega \cosh 2z\}^{1/2} dz \right], \\ v_b(z) &= \left\{ \frac{\Delta - \Omega}{\Delta - \Omega \cosh 2z} \right\}^{1/4} [1] \cosh \left[\int_0^z \{\Delta - \Omega \cosh 2z\}^{1/2} dz \right], \end{aligned}$$

the region for z being

$$|x| \leq \frac{1}{2} \cosh^{-1} \frac{M_1}{2},$$

$$-\pi/2 \leq y \leq 0.$$

The solutions (2.10) are evidently asymptotically multiples of each other when z is real and large. A pair, $v_7(z)$, $v_8(z)$, not subject to this disadvantage is that obtainable by the substitution of iz for s from the functions (2.7). Their forms are explicitly

$$(2.11) \quad v_7(z) = \frac{[1]}{\{\Delta - \Omega \cosh 2z\}^{1/4}} \exp \left[- \int_0^z \{\Delta - \Omega \cosh 2z\}^{1/2} dz \right],$$

$$v_8(z) = \frac{[1]}{\{\Delta - \Omega \cosh 2z\}^{1/4}} \exp \left[\int_0^z \{\Delta - \Omega \cosh 2z\}^{1/2} dz \right].$$

2.4. The characteristic values. If $S_p(\Omega)$ and $C_q(\Omega)$ are a pair of characteristic values, the substitution of the forms (2.8b) into the characteristic equations (17) and (18) shows that each of these values is a root of an equation

$$(2.12) \quad \left[\int_0^{\pi/2} \{\Delta - \Omega \cos 2x\}^{1/2} dx \right] = \frac{n\pi}{2},$$

with the integer n suitably adjusted to p or q as the case may be. To determine this adjustment, it need merely be observed that when $\Omega=0$ the equation reduces to $\Delta=n^2$, and the corresponding Mathieu functions to $\sin nx$ and $\cos nx$. Since these are by definition the forms of $se_n(z, 0)$ and $ce_n(z, 0)$, it must be concluded that $p=n$ and $q=n$, i.e., the form (2.12) is that of the characteristic equation both for $S_n(\Omega)$ and for $C_n(\Omega)$.

The symbol $[\]$ in the equation (2.12) represents a quantity of the order of $\Delta^{-1/2}$ uniformly in σ , which vanishes when $\sigma=0$. Since it like the equation (1) depends analytically upon σ^2 , the equation (2.12) may be written

$$\int_0^{\pi/2} \{\Delta - \Omega \cos 2x\}^{1/2} dx + \sigma^2 O(\Delta^{-1/2}) = \frac{n\pi}{2}.$$

The substitution $x=\pi/2-\zeta$ reduces this to

$$\Delta^{1/2} \{ (1 + \sigma^2)^{1/2} G(h^2, h^2) + \sigma^2 O(\Delta^{-1/2}) \} = \frac{n\pi}{2},$$

where G is the elliptic integral of (26) with $h^2 = 2\sigma^2/(1+\sigma^2)$. Since this value of h^2 is small, the evaluation (26a) gives to the equation the form

$$\Delta^{1/2} \{1 + \sigma^4 O(1) + \sigma^2 O(\Delta^{-1})\} = n,$$

from which it follows that

$$(2.13) \quad \begin{aligned} S_n(\Omega) &= n^2 + \frac{\Omega}{n^2} O(1), \\ C_n(\Omega) &= n^2 + \frac{\Omega}{n^2} O(1), \end{aligned}$$

the quantities indicated by the symbols $O(1)$ being uniformly bounded as to n and Ω while the configuration with which the present chapter deals is maintained.

2.5. The characteristic exponent. The substitution into the formula (25b) of the values given by (2.8b) and (2.9b) yields the evaluation

$$\begin{aligned} \Theta &= [2] \cos [\xi_2] \cos [\xi_2] - 1 \\ &= \cos 2\xi_2 + \sigma^2 O(\Delta^{-1/2}). \end{aligned}$$

Accordingly, from (25a) an asymptotic formula for the characteristic exponent is

$$(2.14) \quad u = \frac{i}{\pi} \cos^{-1} \left\{ \cos \left(\int_0^{\pi/2} 2 \{ \Delta - \Omega \cos 2x \}^{1/2} dx \right) + \frac{\Omega^2}{\Delta^{3/2}} O(1) \right\}.$$

When $\Omega=0$ this reduces to $\mu = i\Delta^{1/2}$, a result which may be verified by actual integration of the differential equation.

Inasmuch as the quantity within the brace in the formula (2.14) does not exceed unity, except possibly for very small ranges of the parameters near those values for which the integral is a multiple of π , it follows that the configuration under consideration in this chapter is predominantly one of stable solutions.*

CHAPTER 3

THE CONFIGURATION III

3.1. Definitions. The parameter configuration contiguous with that of the preceding chapter and designated by III in Figure 1 is to be defined by the relation

$$(3.1) \quad \frac{1}{M_1} \Delta \leq \Omega \leq \Delta - M_2 \Delta^{1/2},$$

* Cf. the Figure 3 in Strutt, loc. cit.

in which M_1 is the constant in (2.1), and M_2 is to be momentarily discussed. The substitutions

$$(3.2) \quad \rho = \frac{\Delta - \Omega}{\Delta^{1/2}}, \quad \sigma^2 = 1 - \frac{\Omega}{\Delta}, \quad s = \frac{-iz}{\sigma}$$

reduce the differential equation (1) in this case to the form (3) with

$$(3.3) \quad \begin{aligned} \chi_0 &\equiv \phi, & \chi_1 &\equiv 0, \\ \phi^2 &\equiv 2(1 - \sigma^2) \frac{\sinh^2 \sigma s}{\sigma^2} - 1. \end{aligned}$$

The parameter ρ is evidently restricted by the relation $\rho \geq M_2$, and since the degree of approximation which the asymptotic formulas yield depends upon the magnitude of ρ , the constant M_2 is in any specific case to be chosen such that representations which are uniformly suitable to the purposes intended are obtained. The secondary parameter is clearly confined to the fixed closed interval

$$(3.4) \quad 0 \leq \sigma^2 \leq 1 - \frac{1}{M_1},$$

in which the lower boundary could in fact more strictly be replaced by $M_2 \Delta^{-1/2}$.

Let z be restricted for the discussion of this configuration to the infinite half-strip R_s given by the formulas

$$(3.5) \quad R_s: \quad -\pi/2 \leq x \leq \pi/2, \quad 0 \leq y.$$

The extension of the solutions from this domain to the entire strip (13) may be accomplished by the use of the identities

$$\begin{aligned} u_a(z) &\equiv u_a(-z) - 2u_a'(0)u_o(-z), \\ u_b(z) &\equiv -u_b(-z) + 2u_b(0)u_e(-z), \end{aligned}$$

and the odd and even characters of $u_o(z)$ and $u_e(z)$. Their extension to general values of z thereupon follows on the lines of §1.6.

3.2. The variables s , Φ and ξ . The region R_s corresponding to R_s is the infinite half-strip

$$(3.6) \quad R_s: \quad 0 \leq s', \quad -\frac{\pi}{2\sigma} \leq s'' \leq \frac{\pi}{2\sigma}.$$

Within this region $\chi_0^2(s)$ has a single zero located on the axis of reals at the point

$$(3.7) \quad s_0' \equiv \frac{1}{\sigma} \sinh^{-1} \frac{\sigma}{\{2(1 - \sigma^2)\}^{1/2}}.$$

Though s'_0 depends upon σ it is both bounded and bounded from zero for all admitted values of the parameters.

The relation between s and the quantity Φ maps R_s upon a corresponding region R_Φ conformally except at the point s'_0 . The shape of R_Φ may be easily determined by observing the values of Φ when s is either real or on the boundaries of R_s . With R_s thought of as cut along the axis of reals from the origin to s'_0 these values for the upper half of R_s are

for $s'' = 0 +$ and $0 \leq s' \leq s'_0$,

$$\Phi = e^{\pi i} \int_{s'_0}^{s'} i \left\{ 1 - 2(1 - \sigma^2) \frac{\sinh^2 \sigma s'}{\sigma^2} \right\}^{1/2} ds';$$

for $s' = 0$ and $0 \leq s'' \leq \pi/(2\sigma)$,

$$\Phi = \Phi(0) + e^{\pi i/2} \int_0^{s''} i \left\{ 2(1 - \sigma^2) \frac{\sin^2 \sigma s''}{\sigma^2} + 1 \right\}^{1/2} ds'';$$

for $s'' = 0$ and $s'_0 \leq s'$,

$$\Phi = \int_{s'_0}^{s'} \left\{ 2(1 - \sigma^2) \frac{\sinh^2 \sigma s'}{\sigma^2} - 1 \right\}^{1/2} ds';$$

for $s'' = \pi/(2\sigma)$ and $0 \leq s'$,

$$\Phi = \Phi\left(\frac{\pi i}{2\sigma}\right) + \int_0^{s'} i \left\{ 2(1 - \sigma^2) \frac{\cosh^2 \sigma s'}{\sigma^2} + 1 \right\}^{1/2} ds'.$$

The map of the lower half of R_s is obtainable by reflection from that of the upper half, since conjugate complex values of s lead to conjugate values of Φ .

Finally since

$$\left| \frac{\sinh \sigma s}{\sigma} \right| > \frac{2}{\pi} |s|,$$

it follows that when $|s|$ is sufficiently large

$$\begin{aligned} \phi &\sim \{2(1 - \sigma^2)\}^{1/2} \frac{\sinh \sigma s}{\sigma}, \\ (3.8) \quad \Phi &\sim \frac{2}{\sigma^2} \{2(1 - \sigma^2)\}^{1/2} \sinh^2 \frac{\sigma s}{2}, \end{aligned}$$

the symbolism designating that the ratio of the members of either relation becomes 1 as $|s| \rightarrow \infty$. From the second relation it follows that when c is any sufficiently large constant the line $s' = c$ maps upon a simple curve in R_Φ . The uniqueness of the correspondence between points of R_s and R_Φ is thereby assured.* Figure 2 indicates the map.

* Cf. Osgood, W. F., *Lehrbuch der Funktionentheorie*, vol. 1, Leipzig, 1912, p. 377.

The variables Φ and ξ differ only by the real factor ρ , whence the domains R_ξ and R_Φ differ only in scale. Figure 3 indicates the relation between R_ξ

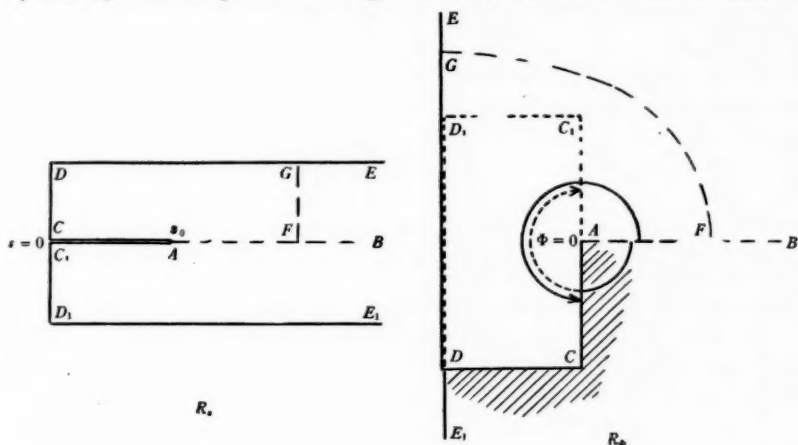


FIG. 2

and R_ξ , each domain being divided into the sub-regions $\Xi^{(i)}$ defined in (7). The lines by which this sub-division is effected need not be determined with

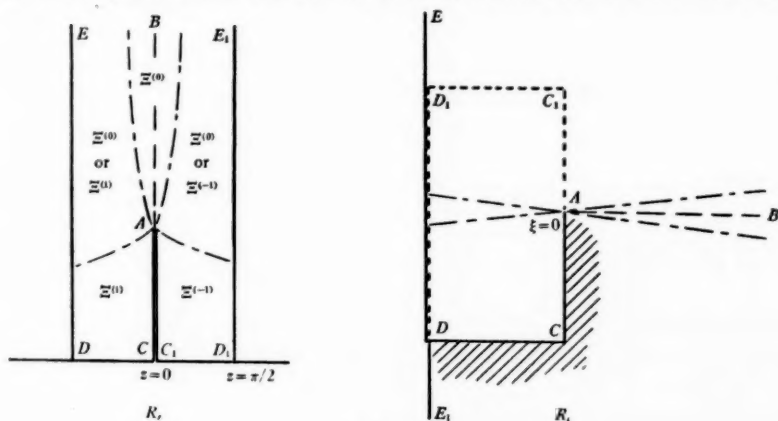


FIG. 3

precision, for due to the overlapping of the regions any displacement of the curves which does not affect the character of the figure is immaterial.

3.3. **Fulfillment of the hypotheses.** The zero of $\chi_0^2(s)$ at s_0' is of the first order. Hence in the hypotheses of §1.2 the values $\nu=1$, $\eta=\omega_1=k=0$ are to

be used. With the value of ϕ given by the formula (3.3) it is found that the functions (6) are in the present instance

$$(3.9) \quad \omega(\phi) = \frac{1}{4} \left\{ -\frac{5}{9} \left(\frac{\phi}{\Phi} \right)^2 + \sigma^2 + \frac{6}{\phi^2} + \frac{5(2 - \sigma^2)}{\phi^4} \right\}, \quad \Psi = \Phi^{1/6} / \phi^{1/2}.$$

Let the region R_s be divided into three parts by the relations

$$(a) \quad |s - s'_0| \leq \delta,$$

$$(b) \quad \delta \leq |s - s'_0| \leq H,$$

$$(c) \quad H \leq |s - s'_0|,$$

with the constants δ and H as specified in the following. It is to be shown that in each of these parts the hypotheses of §1.2 are uniformly fulfilled.

To begin with let H be chosen so large that in the part (c) the formulas (3.8) may be applied. Then it is a matter of simple computation to show that in this part of R_s the hypotheses (i) and (iii) are uniformly fulfilled.

Next let δ be chosen so small that within the part (a), $|\phi^2| \leq \frac{1}{2}$ for all admitted values of σ . Then

$$\left\{ 1 + \frac{\sigma^2 \phi^2}{2 - \sigma^2} \right\}^{-1/2} \equiv 1 - \frac{\sigma^2 \phi^2}{2(2 - \sigma^2)} + \phi^4 O(1),$$

$$(1 + \phi^2)^{-1/2} \equiv 1 - \frac{\phi^2}{2} + \phi^4 O(1),$$

with $O(1)$ designating functions which are uniformly bounded. Since

$$\Phi = \int_0^\phi \frac{\phi}{\phi'} d\phi,$$

whereas from the formula (3.3)

$$\frac{\phi}{\phi'} = \frac{\phi^2}{(2 - \sigma^2)^{1/2}} \left\{ (1 + \phi^2) \left(1 + \frac{\sigma^2 \phi^2}{2 - \sigma^2} \right) \right\}^{-1/2},$$

it is found that

$$\Phi = \frac{\phi^3}{3(2 - \sigma^2)^{1/2}} \left\{ 1 - \frac{3\phi^2}{5(2 - \sigma^2)} + \phi^4 O(1) \right\}.$$

With this evaluation it is seen directly that in the part (a) the functions (3.9) are uniformly bounded.

Lastly in the part (b) the formula (3.3) may be written

$$\chi_s^2(s) = 2 \left\{ (2 - \sigma^2)^{1/2} \frac{\sinh 2\sigma(s - s'_0)}{2\sigma} + \frac{\sinh^2 \sigma(s - s'_0)}{\sigma^2} \right\}.$$

It is evident from this that both

$$\chi_0(s) \quad \text{and} \quad \int_{s_0}^s \chi_0(s) ds$$

are non-vanishing and continuous as functions of the two variables $(s-s_0', \sigma)$ in the closed region determined by (b) and (3.4). Accordingly, they are bounded uniformly in σ and the hypothesis (i) is uniformly fulfilled. Clearly also the functions (3.9) are uniformly bounded and so the requirements of §1.2 upon the differential equation are uniformly met.

3.4. **The forms of the solutions.** Since $\phi^2(s)$ has a simple zero in R , the asymptotic representation of any solution of the differential equation is subject to the Stokes' phenomenon, and ν being 1 the formulas of §1.4 are applicable. From Figure 3 it is seen that the origin $z=0$ may be regarded as lying in the sub-region $\Xi^{(-)}$. Hence with $h=-1$ and the subscript a replaced by 1 the formulas (11a) and (11b) yield the representations of the solutions $u_0(z)$ and $u_e(z)$. It may be observed from Figure 3, however, that the value ξ_1 which corresponds to $z=0$ (at C_1 in the figure) is such that $i\xi_1$ is real and negative, so that any quantity multiplied by $e^{i\xi_1}$ is asymptotically negligible in comparison with the same multiplied by $e^{-i\xi_1}$. With the omission of such negligible terms the formulas obtained are the following:

When z is in $\Xi^{(l)}$, and $|\xi| \geq N$,

$$(3.10) \quad \begin{aligned} u_0(z) &= \frac{1}{2} \left(\frac{\sigma^2}{\rho^2 \phi_1 \phi} \right)^{1/2} \{ K_{0,1}^{-1,l} e^{i\xi} + K_{0,2}^{-1,l} e^{-i\xi} \}, \\ u_e(z) &= \frac{1}{2} \left(\frac{\phi_1}{\phi} \right)^{1/2} \{ K_{e,1}^{-1,l} e^{i\xi} + K_{e,2}^{-1,l} e^{-i\xi} \}, \end{aligned}$$

with coefficients

| l | - 1 | 0 | 1 |
|------------------|-------------------|-------------------|-------------------|
| $K_{0,1}^{-1,l}$ | $e^{-i\xi_1}[1]$ | $e^{-i\xi_1}[1]$ | $-ie^{i\xi_1}[1]$ |
| $K_{0,2}^{-1,l}$ | $-ie^{i\xi_1}[1]$ | $ie^{-i\xi_1}[1]$ | $ie^{-i\xi_1}[1]$ |
| $K_{e,1}^{-1,l}$ | $e^{-i\xi_1}[1]$ | $e^{-i\xi_1}[1]$ | $ie^{i\xi_1}[1]$ |
| $K_{e,2}^{-1,l}$ | $e^{i\xi_1}[1]$ | $ie^{-i\xi_1}[1]$ | $ie^{-i\xi_1}[1]$ |

(3.10a)

When $|\xi| \leq N$,

$$\begin{aligned} u_o(z) &= \left(\frac{\pi i \sigma^2}{6 \rho^2 \phi_1 \phi} \right)^{1/2} \xi^{1/6} e^{-i\xi_1} [\xi^{1/3} J_{-1/3}(\xi) + \xi^{1/3} J_{1/3}(\xi)], \\ (3.10b) \quad u_e(z) &= \left(\frac{\pi i \phi_1}{6 \phi} \right)^{1/2} \xi^{1/6} e^{-i\xi_1} [\xi^{1/3} J_{-1/3}(\xi) + \xi^{1/3} J_{1/3}(\xi)]. \end{aligned}$$

In the original variables

$$\begin{aligned} \frac{\rho \phi}{\sigma} &= \{\Omega \cos 2z - \Delta\}^{1/2}, & \frac{\rho \phi_1}{\sigma} &= e^{-\pi i/2} \{\Delta - \Omega\}^{1/2}, \\ \xi &= -i \int_{y_0}^* \{\Omega \cos 2z - \Delta\}^{1/2} dz, & \xi_1 &= i \int_0^{y_0} \{\Delta - \Omega \cosh 2y\}^{1/2} dy, \end{aligned}$$

with $y_0 = \frac{1}{2} \cosh^{-1} \Delta / \Omega$. Further, it may be noted that since the values of ϕ on the lines AC and AC_1 in Figure 3 differ only in sign, therefore

$$\xi_1 = -\frac{i\rho}{\sigma} \int_A^{C_1} \phi dz = \frac{i\rho}{\sigma} \int_A^C \phi dz,$$

whence the formulas

$$\int_0^* \{\Omega \cos 2z - \Delta\}^{1/2} dz = \begin{cases} i(\xi - \xi_1), & \text{in } \Xi^{(-1)}, \\ i(\xi + \xi_1), & \text{in } \Xi^{(1)} \end{cases}$$

are also valid provided the entire path of integration is taken in each case in the sub-region indicated.

The formulas (11a), (11b) may likewise be drawn upon to give the representations of the solutions $u_a(z)$, $u_b(z)$. If the point corresponding to $z = \pi/2$ is s_2 , the subscript a is to be replaced by 2, and since ξ_2 (at D_1 in Figure 3) lies in the region $\Xi^{(-1)}$, h is again to be taken as -1 . With the omission of asymptotically negligible terms the formulas obtained are the following:

When z is in $\Xi^{(1)}$, and $|\xi| \geq N$,

$$\begin{aligned} u_a(z) &= \frac{1}{2} \left(\frac{\sigma^2}{\rho^2 \phi_2 \phi} \right)^{1/2} \{ K_{\alpha,1}^{-1,l} e^{i\xi} + K_{\alpha,2}^{-1,l} e^{-i\xi} \}, \\ (3.11) \quad u_b(z) &= \frac{1}{2} \left(\frac{\phi_2}{\phi} \right)^{1/2} \{ K_{\beta,1}^{-1,l} e^{i\xi} + K_{\beta,2}^{-1,l} e^{-i\xi} \}, \end{aligned}$$

with coefficients

(3.11a)

| | - 1 | 0 | 1 |
|-----------------------|------------------|-------------------|-------------------|
| $K_{\alpha,1}^{-1,l}$ | $e^{-i\xi_1}[1]$ | $e^{-i\xi_1}[1]$ | $-ie^{i\xi_1}[1]$ |
| $K_{\alpha,2}^{-1,l}$ | $-e^{i\xi_1}[1]$ | $ie^{-i\xi_1}[1]$ | $ie^{-i\xi_1}[1]$ |
| $K_{\beta,1}^{-1,l}$ | $e^{-i\xi_1}[1]$ | $e^{-i\xi_1}[1]$ | $ie^{i\xi_1}[1]$ |
| $K_{\beta,2}^{-1,l}$ | $e^{i\xi_1}[1]$ | $ie^{-i\xi_1}[1]$ | $ie^{-i\xi_1}[1]$ |

When $|\xi| \leq N$,

(3.11b)

$$u_\alpha(z) = \left(\frac{\pi i \sigma^2}{6 \rho^2 \phi_2 \phi} \right)^{1/2} \xi^{1/6} e^{-i\xi_1} [\xi^{1/3} J_{-1/3}(\xi) + \xi^{1/3} J_{1/3}(\xi)],$$

$$u_\beta(z) = \left(\frac{\pi i \phi_2}{6 \phi} \right)^{1/2} \xi^{1/6} e^{-i\xi_1} [\xi^{1/3} J_{-1/3}(\xi) + \xi^{1/3} J_{1/3}(\xi)].$$

Again

$$\frac{\rho \phi_2}{\sigma} = e^{-\pi i/2} \{\Delta + \Omega\}^{1/2},$$

$$\xi_2 = \xi_1 - \int_0^{\pi/2} \{\Delta - \Omega \cos 2x\}^{1/2} dx.$$

Figure 3 shows that the segments $-\pi/2 \leq x \leq 0$ and $0 \leq x \leq \pi/2$ of the axis of reals lie respectively in the sub-regions $\Xi^{(1)}$ and $\Xi^{(-1)}$. The formulas above appropriate to these regions accordingly yield the descriptions of the solutions when z is real. It is found that these formulas are precisely those given in (2.8b) and (2.9b), though it should be noted that with the difference in the definition of the parameter ρ the significance of symbol $[]$ is slightly different in this chapter from that in the preceding one.

The pairs of solutions (3.10) and (3.11) have each the defect that in the region about the upper part of the axis of imaginaries the component solutions are asymptotically multiples of each other. The pair of solutions $u_{-1,1}$, $u_{-1,2}$ given in (9) would be one not subject to this particular shortcoming.

3.5. The solutions of the associated Mathieu equation. If z lies in any of the domains indicated in Figure 4, the point iz lies in the corresponding

sub-region of R , as shown in Figure 3. In accordance with (12) the representations of $iv_0(z)$ and $v_*(z)$ are therefore obtainable in any one of the regions

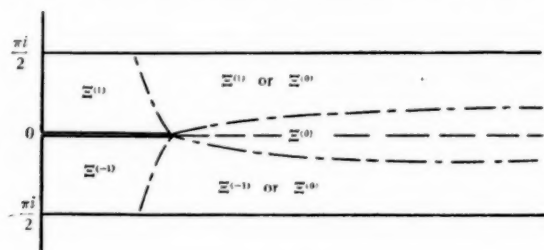


FIG. 4

indicated by the mere substitution in the associated formulas (3.10) of $\bar{\phi}$ and $\bar{\xi}$ in place of ϕ and ξ , the former being the same functions of iz as the latter are of z . Explicitly

$$\frac{\rho\bar{\phi}}{\sigma} = \{\Omega \cosh 2z - \Delta\}^{1/2},$$

$$\bar{\xi} = \int_{x_0}^x \{\Omega \cosh 2z - \Delta\}^{1/2} dz, \quad x_0 = \frac{1}{2} \cosh^{-1} \Delta/\Omega.$$

In particular, for real values of the variable the formulas so obtained are the following:

For $0 \leq x < x_0$, $|\bar{\xi}| \geq N$,

$$(3.12a) \quad v_0(x) = \frac{[1]}{\{(\Delta - \Omega)(\Delta - \Omega \cosh 2x)\}^{1/4}} \sinh \left[\int_0^x \{\Delta - \Omega \cosh 2x\}^{1/2} dx \right],$$

$$v_*(x) = \left\{ \frac{\Delta - \Omega}{\Delta - \Omega \cosh 2x} \right\}^{1/4} [1] \cosh \left[\int_0^x \{\Delta - \Omega \cosh 2x\}^{1/2} dx \right].$$

For $x \leq x_0$, $|\bar{\xi}| \leq N$,

$$(3.12b) \quad v_0(x) = \frac{(2\pi)^{-1/2} |\bar{\xi}|^{1/6} e^{-i\bar{\xi}_1}}{\{(\Delta - \Omega)(\Delta - \Omega \cosh 2x)\}^{1/4}} [|\bar{\xi}|^{1/3} K_{1/3}(|\bar{\xi}|)].$$

For $x_0 \leq x$, $|\bar{\xi}| \leq N$,

$$(3.12c) \quad v_*(x) = \frac{(\pi/6)^{1/2} \bar{\xi}^{1/6} e^{-i\bar{\xi}_1}}{\{(\Delta - \Omega)(\Omega \cosh 2x - \Delta)\}^{1/4}} [\bar{\xi}^{1/3} J_{-1/3}(\bar{\xi}) + \bar{\xi}^{1/3} J_{1/3}(\bar{\xi})].$$

For $x < x_0$, $|\bar{\xi}| \geq N$,

$$(3.12d) \quad v_0(x) = \frac{[1]e^{-i\bar{\xi}_1}}{\{(\Delta - \Omega)(\Omega \cosh 2x - \Delta)\}^{1/4}} \cos \left[\int_{x_0}^x \{\Omega \cosh 2x - \Delta\}^{1/2} dx - \frac{\pi}{4} \right].$$

For the x ranges concerned in the cases (b), (c) and (d) the representation of $v_c(x)$ has been omitted since it is found to differ in appearance from that of $v_0(x)$ only in that the factor $(\Delta - \Omega)^{-1/4}$ is replaced by $(\Delta - \Omega)^{1/4}$. For the range in case (b) the value of $\bar{\xi}$ is imaginary, i.e., $\bar{\xi} = e^{-3\pi i/2} |\bar{\xi}|$, and the relation

$$J_{-1/3}(\bar{\xi}) + J_{1/3}(\bar{\xi}) = \frac{3^{1/2}i}{\pi} K_{1/3}(|\bar{\xi}|)$$

was used.

As already noted in §3.4, a pair of solutions which unlike those above are not asymptotically multiples of each other for large real values of z would be that obtainable in the manner used above from the functions $u_{-1,j}(z)$ described in (9).

3.6. The characteristic values and exponent. The forms of both the exponent μ and the characteristic equations were found in chapter 2 to be determined by the formulas (2.9b). Since these formulas, except for the interpretation of the symbol $[\quad]$, remain valid for the configuration at present under discussion, the deductions of §2.5 and §2.4 require but slight modification to apply to the case in hand. The characteristic exponent is thus given by the formula

$$(3.13) \quad \mu = \frac{i}{\pi} \cos^{-1} \left\{ \cos \int_0^{\pi/2} 2 \{ \Delta - \Omega \cos 2x \}^{1/2} dx + O\left(\frac{\Delta^{1/2}}{\Delta - \Omega}\right) \right\}.$$

The order of the final term within the bracket evidently increases with Ω , from which it is evident that the domain of parameter values for which μ is real, i.e., for which there are unstable solutions, increases in extent as the upper end of the range of values Ω admitted in the configuration of the present chapter is approached.

The characteristic values $S_n(\Omega)$ and $C_n(\Omega)$ are each the root of an equation of the form (2.12) which in the present instance is more explicitly

$$(3.14) \quad \int_0^{\pi/2} \{ \Delta - \Omega \cos 2x \}^{1/2} dx + O\left(\frac{\Delta^{1/2}}{\Delta - \Omega}\right) = \frac{n\pi}{2}.$$

The lower end of the Ω range joins with that of the configuration II, and for such parameter values the formulas (2.13) are again valid as was to be ex-

pected. To obtain formulas valid near the upper end of the range the following process may be used.

Let k_1 be defined by the relation

$$(3.15) \quad \Delta - \Omega = 2^{5/2} k_1 \Omega^{1/2},$$

and in the integral of (3.14) replace x by $\pi/2 - \zeta$. Then the equation becomes

$$(\Delta + \Omega)^{1/2} G(h^2, h^2) + O\left(\frac{\Delta^{1/2}}{\Delta - \Omega}\right) = \frac{n\pi}{2},$$

with G the elliptic integral of (26) and

$$h^2 = \left(1 + \frac{4k_1}{\Omega}\right)^{-1}.$$

For the larger of the admitted values of Ω the ratio k_1/Ω is of the order of $\Delta^{-1/2}$ and h^2 is therefore nearly 1. With the use of the formula (26b) the equation may accordingly be written

$$(3.14b) \quad (2\Omega)^{1/2} - k_1 \log \frac{k_1}{(32\Omega)^{1/2}} + k_1 + k_1 O\left(\frac{k_1}{\Omega^{1/2}} \log \frac{k_1}{\Omega}\right) + O\left(\frac{\Delta^{1/2}}{\Delta - \Omega}\right) = \frac{n\pi}{2}.$$

Recalling (3.15), therefore, it follows that

$$(3.16) \quad \begin{aligned} S_n(\Omega) &= \Omega + 2^{5/2} k_1(n) \Omega^{1/2}, \\ C_n(\Omega) &= \Omega + 2^{5/2} k_1(n) \Omega^{1/2}, \end{aligned}$$

with each $k_1(n)$ a root of an equation of the form (3.14b).

CHAPTER 4

THE CONFIGURATION IV

4.1. **The differential equation.** Let the configuration designated as IV in Figure 1 be defined as that comprising the parameter values (Ω, Δ) in which both are large and

$$(4.1) \quad -M_2 \Omega^{1/2} \leq \Delta - \Omega \leq M_2 \Delta^{1/2},$$

M_2 being the constant in the relation (3.1). Then the substitutions

$$(4.2) \quad \rho = (32\Omega)^{1/2}, \quad \sigma = \frac{\Delta - \Omega}{(32\Omega)^{1/2}}, \quad s = z$$

determine ρ as a large parameter, while the range of values given to σ is

bounded. The differential equation (1) takes the form (3) with the coefficients

$$(4.3) \quad \begin{aligned} \chi_0 &\equiv \frac{1}{2} \sin s, \\ \chi_1 &\equiv \sigma, \end{aligned}$$

in virtue of which the functions (5) are in this case explicitly

$$(4.4) \quad \begin{aligned} k &= -i\sigma, \\ \eta(s) &= 2i\sigma \tan \frac{s}{2}, \\ \phi &= \sin \frac{s}{2} \left\{ \frac{1}{2} \cos \frac{s}{2} + \frac{\sigma}{\rho} \sec \frac{s}{2} \right\}, \\ \Phi &= \frac{1}{2} \sin^2 \frac{s}{2} - \frac{\sigma}{\rho} \log \cos^2 \frac{s}{2}. \end{aligned}$$

Let R_s be chosen as the strip (13). Then in the region R , the coefficient χ_0^2 has a single zero located at the origin and of the second order. It must be shown that with the appropriate values $s_0=0$, $\nu=2$ the requirements of §1.2 are uniformly fulfilled. The hypotheses (i) and (ii) offer no difficulty in this respect, while the consideration of the functions (6) and the hypothesis (iii) may be made as follows.

The relation

$$e^q = \cos^2 \frac{s}{2}$$

defines q , in terms of which

$$\begin{aligned} \Phi &= qe^q \left\{ \frac{e^{-q} - 1}{2q} - \frac{\sigma}{\rho} e^{-q} \right\}, \\ \omega_1 &= \sigma^2 \left\{ 1 - e^{-q} + \frac{2e^q - 2 - qe^q + \frac{2\sigma}{\rho} (1 - e^{-q})}{\Phi} \right\}, \end{aligned}$$

while the various members of the formula

$$\omega(\phi) = \frac{3}{16} \left(\frac{2\phi'}{\phi} - \frac{\phi}{\Phi} \right) \left(\frac{2\phi'}{\phi} + \frac{\phi}{\Phi} \right) - \frac{\phi''}{2\phi}$$

are found to be

$$\frac{\phi''}{\phi} = - \frac{1 + \frac{\sigma}{\rho} e^{-2q}}{1 + \frac{\sigma}{\rho} e^{-q}},$$

$$\frac{2\phi'}{\phi} = \cot \frac{s}{2} - \tan \frac{s}{2} \left\{ \frac{1 - \frac{2\sigma}{\rho} e^{-q}}{1 + \frac{2\sigma}{\rho} e^{-q}} \right\},$$

$$\frac{\phi}{\Phi} = \cot \frac{s}{2} - \tan \frac{s}{2} \left\{ \frac{2\sigma(q-1+e^{-q})}{\rho\Phi(e^{-q}-1)} \right\}.$$

It is to be observed now that q vanishes with s , that $|e^q| \geq \frac{1}{2}$ in R_s , and that the ratio σ/ρ will be uniformly as small as desired if Ω is restricted to remain sufficiently large. It is consequently seen that the brace in the formula for Φ is uniformly bounded from zero and hence that both $\omega(\phi)$ and ω_1 are uniformly bounded in any finite part of R_s . Finally, when $|s|$ is great the asymptotic formulas

$$\phi \sim \frac{\pm i}{2} e^q, \quad \omega(\phi) \sim \frac{-1}{16},$$

$$\omega_1 \sim 2\sigma^2 q, \quad ds \sim \pm i dq$$

are readily checked and in virtue of them the uniform fulfillment of the hypothesis (iii) becomes evident.

4.2. The solutions $u_o(z)$ and $u_s(z)$. The variables Φ and ξ differ only by the real factor ρ , while s and z are identical. Since the values of Φ on the boundaries of R_s are as follows:

for $s' = 0$,

$$\Phi = -\frac{1}{2} \sinh^2 \frac{s''}{2} - \frac{\sigma}{\rho} \log \cosh^2 \frac{s''}{2},$$

for $s' = \pi/2$,

$$\Phi = \left\{ \frac{1}{4} - \frac{\sigma}{\rho} \log \frac{\cosh s''}{2} \right\} + i \left\{ \frac{\sinh s''}{4} + \frac{\sigma}{\rho} \tan^{-1} (\sinh s'') \right\},$$

the map of R_s upon R_ξ is as indicated in Figure 5. The figure shows also the partition of these regions into the sub-regions $\Xi^{(1)}$ defined in (7).

The representation of a pair of solutions $u_1(s)$, $u_2(s)$ which are determined by the initial values

$$\begin{aligned} u_1(0) &= 0, & u_1'(0) &= \left(\frac{i\rho}{4}\right)^{1/2} \left(1 + \frac{2\sigma}{\rho}\right)^{1/4}, \\ u_2(0) &= \left(1 + \frac{2\sigma}{\rho}\right)^{-1/4}, & u_2'(0) &= 0 \end{aligned}$$

is known,* and is expressible in terms of the confluent hypergeometric func-

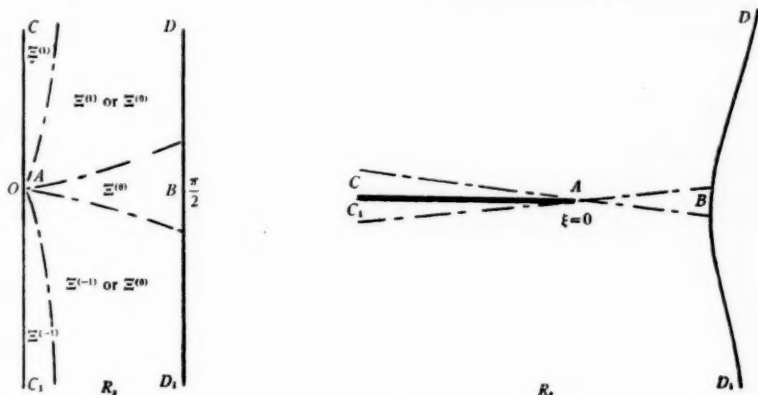


FIG. 5

tions customarily designated by $M_{j,l}$.† With the functions \mathcal{M}_j defined by the formulas

$$(4.5) \quad \begin{aligned} \mathcal{M}_1(\xi, \sigma) &= e^{-3\pi i/8} \xi^{-1/4} \mathcal{M}_{-i\sigma, 1/4}(2i\xi), \\ \mathcal{M}_2(\xi, \sigma) &= e^{-\pi i/8} \xi^{-1/4} \mathcal{M}_{-i\sigma, -1/4}(2i\xi), \end{aligned}$$

it is found thus that the principal solutions, which are evidently mere multiples of u_1 and u_2 , are the following:

For $|\xi| \leq N$,

$$(4.6a) \quad \begin{aligned} u_o(z) &= \left(\frac{2}{\rho}\right)^{1/2} \Psi[M_1(\xi, \sigma)], \\ u_e(z) &= \left(\frac{1}{2}\right)^{1/2} \Psi[M_2(\xi, \sigma)]. \end{aligned}$$

* Paper L₃. See, however, the footnote on p. 646 regarding the differences of notation.

† Cf. Whittaker and Watson, loc. cit., chapter XVI.

On the other hand, when z is not in the neighborhood of the origin the formulas are the following*:

For $|\xi| \geq N$, and z in $\Xi^{(l)}$,

$$(4.6b) \quad \begin{aligned} u_o(z) &= \left(\frac{\pi}{2\phi}\right)^{1/2} (i\rho)^{-3/4} \left\{ \left[\frac{k_{0,1}^{(l)}}{\Gamma(\frac{3}{4} + i\sigma)} \right]_1 (2i\xi)^{i\sigma} e^{i\xi} \right. \\ &\quad \left. + \left[\frac{k_{0,2}^{(l)}}{\Gamma(\frac{3}{4} - i\sigma)} \right]_1 (2i\xi)^{-i\sigma} e^{-i\xi} \right\}, \\ u_e(z) &= \left(\frac{\pi}{2\phi}\right)^{1/2} (i\rho)^{-1/4} \left\{ \left[\frac{k_{e,1}^{(l)}}{\Gamma(\frac{1}{4} + i\sigma)} \right]_1 (2i\xi)^{i\sigma} e^{i\xi} \right. \\ &\quad \left. + \left[\frac{k_{e,2}^{(l)}}{\Gamma(\frac{1}{4} - i\sigma)} \right]_1 (2i\xi)^{-i\sigma} e^{-i\xi} \right\}, \end{aligned}$$

with coefficients

$$(4.6c) \quad \begin{array}{c|ccc} l & -1 & 0 & 1 \\ \hline k_{0,1}^{(l)} & 1 & 1 & -ie^{2\sigma\pi} \\ \hline k_{0,2}^{(l)} & e^{\sigma\pi-3\pi i/4} & e^{-\sigma\pi+3\pi i/4} & e^{-\sigma\pi+3\pi i/4} \\ \hline k_{e,1}^{(l)} & 1 & 1 & ie^{2\sigma\pi} \\ \hline k_{e,2}^{(l)} & e^{\sigma\pi-\pi i/4} & e^{-\sigma\pi+\pi i/4} & e^{-\sigma\pi+\pi i/4} \end{array} ;$$

for use in these formulas it is permissible to write in terms of the original variables

$$(4.7) \quad \begin{aligned} \phi &= \left[\frac{1}{4}\right] \sin z, \quad \xi = (2\Omega)^{1/2} [1](1 - \cos z), \\ e^{i\xi} &= \left(\frac{2}{1 + \cos z}\right)^{i\sigma} e^{(i\rho/4)(1 - \cos z)}, \quad \Psi = \left(\frac{1}{1 + \cos z}\right)^{1/4} [1]. \end{aligned}$$

When z is real the same is true of ϕ, Ψ and ξ , and the last of these is positive. For such values the functions \mathcal{M}_j of (4.5) are real, and the formulas (4.6a) are therefore directly real. From Figure 5 it is seen that such values of z lie in $\Xi^{(0)}$, whence the appropriate formulas (4.6b) reduce to

* The symbol $[]_1$ is used in the sense that $[Q]_1$ denotes a quantity which differs from Q by terms of the order of $(\log \rho)/\rho$ and terms of the order of N^{-1} .

$$(4.6d) \quad \begin{aligned} u_o(x) &= \rho^{-3/4} \left(\frac{2\pi}{\phi} \right)^{1/2} e^{-\sigma\pi/2} \left[\frac{1}{\Gamma_1} \right]_1 \sin \left[\xi + \sigma \log 2\xi - \gamma_1 + \frac{\pi}{8} \right]_1, \\ u_e(x) &= \rho^{-1/4} \left(\frac{2\pi}{\phi} \right)^{1/2} e^{-\sigma\pi/2} \left[\frac{1}{\Gamma_2} \right]_1 \cos \left[\xi + \sigma \log 2\xi - \gamma_2 - \frac{\pi}{8} \right]_1. \end{aligned}$$

The symbols Γ_i and γ_i designate the real values determined by the formulas

$$(4.8) \quad \Gamma(\tfrac{3}{4} \pm i\sigma) = \Gamma_1 e^{\pm i\gamma_1}, \quad \Gamma(\tfrac{1}{4} \pm i\sigma) = \Gamma_2 e^{\pm i\gamma_2},$$

in which the left-hand members are gamma functions.

4.3. The solutions $u_a(z)$ and $u_b(z)$. The solutions of the equation (3) especially associated with the sub-region $\Xi^{(0)}$ which by Figure 5 contains the point $z = \pi/2$, are those described by the following formulas:

For $|\xi| \geq N$, and s in $\Xi^{(1)}$,

$$(4.9a) \quad u_{0,i}(s) = (i\rho)^{-1/4} (2\phi)^{-1/2} \{ B_{i,1}^{(1)}(2i\xi)^{i\sigma} e^{i\xi} + B_{i,2}^{(1)}(2i\xi)^{-i\sigma} e^{-i\xi} \},$$

with coefficients

| l | -1 | 0 | 1 |
|-----------------|---|---------|---|
| $B_{1,1}^{(1)}$ | $[1]_1$ | $[1]_1$ | $[1]_1$ |
| $B_{1,2}^{(1)}$ | $\left[\frac{-2\pi i}{\Gamma(\frac{3}{4} - i\sigma)\Gamma(\frac{1}{4} - i\sigma)} \right]_1$ | 0 | 0 |
| $B_{2,1}^{(1)}$ | 0 | 0 | $\left[\frac{2\pi i e^{2\sigma\pi}}{\Gamma(\frac{3}{4} + i\sigma)\Gamma(\frac{1}{4} + i\sigma)} \right]_1$ |
| $B_{2,2}^{(1)}$ | $[1]_1$ | $[1]_1$ | $[1]_1$ |

For $|\xi| \leq N$,

$$(4.9c) \quad \begin{aligned} u_{0,1} &= i \left(\frac{\pi}{2} \right)^{1/2} e^{-\sigma\pi/2} \Psi \left\{ \frac{2}{\Gamma(\frac{1}{4} - i\sigma)} [\mathcal{M}_1(\xi, \sigma)]_1 \right. \\ &\quad \left. - \frac{e^{\pi i/4}}{\Gamma(\frac{3}{4} - i\sigma)} [\mathcal{M}_2(\xi, \sigma)]_1 \right\}, \\ u_{0,2} &= - \left(\frac{\pi}{2} \right)^{1/2} \Psi \left\{ \frac{2e^{\pi i/4}}{\Gamma(\frac{1}{4} + i\sigma)} [\mathcal{M}_1(\xi, \sigma)]_1 \right. \\ &\quad \left. - \frac{1}{\Gamma(\frac{3}{4} + i\sigma)} [\mathcal{M}_2(\xi, \sigma)]_1 \right\}. \end{aligned}$$

The substitution into the formulas (8b) is simple, the Wronskian having the value $W = (i\rho)^{1/2} [1]_1$, and if the subscript 2 is used to designate evaluations at $z = \pi/2$, it is thus found that we have the following:

For $|\xi| \geq N$, and z in $\Xi^{(1)}$,

$$\begin{aligned} u_\alpha(z) &= \frac{1}{2i\rho\phi_2^{1/2}\phi^{1/2}} \left\{ \left(\frac{\xi}{\xi_2} \right)^{i\sigma} e^{i(\xi-\xi_2)} [1 - \theta_1]_1 \right. \\ &\quad \left. - \left(\frac{\xi}{\xi_2} \right)^{-i\sigma} e^{-i(\xi-\xi_2)} [1 - \theta_2]_1 \right\}, \\ u_\beta(z) &= \frac{\phi_2^{1/2}}{2\phi^{1/2}} \left\{ \left(\frac{\xi}{\xi_2} \right)^{i\sigma} e^{i(\xi-\xi_2)} [1 + \theta_1]_1 \right. \\ &\quad \left. + \left(\frac{\xi}{\xi_2} \right)^{-i\sigma} e^{-i(\xi-\xi_2)} [1 + \theta_2]_1 \right\}, \end{aligned} \quad (4.10a)$$

where

$$\begin{aligned} \theta_1 &= B_{2,1}^{(1)}(-4\xi_2^2)^{i\sigma} e^{2i\xi_2}, \\ \theta_2 &= B_{1,2}^{(1)}(-4\xi_2^2)^{-i\sigma} e^{-2i\xi_2}. \end{aligned}$$

For $|\xi| \leq N$,

$$\begin{aligned} u_\alpha(z) &= 2\pi^{1/2}\rho^{-3/4}e^{-\sigma\pi/2}\Psi \left\{ \frac{2\cos \mathcal{E}_2}{\Gamma_2} [\mathcal{M}_1(\xi, \sigma)]_1 - \frac{\sin \mathcal{E}_1}{\Gamma_1} [\mathcal{M}_2(\xi, \sigma)]_1 \right\}, \\ u_\beta(z) &= \pi^{1/2}\rho^{1/4}e^{-\sigma\pi/2}\Psi \left\{ \frac{\sin \mathcal{E}_2}{\Gamma_2} [\mathcal{M}_1(\xi, \sigma)]_1 + \frac{\cos \mathcal{E}_1}{\Gamma_1} [\mathcal{M}_2(\xi, \sigma)]_1 \right\}, \end{aligned} \quad (4.10b)$$

with

$$\begin{aligned} \mathcal{E}_1 &= \left[\frac{\rho}{4} + \sigma \log \rho - \gamma_1 + \frac{\pi}{8} \right]_1, \\ \mathcal{E}_2 &= \left[\frac{\rho}{4} + \sigma \log \rho - \gamma_2 - \frac{\pi}{8} \right]_1. \end{aligned} \quad (4.11)$$

For real values of z the formulas (4.10b) are directly real, while the appropriate formulas from (4.10a) reduce to

$$\begin{aligned} u_\alpha(x) &= \frac{[2]_1}{\rho\phi^{1/2}} \sin \left[\frac{\rho}{4} \cos x - 2\sigma \log \tan \frac{x}{2} \right]_1, \\ u_\beta(x) &= \frac{[1]_1}{2\phi^{1/2}} \cos \left[\frac{\rho}{4} \cos x - 2\sigma \log \tan \frac{x}{2} \right]_1. \end{aligned} \quad (4.10c)$$

4.4. The solutions of the associated Mathieu equation. The representation of the solutions $iv_\sigma(z)$ and $v_\sigma(z)$, of the "associated" differential equation

The functions within the brackets may be shown to be explicitly real as they should be.

4.5. **The characteristic values.** The values (4.6d) substituted into the characteristic equations (17) and (18) give to the latter the forms

$$(2\Omega)^{1/2} + \frac{\sigma}{2} \log(32\Omega) - \gamma_1 + \frac{\pi}{8} + O(\Omega^{-1/2} \log \Omega) = \frac{n\pi}{2}, \quad (4.13)$$

for an odd Mathieu function,

$$(2\Omega)^{1/2} + \frac{\sigma}{2} \log(32\Omega) - \gamma_2 - \frac{\pi}{8} + O(\Omega^{-1/2} \log \Omega) = \frac{n\pi}{2},$$

for an even Mathieu function.

These equations may be given a somewhat more detailed form when σ is near either the one or the other extreme or the middle of its admitted range of values. The indices of the characteristic values which satisfy the equations with a specific integer n on the right may also be determined as will be shown.

The theory of the gamma function supplies, in particular when $c_1 = 3/4$ and $c_2 = 1/4$, the formulas*

$$\gamma_j = \sigma \frac{\Gamma'(c_j)}{\Gamma(c_j)} + \sum_{r=1}^{\infty} \left(\frac{\sigma}{c_j + r} - \tan^{-1} \frac{\sigma}{c_j + r} \right),$$

$$(4.14) \quad \log \Gamma(c_j + i\sigma) = \frac{1}{2} \log 2\pi + (c_j - \frac{1}{2} + i\sigma) \log(c_j + i\sigma) - (c_j + i\sigma) + O\left(\frac{1}{|\sigma|}\right),$$

$$\Gamma(\zeta)\Gamma(1-\zeta) = \pi \csc \pi\zeta,$$

and from the first of these it is readily seen that with Ω fixed the left members of the equations (4.13) vary monotonically with σ so that the roots for any integer n are unique.

When σ is near the upper end of its admitted range of values, it is large and positive, and the second of the formulas (4.14) gives the evaluations

$$\gamma_j = \sigma \log \sigma - \sigma + (2c_j - 1)(\pi/4) + O(1/\sigma).$$

Both the equations (4.13) thus become

$$(4.13a) \quad (2\Omega)^{1/2} - \frac{\sigma}{2} \log \frac{\sigma^2}{32\Omega} + \sigma + O(\Omega^{1/2} \log \Omega) + O\left(\frac{\Omega^{1/2}}{\Delta - \Omega}\right) = \frac{n\pi}{2},$$

* Cf. Nielsen, N., *Handbuch der Theorie der Gammafunktion*, Leipzig, 1906, p. 23 and pp. 94 and 209.

which is, therefore, the form of the characteristic equations when Ω is near the lower end of the range of values admitted for it in the present configuration. Since for these values the configurations of the present and the preceding chapter abut, the indices of the characteristic values concerned may be determined by a comparison of the equations (4.13a) and (3.16), k_1 in the latter having been defined precisely as σ is in the former. With a given value of n the roots of the equations (4.13) are thus seen to be precisely $S_n(\Omega)$ and $C_n(\Omega)$ respectively.

Near the middle of its range σ is small, and the left members of the equations (4.13) are essentially represented by the early terms of their expansions in powers of σ . Thus the equations become

$$\left\{ \left(n + \frac{1}{2} - c_j \right) \frac{\pi}{2} - (2\Omega)^{1/2} + O(\Omega^{-1/2} \log \Omega) \right\} + \sigma \left\{ \frac{\Gamma'(c_j)}{\Gamma(c_j)} - \frac{1}{2} \log (32\Omega) \right\} + O(\sigma^2) = 0,$$

the values of n concerned being such as make the initial term small. The formulas which are valid in this case, i.e., when Δ and Ω are nearly equal, are thus

$$\begin{aligned} S_n(\Omega) &= \Omega + (32\Omega)^{1/2} \left\{ \frac{(n - \frac{1}{4})\pi - (8\Omega)^{1/2}}{\log(32\Omega) - 2\Gamma'(\frac{3}{4})/\Gamma(\frac{3}{4})} \right\} + O(1), \\ C_n(\Omega) &= \Omega + (32\Omega)^{1/2} \left\{ \frac{(n + \frac{1}{4})\pi - (8\Omega)^{1/2}}{\log(32\Omega) - 2\Gamma'(\frac{1}{4})/\Gamma(\frac{1}{4})} \right\} + O(1). \end{aligned} \quad (4.13b)$$

In particular, the values of Ω for which $\sigma=0$ is a root, i.e., for which there is a characteristic value equal to Ω , are found to be as follows:

$$\begin{aligned} \text{If } S_n(\Omega) = \Omega, \text{ then } (2\Omega)^{1/2} &= \left(n - \frac{1}{4} \right) \frac{\pi}{2} + O\left(\frac{\log n}{n} \right), \\ \text{If } C_n(\Omega) = \Omega, \text{ then } (2\Omega)^{1/2} &= \left(n + \frac{1}{4} \right) \frac{\pi}{2} + O\left(\frac{\log n}{n} \right)^*. \end{aligned} \quad (4.15)$$

* These values were considered by Goldstein, S., in *A note on certain approximate solutions of linear differential equations*, etc., Proceedings of the London Mathematical Society, (2), vol. 28 (1928), p. 87, where the results are stated in the following form:

$$\begin{aligned} \text{If } S_n(\Omega) = \Omega, \text{ then } \begin{cases} 2^{1/2} \cos(8\Omega)^{1/2} \sim (-1)^n, \\ 2^{1/2} \sin(8\Omega)^{1/2} \sim (-1)^{n+1}, \end{cases} \\ \text{If } C_n(\Omega) = \Omega, \text{ then } \begin{cases} 2^{1/2} \cos(8\Omega)^{1/2} \sim (-1)^n, \\ 2^{1/2} \sin(8\Omega)^{1/2} \sim (-1)^n. \end{cases} \end{aligned}$$

Finally near the lower end of its permitted range σ is large but negative, and the second of formulas (4.14) gives

$$\gamma_j = -\sigma + \sigma \log |\sigma| - (2c_j - 1) \frac{\pi}{4} + O\left(\frac{1}{|\sigma|}\right).$$

The characteristic equations (4.13) accordingly become respectively

$$(2\Omega)^{1/2} + \frac{\sigma}{2} \log \frac{32\Omega}{\sigma^2} + \sigma + \frac{\pi}{4} + O(\Omega^{-1/2} \log \Omega) + O\left(\frac{\Omega^{1/2}}{\Omega - \Delta}\right) = \frac{n\pi}{2},$$

for the characteristic value $S_n(\Omega)$;

(4.13c)

$$(2\Omega)^{1/2} + \frac{\sigma}{2} \log \frac{32\Omega}{\sigma^2} + \sigma - \frac{\pi}{4} + O(\Omega^{-1/2} \log \Omega) + O\left(\frac{\Omega^{1/2}}{\Omega - \Delta}\right) = \frac{n\pi}{2},$$

for the characteristic value $C_n(\Omega)$.

These are, therefore, the forms which are valid when Ω is near the upper end of its permitted range of values, or, in other words, when Δ is near the lower end of its possible range.

4.6. The characteristic exponent. The formulas (4.6d) and (4.10b) yield for the evaluation of Θ in (25b)

$$\Theta = 4e^{-\sigma\pi} \left[\frac{\pi}{\Gamma_1 \Gamma_2} \right]_1 \cos \mathcal{E}_1 \cos \mathcal{E}_2 - 1,$$

where \mathcal{E}_1 and \mathcal{E}_2 are as defined in (4.11). The third of the formulas (4.14) may be made to give further

$$\frac{\pi}{\Gamma_1 \Gamma_2} = \left\{ \frac{\cosh 2\sigma\pi}{2} \right\}^{1/2},$$

whence

$$(4.16) \quad \Theta = 2 \{1 + e^{-4\sigma\pi}\}^{1/2} [1]_1 \cos \mathcal{E}_2 \cos \mathcal{E}_1 - 1,$$

and the characteristic exponent is obtainable from the appropriate formula (25a).

The (Ω, Δ) sub-regions of the domain IV of Figure 1 which comprise parameter values for which the differential equation has stable solutions are those for which the value of Θ is less than unity. It is evident from the formula (4.16) that these sub-regions become more and more attenuated as σ decreases, i.e., as the right-hand boundary of the configuration IV is approached.

CHAPTER 5

THE CONFIGURATION V

5.1. Preliminaries. Abutting the configuration of the preceding chapter is that denoted by V in Figure 1, in which Ω is taken to be large and

$$(5.1) \quad 0 \leq \Delta \leq \Omega - M_2 \Omega^{1/2}.$$

In this case the substitutions

$$(5.2) \quad \rho = \frac{\Omega - \Delta}{\Omega^{1/2}}, \quad \sigma^2 = 1 - \frac{\Delta}{\Omega}, \quad s = \frac{z}{\sigma}$$

reduce the differential equation (1) to the form (3) with

$$(5.3) \quad \chi_0^2(s, \sigma) \equiv \frac{2 \sin^2 \sigma s}{\sigma^2} - 1, \\ \chi_1 \equiv 0.$$

The parameter ρ is bounded below by the constant M_2 while σ^2 is confined to the fixed closed range $0 \leq \sigma^2 \leq 1$, its smallest possible value being in fact $M_2 \Omega^{-1/2}$.

With the strip (13) chosen as R_s , the region R_s is

$$(5.4) \quad R_s: \quad 0 \leq s' \leq \frac{\pi}{2\sigma},$$

and within this χ_0^2 admits just one zero which is simple and is located on the axis of reals at the point

$$s_0' = \frac{1}{\sigma} \sin^{-1} \frac{\sigma}{2^{1/2}}.$$

The position of s_0' varies with σ but is restricted to the fixed interval $(2^{-1/2}, \pi/4)$.

If R_s is thought of as cut along the axis of reals from $s=0$ to $s=s_0'$, the values of Φ on its boundaries are the following:

For $s''=0+$, $0 \leq s' \leq s_0'$,

$$\Phi = e^{\pi i} \int_{s'}^{s_0'} i \left\{ 1 - \frac{2 \sin^2 \sigma s'}{\sigma^2} \right\}^{1/2} ds'.$$

For $s'=0$, $s'' \geq 0$,

$$\Phi = \Phi(0) + e^{\pi i/2} \int_0^{s''} i \left\{ \frac{2 \sinh^2 \sigma s''}{\sigma^2} + 1 \right\}^{1/2} ds'.$$

For $s' = \pi/(2\sigma)$, $s'' \geq 0$,

$$\Phi = \Phi\left(\frac{\pi}{2\sigma}\right) + e^{\pi i/2} \int_0^{s''} \left\{ \frac{2 \cosh^2 \sigma s''}{\sigma^2} - 1 \right\}^{1/2} ds''.$$

The maps of R_s upon R_Φ , and hence of R_s upon R_t , are thus revealed, the latter being as indicated in Figure 7:

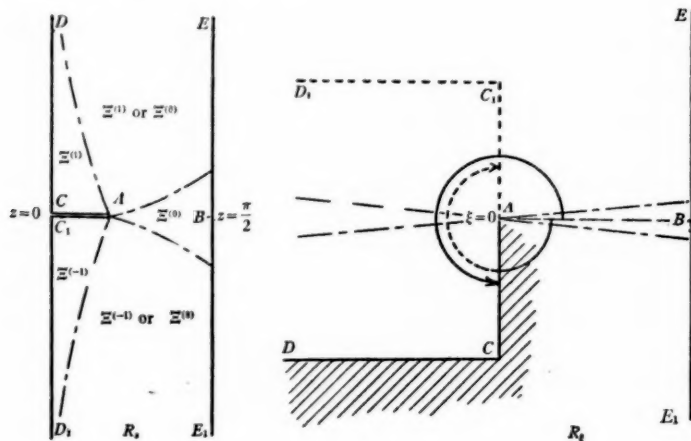


FIG. 7

5.2. The hypotheses. The discussion by which the uniform fulfillment of the requirements of §1.2 by the present differential equation may be established, will be omitted as to detail inasmuch as it proceeds almost entirely like that of §3.3. In virtue of the values (5.3) the functions (6) are in this case explicitly

$$\omega(\phi) = \frac{1}{4} \left\{ -\frac{5}{9} \left(\frac{\phi}{\Phi} \right)^2 - \sigma^2 + \frac{6(1-\sigma^2)}{\phi^2} + \frac{5(2-\sigma^2)}{\phi^4} \right\},$$

$$\Psi = \Phi^{1/6} / \phi^{1/2}, \quad \omega_1 \equiv 0.$$

When $|s-s_0'|$ is great the formulas

$$\phi \sim 2^{1/2} \frac{\sin \sigma s}{\sigma}, \quad \Phi \sim \frac{2^{3/2}}{\sigma^2} \sin^2 \frac{\sigma s}{2}$$

may be used, while for intermediate values the first of the formulas (5.3) may be written

$$\phi^2 = 2(1 - \sigma^2) \frac{\sin^2 \sigma(s - s_0')}{\sigma^2} + (2 - \sigma^2) \frac{\sin 2\sigma(s - s_0')}{\sigma}.$$

For small values of $|s - s_0'|$ it may be shown that

$$\Phi = \frac{\phi^3}{3(2 - \sigma^2)^{1/2}} \left\{ 1 - \frac{3(1 - \sigma^2)}{5(2 - \sigma^2)} \phi^2 + \phi^4 O(1) \right\},$$

and with these formulas at hand the arguments of §3.3 may be paralleled.

5.3. **The solutions relative to $z=0$.** The point $z=0$ may, as is seen from Figure 7, be regarded as lying in the sub-region $\Xi^{(1)}$. Moreover, the zero of χ_0^2 being simple the formulas of §1.4 are applicable, with $h=1$ as the appropriate value. The formulas (11b) and (11a) thus become, in the manner now familiar, the following:

When $|\xi| \leq N$,

$$(5.5a) \quad \begin{aligned} u_o(z) &= \left\{ \frac{\pi \sigma^2 i}{6 \rho^2 \phi_1 \phi} \right\}^{1/2} \xi^{1/6} e^{i\xi_1} [\xi^{1/3} J_{-1/3}(\xi) + \xi^{1/3} J_{1/3}(\xi)], \\ u_s(z) &= \left\{ \frac{\pi \phi_1}{6 i \phi} \right\}^{1/2} \xi^{1/6} e^{i\xi_1} [\xi^{1/3} J_{-1/3}(\xi) + \xi^{1/3} J_{1/3}(\xi)]; \end{aligned}$$

and when z lies in $\Xi^{(1)}$, and $|\xi| \geq N$,

$$(5.5b) \quad \begin{aligned} u_o(z) &= \frac{1}{2} \left\{ \frac{\sigma^2}{\rho^2 \phi_1 \phi} \right\}^{1/2} \{ K_{0,1}^{(1)} e^{i\xi} + K_{0,2}^{(1)} e^{-i\xi} \}, \\ u_s(z) &= \frac{1}{2} \left\{ \frac{\phi_1}{\phi} \right\}^{1/2} \{ K_{s,1}^{(1)} e^{i\xi} + K_{s,2}^{(1)} e^{-i\xi} \}, \end{aligned}$$

with coefficients

| l | - 1 | 0 | 1 |
|-----------------|--------------------|-------------------|--------------------|
| $K_{0,1}^{(1)}$ | $e^{i\xi_1}[1]$ | $e^{i\xi_1}[1]$ | $-ie^{-i\xi_1}[1]$ |
| $K_{0,2}^{(1)}$ | $-e^{-i\xi_1}[1]$ | $ie^{i\xi_1}[1]$ | $ie^{i\xi_1}[1]$ |
| $K_{s,1}^{(1)}$ | $-ie^{i\xi_1}[1]$ | $-ie^{i\xi_1}[1]$ | $e^{-i\xi_1}[1]$ |
| $K_{s,2}^{(1)}$ | $-ie^{-i\xi_1}[1]$ | $e^{i\xi_1}[1]$ | $e^{i\xi_1}[1]$ |

The symbols involved would have the evaluations

$$\begin{aligned}\frac{\rho\phi}{\sigma} &= \{\Delta - \Omega \cos 2z\}^{1/2}, & \frac{\rho\phi_1}{\sigma} &= e^{\pi i/2} \{\Omega - \Delta\}^{1/2}, \\ \xi &= \int_{x_0}^x \{\Delta - \Omega \cos 2z\}^{1/2} dz, & x_0 &= \frac{1}{2} \cos^{-1} \Delta/\Omega, \\ \xi_1 &= e^{3\pi i/2} \int_0^{x_0} \{\Omega \cos 2x - \Delta\}^{1/2} dx.\end{aligned}$$

When z is real and less than x_0 the relation $\xi = |\xi| e^{3\pi i/2}$ is valid and hence

$$J_{-1/3}(\xi) + J_{1/3}(\xi) = \frac{3^{1/2}}{\pi i} K_{1/3}(|\xi|).$$

The formulas given in (5.5) thus reduce when the variable is real to the following:

When $0 \leq x < x_0$, and $|\xi| \geq N$,

$$\begin{aligned}(5.6a) \quad u_o(x) &= \frac{[1]}{\{(\Omega - \Delta)(\Omega \cos 2x - \Delta)\}^{1/4}} \sinh \left[\int_0^x \{\Omega \cos 2x - \Delta\}^{1/2} dx \right], \\ u_e(x) &= \left\{ \frac{\Omega - \Delta}{\Omega \cos 2x - \Delta} \right\}^{1/4} [1] \cosh \left[\int_0^x \{\Omega \cos 2x - \Delta\}^{1/2} dx \right].\end{aligned}$$

When $x \leq x_0$, and $|\xi| \leq N$,

$$\begin{aligned}(5.6b) \quad u_o(x) &= \frac{|\xi|^{1/6} e^{i\xi_1}}{\{4\pi^2(\Omega - \Delta)(\Omega \cos 2x - \Delta)\}^{1/4}} [|\xi|^{1/3} K_{1/3}(|\xi|)], \\ &\text{with } |\xi| = \int_x^{x_0} \{\Omega \cos 2x - \Delta\}^{1/2} dx.\end{aligned}$$

When $x_0 \leq x$, and $|\xi| \leq N$,

$$(5.6c) \quad u_o(x) = \frac{(\pi/6)^{1/2} \xi^{1/6} e^{i\xi_1}}{\{(\Omega - \Delta)(\Delta - \Omega \cos 2x)\}^{1/4}} [\xi^{1/3} J_{-1/3}(\xi) + \xi^{1/3} J_{1/3}(\xi)].$$

When $x_0 < x \leq \pi/2$, and $|\xi| \geq N$,

$$\begin{aligned}(5.6d) \quad u_o(x) &= \frac{[1]e^{i\xi_1}}{\{(\Omega - \Delta)(\Delta - \Omega \cos 2x)\}^{1/4}} \\ &\cdot \sin \left[\frac{\pi}{4} + \int_{x_0}^x \{\Delta - \Omega \cos 2x\}^{1/2} dx \right].\end{aligned}$$

In the cases (b), (c) and (d) the representation of $u_\alpha(x)$ may be formally obtained from that of $u_\alpha(x)$ by replacing the factor $(\Omega - \Delta)^{-1/4}$ by $(\Omega - \Delta)^{1/4}$.

5.4. The solutions relative to $z = \pi/2$. The formulas (11) with the subscripts α replaced by 2, where the latter denote values corresponding to $z = \pi/2$, may be made to yield also the solutions $u_\alpha(z)$ and $u_\beta(z)$. Since the point $z = \pi/2$ lies in the sub-region $\Xi^{(0)}$ the value $h = 0$ is appropriate and the formulas obtained are the following:

When $|\xi| \leq N$,

$$\begin{aligned} u_\alpha(z) &= \left(\frac{2\pi\sigma^2}{3\rho^2\phi_2\phi} \right)^{1/2} \xi^{1/6} \left\{ \cos \left(\xi_2 - \frac{\pi}{12} \right) [\xi^{1/3} J_{1/3}(\xi)] \right. \\ &\quad \left. - \sin \left(\xi_2 + \frac{\pi}{12} \right) [\xi^{1/3} J_{-1/3}(\xi)] \right\}, \\ u_\beta(z) &= \left(\frac{2\pi\phi_2}{3\phi} \right)^{1/2} \xi^{1/6} \left\{ \sin \left(\xi_2 - \frac{\pi}{12} \right) [\xi^{1/3} J_{1/3}(\xi)] \right. \\ &\quad \left. + \cos \left(\xi_2 + \frac{\pi}{12} \right) [\xi^{1/3} J_{-1/3}(\xi)] \right\}. \end{aligned} \quad (5.7a)$$

When z is in $\Xi^{(1)}$, and $|\xi| \geq N$,

$$\begin{aligned} u_\alpha(z) &= \frac{1}{2} \left(\frac{\sigma^2}{\rho^2\phi_2\phi} \right)^{1/2} \{ K_{\alpha,1}^{(1)} e^{i\xi} + K_{\alpha,2}^{(1)} e^{-i\xi} \}, \\ u_\beta(z) &= \frac{1}{2} \left(\frac{\phi_2}{\phi} \right)^{1/2} \{ K_{\beta,1}^{(1)} e^{i\xi} + K_{\beta,2}^{(1)} e^{-i\xi} \}, \end{aligned} \quad (5.7b)$$

with coefficients

| l | -1 | 0 | 1 |
|----------------------|--|-------------------|---|
| $K_{\alpha,1}^{(1)}$ | $-ie^{-i\xi_2}[1]$ | $-ie^{i\xi_2}[1]$ | $2e^{-3\pi i/4} \left[\cos \left(\xi_2 - \frac{\pi}{4} \right) \right]$ |
| $K_{\alpha,2}^{(1)}$ | $2e^{3\pi i/4} \left[\cos \left(\xi_2 - \frac{\pi}{4} \right) \right]$ | $ie^{i\xi_2}[1]$ | $ie^{i\xi_2}[1]$ |
| $K_{\beta,1}^{(1)}$ | $2e^{3\pi i/4} \left[\sin \left(\xi_2 - \frac{\pi}{4} \right) \right]$ | $e^{i\xi_2}[1]$ | $e^{i\xi_2}[1]$ |
| $K_{\beta,2}^{(1)}$ | $e^{-i\xi_2}[1]$ | $e^{-i\xi_2}[1]$ | $2e^{-3\pi i/4} \left[\sin \left(\xi_2 - \frac{\pi}{4} \right) \right]$ |

(5.7c)

In terms of the original variables

$$\frac{\rho\phi_2}{\sigma} = \{\Omega + \Delta\}^{1/2},$$

$$\xi_2 = \int_{x_0}^{\pi/2} \{\Delta - \Omega \cos 2x\}^{1/2} dx.$$

The forms obtained for real values of the variable are the following:
When $0 \leq x < x_0$, and $|\xi| \geq N$,

$$(5.8a) \quad u_\alpha(x) = \frac{-\left[\cos\left(\frac{\pi}{4} - \xi_2\right)\right]}{\{(\Omega + \Delta)(\Omega \cos 2x - \Delta)\}^{1/4}} \exp\left(\int_x^{x_0} \{\Omega \cos 2x - \Delta\}^{1/2} dx\right),$$

$$u_\beta(x) = \left\{\frac{\Omega + \Delta}{\Omega \cos 2x - \Delta}\right\}^{1/4} \left[\sin\left(\frac{\pi}{4} - \xi_2\right)\right] \exp\left(\int_x^{x_0} \{\Omega \cos 2x - \Delta\}^{1/2} dx\right).$$

When $x \leq x_0$, and $|\xi| \leq N$,

$$(5.8b) \quad u_\alpha(x) = \frac{-(2\pi/3)^{1/2} |\xi|^{1/6}}{\{(\Omega + \Delta)(\Omega \cos 2x - \Delta)\}^{1/4}} \left\{ \cos\left(\xi_2 - \frac{\pi}{12}\right) [|\xi|^{1/3} I_{1/3}(|\xi|)] \right. \\ \left. + \sin\left(\xi_2 + \frac{\pi}{12}\right) [|\xi|^{1/3} I_{-1/3}(|\xi|)] \right\},$$

$$u_\beta(x) = \left(\frac{2\pi}{3}\right)^{1/2} |\xi|^{1/6} \left\{ \frac{\Omega + \Delta}{\Omega \cos 2x - \Delta} \right\}^{1/4} \left\{ \sin\left(\xi_2 - \frac{\pi}{12}\right) [|\xi|^{1/3} I_{1/3}(|\xi|)] \right. \\ \left. - \cos\left(\xi_2 + \frac{\pi}{12}\right) [|\xi|^{1/3} I_{-1/3}(|\xi|)] \right\}.$$

When $x_0 \leq x$, and $|\xi| \leq N$,

$$(5.8c) \quad u_\alpha(x) = \frac{(2\pi/3)^{1/2} \xi^{1/6}}{\{(\Omega + \Delta)(\Delta - \Omega \cos 2x)\}^{1/4}} \left\{ \cos\left(\xi_2 - \frac{\pi}{12}\right) [\xi^{1/3} J_{1/3}(\xi)] \right. \\ \left. - \sin\left(\xi_2 + \frac{\pi}{12}\right) [\xi^{1/3} J_{-1/3}(\xi)] \right\},$$

$$u_\beta(x) = \left(\frac{2\pi}{3}\right)^{1/2} \xi^{1/6} \left\{ \frac{\Omega + \Delta}{\Delta - \Omega \cos 2x} \right\}^{1/4} \left\{ \sin\left(\xi_2 - \frac{\pi}{12}\right) [\xi^{1/3} J_{1/3}(\xi)] \right. \\ \left. + \cos\left(\xi_2 + \frac{\pi}{12}\right) [\xi^{1/3} J_{-1/3}(\xi)] \right\}.$$

When $x_0 < x \leq \pi/2$, and $|\xi| \geq N$,

$$(5.8d) \quad \begin{aligned} u_\alpha(x) &= \frac{[1]}{\{(\Omega + \Delta)(\Delta - \Omega \cos 2x)\}^{1/4}} \sin \left[\int_{\pi/2}^x \{\Delta - \Omega \cos 2x\}^{1/2} dx \right], \\ u_\beta(x) &= \left\{ \frac{\Omega + \Delta}{\Delta - \Omega \cos 2x} \right\}^{1/4} [1] \cos \left[\int_{\pi/2}^x \{\Delta - \Omega \cos 2x\}^{1/2} dx \right]. \end{aligned}$$

5.5. **The solutions of the associated equation.** The positive axis of imaginaries in Figure 7 lies in the sub-region $\Xi^{(1)}$. The formulas (5.5) appropriate to this region are to be used, therefore, in obtaining the solutions of the equation (2) for real values of the variable by the substitutions (12). The formulas thus found are

$$(5.9) \quad \begin{aligned} v_o(x) &= \frac{[1]}{\{(\Omega - \Delta)(\Omega \cosh 2x - \Delta)\}^{1/4}} \sin \left[\int_0^x \{\Omega \cosh 2x - \Delta\}^{1/2} dx \right], \\ v_e(x) &= \left\{ \frac{\Omega - \Delta}{\Omega \cosh 2x - \Delta} \right\}^{1/4} [1] \cos \left[\int_0^x \{\Omega \cosh 2x - \Delta\}^{1/2} dx \right]. \end{aligned}$$

5.6. **The characteristic values and exponent.** The forms (5.6d) show that the characteristic values for both even and odd Mathieu functions are in this case determined by equations

$$(5.10a) \quad \left[\frac{\pi}{4} + \int_{x_0}^{\pi/2} \{\Delta - \Omega \cos 2x\}^{1/2} dx \right] = \frac{n\pi}{2},$$

the proper correlation of the indices of the roots with the integer n being duly regarded.

If k_1 is defined in terms of Δ and Ω by the same formula as is the σ of chapter 4, i.e., by (3.15), the substitutions

$$\cos x = k \sin \xi, \quad k^2 = 1 + \frac{4k_1}{(2\Omega)^{1/2}}$$

reduce the equation (5.10a) to the form

$$(5.10b) \quad \frac{\pi}{4} + (2\Omega)^{1/2} k^2 G(1, k^2) + O\left(\frac{\Omega^{1/2}}{\Omega - \Delta}\right) = \frac{n\pi}{2},$$

in which G is the elliptic integral of (26). In the range of transition from the configuration of chapter 4 to that of the present chapter, k_1 is negative and k^2 accordingly little less than unity. The evaluation of (5.10b) to the form

$$\frac{\pi}{4} + (2\Omega)^{1/2} + \frac{k_1}{2} \log \frac{32\Omega}{k_1^2} + k_1 + O(\Omega^{-1/2} \log \Omega) + O\left(\frac{\Omega^{1/2}}{\Omega - \Delta}\right) = \frac{n\pi}{2}$$

may, therefore, be obtained by the use of (26b), and a comparison of the result with the equations (4.13c) shows that the characteristic values which occur as the roots of equations representable by (5.10a) are respectively $S_n(\Omega)$ and $C_{n-1}(\Omega)$. In other words the characteristic equations are as follows:

$$(5.11) \quad \int_{x_0}^{\pi/2} \{\Delta - \Omega \cos 2x\}^{1/2} dx + O\left(\frac{\Omega^{1/2}}{\Omega - \Delta}\right) = \left(n - \frac{1}{2}\right) \frac{\pi}{2},$$

for the characteristic value $S_n(\Omega)$;

$$\int_{x_0}^{\pi/2} \{\Delta - \Omega \cos 2x\}^{1/2} dx + O\left(\frac{\Omega^{1/2}}{\Omega - \Delta}\right) = \left(n + \frac{1}{2}\right) \frac{\pi}{2},$$

for the characteristic value $C_n(\Omega)$.

In the consideration of the characteristic exponent the formulas (5.6d) and (5.8a) in conjunction with (25b) are found to lead to the evaluation

$$(5.12) \quad \Theta = e^{2i\xi_1} [\cos 2\xi_2] - 1,$$

and with this the value of μ is given by the formula (25a). Since the right-hand member of (5.12) can be exceeded by unity only when the cosine is very small, it is evident that the unstable solutions greatly predominate in the present configuration.

An evaluation of the several elliptic integrals involved may be made to show that the transition from the formula (4.16) to (5.12) is a continuous one.

CHAPTER 6

THE CONFIGURATION VI

6.1. **Remarks.** The configuration designated by VI in Figure 1 is to be that in which Δ is negative and

$$(6.1) \quad -\{\Omega - M_2\Omega^{1/2}\} \leq \Delta \leq 0.$$

It clearly differs from that of the preceding chapter only in the sign of Δ . The distinction between the two configurations is indeed largely an artificial one, entered into primarily for the purpose of utilizing the discussion of §5.2 without modification when parameter values admitted by (6.1) are concerned. For in this latter case the substitutions

$$(6.2) \quad \rho = \frac{\Omega + \Delta}{\Omega^{1/2}} e^{3\pi i/2}, \quad \sigma^2 = 1 + \frac{\Delta}{\Omega}, \quad s = \frac{1}{\sigma} \left(\frac{\pi}{2} - z \right)$$

transform the differential equation (1) into the form (3) with precisely the coefficients (5.3), with σ restricted precisely as in the earlier case. The de-

ductions of §5.2, therefore, serve again to show that the requirements of the general theory are uniformly fulfilled.

By their definitions the intermediate variables s , ϕ , and Φ , and the parameters σ and ρ , differ from the corresponding quantities in chapter 5. The ultimate variables z and ξ are, however, found to have the same relation to each other, so that Figure 7 continues to remain valid in the present configuration. It is found as a consequence that the various formulas deduced in §5.3, §5.4, and §5.5 apply also in the present instance, provided they are expressed entirely in terms of the original variables z , Δ , and Ω .

6.2. The characteristic values and exponent. With the prevailing forms of the solutions exactly those of chapter 5 the characteristic equations of course remain of the form (5.11). It is of interest, however, to obtain from these equations more explicit formulas which are valid near the lower end of the admitted range of values for Δ . For such values h^2 , which may be written

$$h^2 = \frac{\Omega - |\Delta|}{2\Omega},$$

is small of the order of $\Omega^{-1/2}$, and in the equation (5.10b) the evaluation given by (26a) is appropriate. The equation thus becomes

$$(2\Omega)^{1/2} h^2 \left\{ 1 + \frac{h^2}{8} + h^4 O(1) \right\} = 2n - 1.$$

It is evident that the integers n concerned are those of a bounded set, the equation being expressible for such n in the form

$$\Omega + \Delta = (2n - 1)(2\Omega)^{1/2} + O(1).$$

Inasmuch as the characteristic equations represented by (5.10b) were found to be those for $S_n(\Omega)$ and $C_{n-1}(\Omega)$, it follows that for the algebraically smaller of the presently admitted values of Δ the characteristic values are described by formulas

$$(6.3) \quad \begin{aligned} S_n(\Omega) &= -\Omega + (2n - 1)(2\Omega)^{1/2} + O(1), \\ C_n(\Omega) &= -\Omega + (2n + 1)(2\Omega)^{1/2} + O(1). \end{aligned}$$

The characteristic exponent is again given by (25a) and (5.12).

CHAPTER 7

THE CONFIGURATION VII

7.1. The transformed differential equation. When Δ is large and negative and

$$(7.1) \quad -M_2 |\Delta|^{1/2} \leq \Omega + \Delta \leq M_2 \Omega^{1/2},$$

the configuration is that designated by VII, Figure 1. In this case the substitutions

$$(7.2) \quad \rho = (32\Omega)^{1/2} e^{-\pi i/2}, \quad \sigma = \frac{\Omega + \Delta}{(32\Omega)^{1/2}}, \quad s = \frac{\pi}{2} - z$$

bring the differential equation (1) into the form (3) with

$$(7.3) \quad \begin{aligned} \chi_0(s, \sigma) &\equiv \frac{1}{2} \sin s, \\ \chi_1 &\equiv i\sigma. \end{aligned}$$

The transformed equation thus differs from that obtained in chapter 4 only to the extent that σ is replaced by $i\sigma$. The formulas for ϕ and Φ given in (4.4) are adaptable to the present case by the substitution of $-\sigma/|\rho|$ in place of σ/ρ , a change which is easily seen to affect in no way the validity of the arguments of §4.1. That the differential equation in the present instance uniformly satisfies the hypotheses of §1.2 may, therefore, be accepted without further consideration.

The regions R_s and R_ϕ which correspond to the strip (13) are, both as to outline and relative orientation, precisely like the z and ξ regions shown in Figure 5. Since under the relations (7.2) the region R_s is a reflection of R_s in the point $s = \pi/4$, whereas R_ϕ is obtainable from R_s by a rotation besides the change of scale, the figure which relates the ultimate regions R_s and R_ϕ for the chapter at hand is as indicated in Figure 8. The division of these regions into the sub-regions $\Xi^{(i)}$ is also as shown.

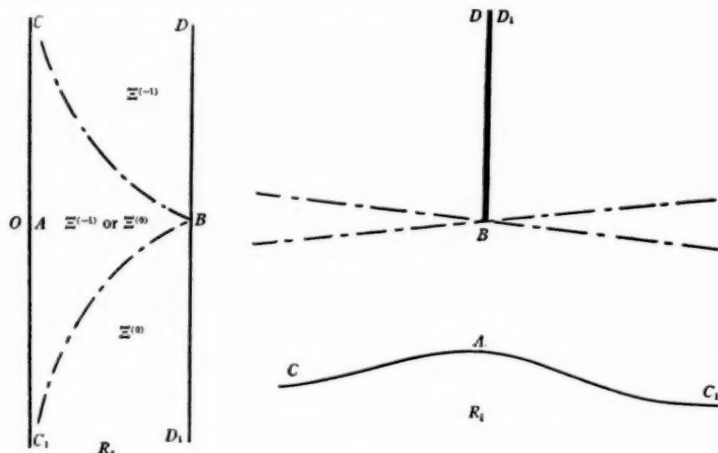


FIG. 8

7.2. The solutions. The origin $z=0$ corresponds to $s_1=\pi/2$ and lies in the sub-region $\Xi^{(0)}$. The principal solutions relative to this point may accordingly be deduced by the substitution of the values (4.9) (with σ replaced by $i\sigma$) into the formulas (8b), the subscript a being taken as 1. To this extent the process coincides with that by which the forms (4.10) were deduced. In the present instance, however, certain terms may be dropped from the resulting formulas for, as may be seen from Figure 8, the quantity $i\xi_1$ is real and positive and $e^{-i\xi_1}$ therefore asymptotically negligible in comparison with $e^{i\xi_1}$. It is found thus that the following formulas hold:

When z is (anywhere) in R_s , and $|\xi| \geq N$,

$$(7.4a) \quad \begin{aligned} u_o(z) &= \frac{-1}{2i\rho\phi_1^{1/2}\phi^{1/2}} \left\{ \left(\frac{\xi}{\xi_1} \right)^{-\sigma} e^{i(\xi-\xi_1)} [1]_1 - \left(\frac{\xi}{\xi_1} \right)^{\sigma} e^{-i(\xi-\xi_1)} [1]_1 \right\}, \\ u_s(z) &= \frac{\phi_1^{1/2}}{2\phi^{1/2}} \left\{ \left(\frac{\xi}{\xi_1} \right)^{-\sigma} e^{i(\xi-\xi_1)} [1]_1 + \left(\frac{\xi}{\xi_1} \right)^{\sigma} e^{-i(\xi-\xi_1)} [1]_1 \right\}; \end{aligned}$$

when $|\xi| \leq N$,

$$(7.4b) \quad \begin{aligned} u_o(z) &= \left(\frac{\pi}{\phi_1\phi} \right)^{1/2} \frac{(2i\xi_1)^{-\sigma} e^{i\xi_1} (i\xi)^{1/4}}{2i\rho} \left\{ \frac{2e^{\pi i/4}}{\Gamma(\frac{3}{4}-\sigma)} [\mathcal{M}_1(\xi, i\sigma)]_1 \right. \\ &\quad \left. - \frac{1}{\Gamma(\frac{3}{4}-\sigma)} [\mathcal{M}_2(\xi, i\sigma)] \right\}, \\ u_s(z) &= - \left(\frac{\pi\phi_1}{\phi} \right)^{1/2} \frac{(2i\xi_1)^{-\sigma} e^{i\xi_1} (i\xi)^{1/4}}{2} \left\{ \frac{2e^{\pi i/4}}{\Gamma(\frac{3}{4}-\sigma)} [\mathcal{M}_1(\xi, i\sigma)]_1 \right. \\ &\quad \left. - \frac{1}{\Gamma(\frac{3}{4}-\sigma)} [\mathcal{M}_2(\xi, i\sigma)]_1 \right\}. \end{aligned}$$

In these as in subsequent formulas any term is to be omitted if σ is such that the gamma function involved is infinite.

The point $z=\pi/2$ corresponds to $s_2=0$ and the principal solutions relative to this point are therefore to be obtained precisely as were the solutions of §4.2. The formulas found are as follows:

When $|\xi| \leq N$,

$$(7.5a) \quad \begin{aligned} u_a(z) &= - \left(\frac{2}{i\rho} \right)^{1/2} \Psi[(i\xi)^{-1/4} M_{\sigma, 1/4}(2i\xi)], \\ u_b(z) &= \left(\frac{1}{2} \right)^{1/2} \Psi[(i\xi)^{-1/4} M_{\sigma, -1/4}(2i\xi)]; \end{aligned}$$

when $|\xi| \geq N$, and z is in $\Xi^{(l)}$,

$$(7.5b) \quad \begin{aligned} u_a(z) &= - \left(\frac{\pi}{2\phi} \right)^{1/2} (i\rho)^{-1/4} \left\{ \left[\frac{1}{\Gamma(\frac{3}{4} - \sigma)} \right]_1 (2i\xi)^{-\sigma} e^{i\xi} \right. \\ &\quad \left. + \left[\frac{h_a^{(l)}}{\Gamma(\frac{3}{4} + \sigma)} \right]_1 (2i\xi)^{\sigma} e^{-i\xi} \right\}, \\ u_\beta(z) &= \left(\frac{\pi}{2\phi} \right)^{1/2} (i\rho)^{-1/4} \left\{ \left[\frac{1}{\Gamma(\frac{3}{4} - \sigma)} \right]_1 (2i\xi)^{-\sigma} e^{i\xi} \right. \\ &\quad \left. + \left[\frac{h_\beta^{(l)}}{\Gamma(\frac{3}{4} + \sigma)} \right]_1 (2i\xi)^{\sigma} e^{-i\xi} \right\}, \end{aligned}$$

with coefficients given by the table

| l | -1 | 0 |
|-----------------|-------------------------|--------------------------|
| $h_a^{(l)}$ | $e^{(\sigma-3/4)\pi i}$ | $e^{-(\sigma-3/4)\pi i}$ |
| $h_\beta^{(l)}$ | $e^{(\sigma-1/4)\pi i}$ | $e^{-(\sigma-1/4)\pi i}$ |

In terms of the original variables

$$\begin{aligned} \phi &= \frac{\cos z}{4} - \frac{\Omega + \Delta}{32\Omega} \tan \left(\frac{\pi}{4} - \frac{z}{2} \right), \\ i\xi &= (2\Omega)^{1/2} (1 - \sin z) + \frac{\Omega + \Delta}{(32\Omega)^{1/2}} \log \frac{1 + \sin z}{2}, \end{aligned}$$

which permits the abbreviated relations

$$\begin{aligned} \phi &= \left[\frac{\cos z}{4} \right], & \phi_1 &= \left[\frac{1}{4} \right], \\ i\xi &= \frac{|\rho|}{4} [1 - \sin z], & i\xi_1 &= \frac{|\rho|}{4} [1], \\ e^{i\xi} &= \left(\frac{1 + \sin z}{2} \right)^{\sigma} e^{(|\rho|/4)(1 - \sin z)} & e^{i\xi_1} &= \left(\frac{1}{2} \right)^{\sigma} e^{|\rho|/4}. \end{aligned}$$

The specialization of the various formulas to the case in which the variable is real may be made as usual, it being noted that then $i\xi = |\xi|$. The representations which result are as follows:

When $|\xi| \geq N$

$$(7.6a) \quad u_o(x) = \left(\frac{\sec x}{2\Omega}\right)^{1/2} [1]_1 \sinh \left[(2\Omega)^{1/2} \sin x + \sigma \log \frac{1 + \sin x}{1 - \sin x} \right]_1,$$

$$u_e(x) = (\sec x)^{1/2} [1]_1 \cosh \left[(2\Omega)^{1/2} \sin x + \sigma \log \frac{1 + \sin x}{1 - \sin x} \right]_1;$$

when $|\xi| \leq N$,

$$(7.6b) \quad u_o(x) = \frac{-(2\pi)^{1/2} e^{(2\Omega)^{1/2}}}{(32\Omega)^{\sigma/2+3/8} (1 + \sin x)^{1/4}} \left\{ \frac{2}{\Gamma(\frac{3}{4} - \sigma)} [|\xi|^{-1/4} M_{\alpha, 1/4}(2|\xi|)]_1 \right. \\ \left. - \frac{1}{\Gamma(\frac{3}{4} - \sigma)} [|\xi|^{-1/4} M_{\alpha, -1/4}(2|\xi|)]_1 \right\},$$

$$u_e(x) = \frac{-(2\pi)^{1/2} e^{(2\Omega)^{1/2}}}{(32\Omega)^{\sigma/2-1/8} (1 + \sin x)^{1/4}} \left\{ \frac{2}{\Gamma(\frac{3}{4} - \sigma)} [|\xi|^{-1/4} M_{\alpha, 1/4}(2|\xi|)]_1 \right. \\ \left. - \frac{1}{\Gamma(\frac{3}{4} - \sigma)} [|\xi|^{-1/4} M_{\alpha, -1/4}(2|\xi|)]_1 \right\},$$

the symbols M representing the confluent hypergeometric functions which occur in the formulas (4.5).

When $|\xi| \geq N$,

$$(7.7a) \quad u_a(x) = \frac{-(2\pi \sec x)^{1/2}}{(32\Omega)^{\sigma/2+3/8}} \left(\frac{1 + \sin x}{1 - \sin x} \right)^{\sigma} \left[\frac{1}{\Gamma(\frac{3}{4} - \sigma)} \right]_1 e^{(2\Omega)^{1/2} (1 - \sin x)},$$

$$u_b(x) = \frac{(2\pi \sec x)^{1/2}}{(32\Omega)^{\sigma/2+1/8}} \left(\frac{1 + \sin x}{1 - \sin x} \right)^{\sigma} \left[\frac{1}{\Gamma(\frac{3}{4} - \sigma)} \right]_1 e^{(2\Omega)^{1/2} (1 - \sin x)}.$$

When $|\xi| \leq N$,

$$(7.7b) \quad u_a(x) = \frac{-1}{(8\Omega)^{1/2} (1 + \sin x)^{1/4}} [|\xi|^{-1/4} M_{\alpha, 1/4}(2|\xi|)],$$

$$u_b(x) = \frac{1}{(1 + \sin x)^{1/4}} [|\xi|^{-1/4} M_{\alpha, -1/4}(2|\xi|)].$$

The solutions of the associated Mathieu equation, as obtained from the forms (7.4a) by the method of §1.5, are for real values of the variable represented thus:

$$(7.8) \quad v_o(x) = \frac{[1]_1}{(2\Omega \cosh x)^{1/2}} \sin [(2\Omega)^{1/2} \sinh x - 2\sigma \tan^{-1}(\sinh x)]_1,$$

$$v_e(x) = \frac{[1]_1}{(\cosh x)^{1/2}} \cos [(2\Omega)^{1/2} \sinh x - 2\sigma \tan^{-1}(\sinh x)]_1.$$

7.3. The characteristic values and exponent. The characteristic equations (17) and (18) may obviously if desired be rewritten in the forms $u_a(0)=0$, $u_p(0)=0$ and $u_p'(0)=0$, $u_a'(0)=0$. It accordingly follows from the formulas (7.7a) that any characteristic value must be a root of the one or the other of the equations

$$(7.9) \quad \left[\frac{1}{\Gamma(\frac{3}{4} - \sigma)} \right]_1 = 0, \quad \left[\frac{1}{\Gamma(\frac{1}{4} - \sigma)} \right]_1 = 0.$$

If σ is not positive the relations (7.9) are manifestly impossible. Hence no characteristic values exist when $\Delta \leq -\Omega$, a fact which may be simply concluded from a direct perusal of the differential equation. When σ is positive and of suitable magnitude, on the other hand, a relation (7.9) may be satisfied in virtue of the gamma function becoming infinite. The appropriate values are clearly those for which

$$\left[\sigma + \frac{1}{4} \right]_1 = \frac{n}{2},$$

whence the characteristic equations are found to be of the form

$$\Delta = -\Omega + (2n-1)(2\Omega)^{1/2} - O(\log \Omega).$$

This result when compared with the formulas (6.3), with which it must be in accord for suitable values of Δ and Ω , shows that the characteristic values in the present configuration are given by formulas

$$(7.10) \quad \begin{aligned} S_n(\Omega) &= -\Omega + (2n-1)(2\Omega)^{1/2} + O(\log \Omega), \\ C_n(\Omega) &= -\Omega + (2n+1)(2\Omega)^{1/2} + O(\log \Omega). \end{aligned}$$

Finally the computation of the characteristic exponent depends only upon the evaluation of the quantity Θ given in (25b). This evaluation from the forms (7.6b) and (7.7a) is found in the present case to be

$$(7.11) \quad \Theta = \frac{\pi e^{(8\Omega)^{1/2}}}{(32\Omega)^\sigma} \left[\frac{1}{\Gamma(\frac{1}{4} - \sigma)\Gamma(\frac{3}{4} - \sigma)} \right]_1 - 1.$$

CHAPTER 8

THE CONFIGURATION VIII

8.1. The change of variables. The configuration numbered VIII in Figure 1 is to be defined as that in which Δ is negative and numerically large, while

$$(8.1) \quad -|\Delta| + M_2|\Delta|^{1/2} \leq -\Omega \leq \frac{-1}{M_1}|\Delta|.$$

The substitutions for the transformation of the equation (1) are to be

$$(8.2) \quad \rho = \frac{|\Delta| - \Omega}{|\Delta|^{1/2}} e^{\pi i/2}, \quad \sigma^2 = 1 - \frac{\Omega}{|\Delta|}, \quad s = \frac{i}{\sigma} \left(\frac{\pi}{2} - z \right),$$

in which case the resulting equation of the form (3) has precisely the coefficients (3.3). The value of σ is again confined to the range (3.4), and if the half-strip

$$(8.3) \quad R_s: \quad \begin{aligned} 0 &\leq x \leq \pi, \\ 0 &\leq y, \end{aligned}$$

is chosen as the domain of z , the corresponding region R_s is precisely that of (3.6). In terms of s , therefore, the present equation coincides entirely with that of chapter 3. The hypotheses are in consequence uniformly fulfilled and Figure 2 again applies. The latter evidently leads in the present instance to Figure 9.

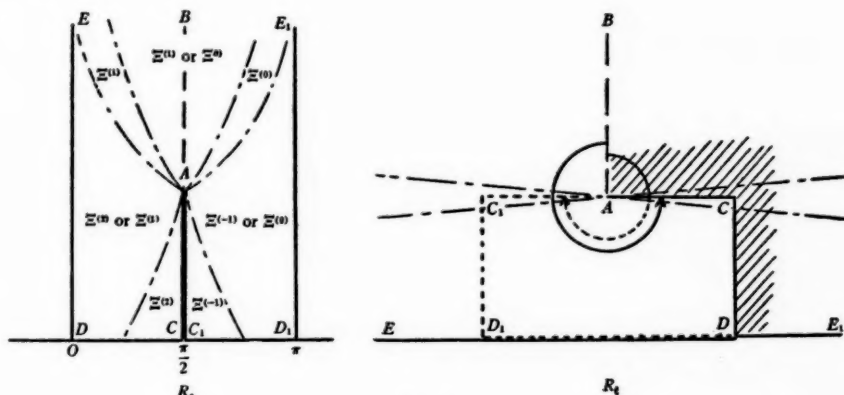


FIG. 9

The extension of the representations which are to be obtained from R_s into the entire strip (13) may be made directly by observing that $u_a(z)$ and $u_b(z)$ are respectively odd and even as functions of the variable $(z - \pi/2)$, and by applying the identities (14a) and (14b) to the formulas for $u_o(z)$ and $u_e(z)$. With this accomplished the further considerations of §1.6 are, of course, applicable.

8.2. The solutions. The zero of χ_0^2 is of the first order, and, as may be seen from Figure 9, both the points $z=0$ and $z=\pi/2$ lie in the sub-region $\Xi^{(2)}$. The formulas (11a) and (11b) may, therefore, be drawn upon, with $h=2$, and lead to the formulas which follow.

When z is in $\Xi^{(1)}$, and $|\xi| \geq N$,

$$u_o(z) = \frac{1}{2} \left(\frac{\sigma^2}{\rho^2 \phi_1 \phi} \right)^{1/2} \{ K_{0,1}^{2,l} e^{i\xi} + K_{0,2}^{2,l} e^{-i\xi} \}, \quad (8.4a)$$

$$u_e(z) = \frac{1}{2} \left(\frac{\phi_1}{\phi} \right)^{1/2} \{ K_{e,1}^{2,l} e^{i\xi} + K_{e,2}^{2,l} e^{-i\xi} \},$$

$$u_a(z) = \frac{1}{2} \left(\frac{\sigma^2}{\rho^2 \phi_2 \phi} \right)^{1/2} \{ K_{a,1}^{2,l} e^{i\xi} + K_{a,2}^{2,l} e^{-i\xi} \}, \quad (8.5a)$$

$$u_\beta(z) = \frac{1}{2} \left(\frac{\phi_2}{\phi} \right)^{1/2} \{ K_{\beta,1}^{2,l} e^{i\xi} + K_{\beta,2}^{2,l} e^{-i\xi} \},$$

with coefficients

| l | -1 | 0 | 1 | 2 |
|-----------------|-------------------|------------------|------------------|------------------|
| $K_{0,1}^{2,l}$ | $[i]e^{i\xi_1}$ | $[i]e^{i\xi_1}$ | $[1]e^{-i\xi_1}$ | $[1]e^{-i\xi_1}$ |
| $K_{0,2}^{2,l}$ | $[-i]e^{-i\xi_1}$ | $[-1]e^{i\xi_1}$ | $[-1]e^{i\xi_1}$ | $[-1]e^{i\xi_1}$ |
| $K_{e,1}^{2,l}$ | $[-i]e^{i\xi_1}$ | $[-i]e^{i\xi_1}$ | $[1]e^{-i\xi_1}$ | $[1]e^{-i\xi_1}$ |
| $K_{e,2}^{2,l}$ | $[-i]e^{i\xi_1}$ | $[1]e^{i\xi_1}$ | $[1]e^{i\xi_1}$ | $[1]e^{i\xi_1}$ |

| l | -1 | 0 | 1 | 2 |
|---------------------|-------------------|---|---|------------------|
| $K_{a,1}^{2,l}$ | $[i]e^{i\xi_2}$ | $[i]e^{i\xi_2}$ | $[1]e^{-i\xi_2}$ | $[1]e^{-i\xi_2}$ |
| $K_{a,2}^{2,l}$ | $[-i]e^{-i\xi_2}$ | $[-2e^{\pi i/4} \cos(\xi_2 - \frac{\pi}{4})]$ | $[-2e^{\pi i/4} \cos(\xi_2 - \frac{\pi}{4})]$ | $[-1]e^{i\xi_2}$ |
| $K_{\beta,1}^{2,l}$ | $[-i]e^{i\xi_2}$ | $[-i]e^{i\xi_2}$ | $[1]e^{-i\xi_2}$ | $[1]e^{-i\xi_2}$ |
| $K_{\beta,2}^{2,l}$ | $[-i]e^{i\xi_2}$ | $[2e^{-\pi i/4} \cos(\xi_2 + \frac{\pi}{4})]$ | $[2e^{-\pi i/4} \cos(\xi_2 + \frac{\pi}{4})]$ | $[1]e^{i\xi_2}$ |

When $|\xi| \leq N$,

$$(8.4b) \quad \begin{aligned} u_o(z) &= \left(\frac{\pi \sigma^2}{6 \rho^2 \phi_1 \phi} \right)^{1/2} \xi^{1/6} e^{i\xi_1 + 3\pi i/4} [\xi^{1/3} J_{-1/3}(\xi) + \xi^{1/3} J_{-1/3}(\xi)], \\ u_s(z) &= \left(\frac{\pi \phi_1}{6 \phi} \right)^{1/2} \xi^{1/6} e^{i\xi_1 - \pi i/4} [\xi^{1/3} J_{-1/3}(\xi) + \xi^{1/3} J_{1/3}(\xi)], \end{aligned}$$

$$(8.5b) \quad \begin{aligned} u_a(z) &= - \left(\frac{2\pi \sigma^2}{3 \rho^2 \phi_2 \phi} \right)^{1/2} \xi^{1/6} \left\{ e^{\pi i/6} \sin \left(\xi_2 + \frac{\pi}{12} \right) [\xi^{1/3} J_{-1/3}(\xi)] \right. \\ &\quad \left. + e^{-\pi i/6} \cos \left(\xi_2 - \frac{\pi}{12} \right) [\xi^{1/3} J_{1/3}(\xi)] \right\}, \\ u_\beta(z) &= \left(\frac{2\pi \phi_2}{3 \phi} \right)^{1/2} \xi^{1/6} \left\{ e^{-\pi i/3} \cos \left(\xi_2 + \frac{\pi}{12} \right) [\xi^{1/3} J_{-1/3}(\xi)] \right. \\ &\quad \left. + e^{\pi i/3} \sin \left(\xi_2 - \frac{\pi}{12} \right) [\xi^{1/3} J_{1/3}(\xi)] \right\}. \end{aligned}$$

For use in these formulas,

$$\begin{aligned} \frac{\rho \phi}{\sigma} &= - \{ |\Delta| + \Omega \cos 2z \}^{1/2}, \\ \frac{\rho \phi_1}{\sigma} &= - \{ |\Delta| + \Omega \}^{1/2}, & \frac{\rho \phi_2}{\sigma} &= - \{ |\Delta| - \Omega \}^{1/2}, \\ \xi &= \int_{z_0}^z \{ \Delta - \Omega \cos 2z \}^{1/2} dz, & z_0 &= \frac{1}{2} \cos^{-1} \Delta / \Omega, \\ \xi_2 &= \int_0^{y_0} \{ |\Delta| - \Omega \cosh 2y \}^{1/2} dy, & y_0 &= \frac{1}{2} \cosh^{-1} |\Delta| / \Omega, \\ \xi_1 &= \xi_2 - i \int_0^{\pi/2} \{ |\Delta| + \Omega \cos 2x \}^{1/2} dx. \end{aligned}$$

It is found that for all real values of z on the interval $(0, \pi)$ the respective formulas are

$$(8.4c) \quad \begin{aligned} u_o(x) &= \frac{[1]}{\{ (|\Delta| + \Omega)(|\Delta| + \Omega \cos 2x) \}^{1/4}} \sinh \left[\int_0^x \{ |\Delta| + \Omega \cos 2x \}^{1/2} dx \right], \\ u_s(x) &= \left\{ \frac{|\Delta| + \Omega}{|\Delta| + \Omega \cos 2x} \right\}^{1/4} [1] \cosh \left[\int_0^x \{ |\Delta| + \Omega \cos 2x \}^{1/2} dx \right], \end{aligned}$$

$$(8.5c) \quad \begin{aligned} u_a(x) &= \frac{[1]}{\{(|\Delta| - \Omega)(|\Delta| + \Omega \cos 2x)\}^{1/4}} \sinh \left[\int_{\pi/2}^x \{|\Delta| + \Omega \cos 2x\}^{1/2} dx \right], \\ u_b(x) &= \left\{ \frac{|\Delta| - \Omega}{|\Delta| + \Omega \cos 2x} \right\}^{1/4} [1] \cosh \left[\int_{\pi/2}^x \{|\Delta| + \Omega \cos 2x\}^{1/2} dx \right]. \end{aligned}$$

The axis of imaginaries in Figure 9 likewise lies in the sub-region $\Xi^{(2)}$, and the formulas for the solutions of the associated Mathieu equation are accordingly found to be

$$(8.6) \quad \begin{aligned} v_o(x) &= \frac{[1]}{\{(|\Delta| + \Omega)(|\Delta| + \Omega \cosh 2x)\}^{1/4}} \sin \left[\int_0^x \{|\Delta| - \Omega \cosh 2x\}^{1/2} dx \right], \\ v_e(x) &= \left\{ \frac{|\Delta| + \Omega}{|\Delta| + \Omega \cosh 2x} \right\}^{1/4} [1] \cos \left[\int_0^x \{|\Delta| + \Omega \cosh 2x\}^{1/2} dx \right]. \end{aligned}$$

8.3. The characteristic exponent. It is evident that the present configuration admits no characteristic values. The formulas (8.4c) and (8.5c) yield the evaluation

$$\Theta = [2] \cosh^2 \left[\int_0^{\pi/2} \{|\Delta| + \Omega \cos 2x\}^{1/2} dx \right] - 1,$$

and the formula (25a) accordingly gives the characteristic exponent in the form

$$(8.7) \quad \mu = \left[\frac{2}{\pi} \int_0^{\pi/2} \{|\Delta| + \Omega \cos 2x\}^{1/2} dx \right].$$

Clearly, the configuration is one of unstable solutions.

CHAPTER 9

THE CONFIGURATION IX

9.1. The differential equation. In this final configuration to be considered, i.e., IX of Figure 1, the parameter Δ is large and negative while

$$(9.1) \quad -\frac{1}{M_1} |\Delta| \leq -\Omega \leq 0.$$

The variable is to be restricted to any region in which a relation (2.4a) is fulfilled with some constant M_1 , and this constant is that which figures in (9.1). The substitutions

$$(9.2) \quad \rho = |\Delta|^{1/2} e^{-i/2}, \quad \sigma^2 = \frac{\Omega}{|\Delta|}, \quad s = \frac{\pi}{2} - z$$

reduce the differential equation to the form (3) with the coefficients (2.3), the parameter σ being confined as in (2.5). As was remarked in chapter 2, the Stokes' phenomenon is absent and a single formula serves to describe a solution over the entire strip given by (2.4a) and (2.4b).

9.2. The solutions. The solutions (2.7) apply to the present differential equation (3) and may be used in the formulas (8b). It is found thus that

$$(9.3) \quad \begin{aligned} u_o(z) &= \frac{i}{2} \left(\frac{1}{\rho^2 \phi_1 \phi} \right)^{1/2} \{ e^{i(\xi - \xi_1)} [1] - e^{-i(\xi - \xi_1)} [1] \}, \\ u_e(z) &= \frac{1}{2} \left(\frac{\phi_1}{\phi} \right)^{1/2} \{ e^{i(\xi - \xi_1)} [1] + e^{-i(\xi - \xi_1)} [1] \}, \end{aligned}$$

$$(9.4) \quad \begin{aligned} u_a(z) &= \frac{i}{2} \left(\frac{1}{\rho^2 \phi_2 \phi} \right)^{1/2} \{ e^{i\xi} [1] - e^{-i\xi} [1] \}, \\ u_g(z) &= \frac{1}{2} \left(\frac{\phi_2}{\phi} \right)^{1/2} \{ e^{i\xi} [1] + e^{-i\xi} [1] \}, \end{aligned}$$

with the symbols evaluated by the relations

$$\begin{aligned} \rho\phi &= i \{ |\Delta| + \Omega \cos 2z \}^{1/2}, \\ \rho\phi_1 &= i \{ |\Delta| + \Omega \}^{1/2}, \quad \rho\phi_2 = i \{ |\Delta| - \Omega \}^{1/2}, \\ i\xi &= \int_{\pi/2}^z \{ |\Delta| + \Omega \cos 2z \}^{1/2} dz, \\ i(\xi - \xi_1) &= \int_0^z \{ |\Delta| + \Omega \cos 2z \}^{1/2} dz. \end{aligned}$$

For real values of z these formulas are found to reduce precisely to the forms (8.4c) and (8.5c), while the forms which describe the solutions of the equation (2) are again found to be those of (8.6). As in the case of chapter 2 the conclusion is possible that the symbols $[]$ may be dropped from the formulas when $\Omega=0$.

Lastly, the formula for the characteristic exponent is that already given in (8.7), and there are, of course, no characteristic values.

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THE MOVING TRIHEDRON*

BY

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1. **Introduction.** A classical method of studying the metric differential geometry of curves and surfaces in three-dimensional space is based upon the use of a moving trihedron. A trihedron of reference is associated with an ordinary point of the curve, or surface, under consideration, and then the point is allowed to vary over the whole, or a suitably restricted portion, thereof. The theory which thus originates is particularly powerful in solving problems concerning two curves, or two surfaces, whose points are in one-to-one correspondence.

The theory of the moving trihedron in the study of curves, as outlined in §2 below, is due to Professor G. A. Bliss, who employed it effectively in his lectures on metric differential geometry at the University of Chicago. It was later also used by the author, to whom the extension to surfaces in the third and fourth sections is due. The essentially new feature of the treatment both for curves and for surfaces is found in the *recursion formulas* upon which the discussion rests. As these do not seem to have appeared elsewhere in the literature, the following exposition is designed to exhibit them and deduce some of their consequences.

2. **Curves.** The method of the moving trihedron as employed in the theory of curves will now be explained. Let us first of all establish an orthogonal cartesian coordinate system, which will be designated hereinafter as the *fixed* coordinate system. Referred to this system let the parametric equations of a real proper non-rectilinear analytic curve C be

$$(1) \quad x = x(s), \quad y = y(s), \quad z = z(s),$$

the parameter s being the arc length measured from some fixed point to the ordinary point $P(x, y, z)$ of C . Further, let us consider a point Q whose coordinates X, Y, Z are given as functions of s by equations of the form

$$(2) \quad X = X(s), \quad Y = Y(s), \quad Z = Z(s).$$

If these three functions of s are all constant, the point Q is fixed, relative to the fixed coordinate system, when the point P varies on the curve C . This case will be excluded hereinafter, unless the contrary is indicated. Then as s varies, the point P moves along the curve C , and the point Q traces a curve

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C_1 represented by the parametric equations (2). The points P, Q of the curves C, C_1 are in one-to-one correspondence, corresponding points being those associated with the same value of the parameter s .

At a point P of a curve C there is the *local coordinate system* with its origin at P , with the ξ -axis along the tangent, the η -axis along the principal normal, and the ζ -axis along the binormal. The equations of transformation between the coordinates X, Y, Z of the point Q that corresponds to P and the local coordinates ξ, η, ζ of Q are

$$\begin{aligned} X &= x + \alpha\xi + l\eta + \lambda\zeta, \\ Y &= y + \beta\xi + m\eta + \mu\zeta, \\ Z &= z + \gamma\xi + n\eta + \nu\zeta, \end{aligned} \quad (3)$$

wherein α, β, γ are the direction cosines, in the fixed coordinate system, of the tangent; l, m, n are those of the principal normal; and λ, μ, ν those of the binormal, of the curve C at the point P .

When the point P moves along the curve C , the local trihedron of C at P also moves, of course, and hence is appropriately called the *moving trihedron of the curve C* . The local coordinate system associated with the moving trihedron will be designated hereinafter as the *moving coordinate system*. The local coordinates ξ, η, ζ of the point Q corresponding to P are themselves functions of s . If these functions are constants, the point Q is rigidly attached to the moving trihedron, so that the motion of Q relative to the moving trihedron is zero.

For the purpose of investigating the relations of the curves C, C_1 , it is convenient to know the direction cosines of the tangent, principal normal, and binormal of C_1 referred to the moving trihedron of C . In order to calculate these, some analytical consequences of equations (3) will next be deduced. If equations (3) are differentiated with respect to s , the results can be reduced, by means of the well known Frenet formulas, to

$$\begin{aligned} X' &= \alpha A_1 + l B_1 + \lambda C_1, \\ Y' &= \beta A_1 + m B_1 + \mu C_1, \\ Z' &= \gamma A_1 + n B_1 + \nu C_1 \quad (X' = dX/ds, \dots), \end{aligned} \quad (4)$$

wherein the coefficients A_1, B_1, C_1 are defined by the formulas

$$A_1 = 1 - \frac{\eta}{\rho} + \xi', \quad B_1 = \frac{\xi}{\rho} + \frac{\zeta}{\tau} + \eta', \quad C_1 = -\frac{\eta}{\tau} + \zeta', \quad (5)$$

and $1/\rho, 1/\tau$ are respectively the curvature and torsion of the curve C at

the point P . A second differentiation and reduction by the Frenet formulas lead to

$$(6) \quad X'' = \alpha A_2 + lB_2 + \lambda C_2$$

and similar formulas for Y'' , Z'' , in which A_2 , B_2 , C_2 are defined by

$$(7) \quad A_2 = -\frac{B_1}{\rho} + A_1', \quad B_2 = \frac{A_1}{\rho} + \frac{C_1}{\tau} + B_1', \quad C_2 = -\frac{B_1}{\tau} + C_1'.$$

Repetition of the process gives

$$(8) \quad X''' = \alpha A_3 + lB_3 + \lambda C_3$$

and similar formulas for Y''' , Z''' , in which A_3 , B_3 , C_3 are defined by

$$(9) \quad A_3 = -\frac{B_2}{\rho} + A_2', \quad B_3 = \frac{A_2}{\rho} + \frac{C_2}{\tau} + B_2', \quad C_3 = -\frac{B_2}{\tau} + C_2'.$$

An easy induction would yield

$$(10) \quad X^{(n)} = \alpha A_n + lB_n + \lambda C_n$$

and similar formulas for $Y^{(n)}$, $Z^{(n)}$, in which the coefficients A_n , B_n , C_n are given by the *recursion formulas*

$$(11) \quad A_n = -\frac{B_{n-1}}{\rho} + A_{n-1}', \quad B_n = \frac{A_{n-1}}{\rho} + \frac{C_{n-1}}{\tau} + B_{n-1}', \quad C_n = -\frac{B_{n-1}}{\tau} + C_{n-1}'.$$

It should be observed that A_n , B_n , C_n are the components in the moving coordinate system of that vector whose components in the fixed coordinate system are the derivatives $X^{(n)}$, $Y^{(n)}$, $Z^{(n)}$. Such a vector may be called a *derivative vector*. The components A_n , B_n , C_n are not themselves actually derivatives, but they behave in some respects like derivatives.

Some additional formulas will now be established. Let us make the convention that the arc length s_1 of the curve C_1 , measured from some fixed point thereon, shall be an increasing function of the arc length s of C . Then squaring and adding equations (4), and taking the positive square root, we find

$$(12) \quad \frac{ds_1}{ds} = (\sum X'^2)^{1/2} = (\sum A_1^2)^{1/2},$$

the summation being for cyclical permutations. Easy calculations now yield

$$(13) \quad \begin{aligned} \frac{ds}{ds_1} &= \frac{1}{(\sum A_1^2)^{1/2}}, \\ \frac{d^2s}{ds_1^2} &= -\frac{\sum A_1 A_2}{(\sum A_1^2)^2}. \end{aligned}$$

Formulas for higher derivatives of s with respect to s_1 could be calculated but will not be needed in what is to follow.

Elementary calculus supplies the formulas

$$(14) \quad \begin{aligned} \frac{dX}{ds_1} &= X' \frac{ds}{ds_1}, \\ \frac{d^2X}{ds_1^2} &= X'' \left(\frac{ds}{ds_1} \right)^2 + X' \frac{d^2s}{ds_1^2}, \\ \frac{d^3X}{ds_1^3} &= X''' \left(\frac{ds}{ds_1} \right)^3 + 3X'' \frac{ds}{ds_1} \frac{d^2s}{ds_1^2} + X' \frac{d^3s}{ds_1^3}, \end{aligned}$$

and similar ones for the derivatives of Y, Z . The second of (14) can be reduced to

$$(15) \quad \frac{d^2X}{ds_1^2} = \alpha L + lM + \lambda N,$$

where L, M, N are defined by

$$(16) \quad \begin{aligned} L &= \frac{1}{(\sum A_1^2)^2} (A_2 \sum A_1^2 - A_1 \sum A_1 A_2), \\ M &= \frac{1}{(\sum A_1^2)^2} (B_2 \sum A_1^2 - B_1 \sum A_1 A_2), \\ N &= \frac{1}{(\sum A_1^2)^2} (C_2 \sum A_1^2 - C_1 \sum A_1 A_2). \end{aligned}$$

Direct calculation results in

$$(17) \quad \frac{dY}{ds_1} \frac{d^2Z}{ds_1^2} - \frac{d^2Y}{ds_1^2} \frac{dZ}{ds_1} = \left(\frac{ds}{ds_1} \right)^3 (\alpha P + lQ + \lambda R),$$

where P, Q, R are defined by

$$(18) \quad P = B_1 C_2 - B_2 C_1, \quad Q = C_1 A_2 - C_2 A_1, \quad R = A_1 B_2 - A_2 B_1.$$

Finally, the curvature $1/\rho_1$ and the torsion $1/\tau_1$ at a point of the curve C_1 can without difficulty be shown to be given by the formulas

$$(19) \quad \begin{aligned} \frac{1}{\rho_1^2} &= \sum L^2 = \frac{\sum P^2}{(\sum A_1^2)^3}, \\ \frac{1}{\tau_1} &= - \frac{1}{\sum P^2} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}. \end{aligned}$$

The direction cosines of the tangent, principal normal, and binormal at a point Q of the curve C_1 , referred to the moving trihedron of the curve C at the corresponding point P , can now be found by the familiar equations of transformation of direction cosines. For example, the *direction cosines of the tangent* of C_1 , referred to the fixed coordinate system, are known to be

$$(20) \quad \frac{dX}{ds_1}, \quad \frac{dY}{ds_1}, \quad \frac{dZ}{ds_1}.$$

Therefore, by equations (4) and the first of (14), the direction cosines of the tangent referred to the moving trihedron are found to be

$$(21) \quad A_1 \frac{ds}{ds_1}, \quad B_1 \frac{ds}{ds_1}, \quad C_1 \frac{ds}{ds_1}.$$

Similarly, the *direction cosines of the principal normal* of C_1 in the fixed coordinate system are known to be

$$(22) \quad \rho_1 \frac{d^2X}{ds_1^2}, \quad \rho_1 \frac{d^2Y}{ds_1^2}, \quad \rho_1 \frac{d^2Z}{ds_1^2},$$

and in the moving coordinate system are found to be

$$(23) \quad \rho_1 L, \quad \rho_1 M, \quad \rho_1 N.$$

Finally, the *direction cosines of the binormal* of C_1 in the fixed coordinate system are known to be

$$(24) \quad \rho_1 \left(\frac{dY}{ds_1} \frac{d^2Z}{ds_1^2} - \frac{d^2Y}{ds_1^2} \frac{dZ}{ds_1} \right)$$

and two similar expressions; hence these direction cosines in the moving coordinate system are

$$(25) \quad \rho_1 \left(\frac{ds}{ds_1} \right)^3 P, \quad \rho_1 \left(\frac{ds}{ds_1} \right)^3 Q, \quad \rho_1 \left(\frac{ds}{ds_1} \right)^3 R.$$

The *direction cosines of the tangent, principal normal, and binormal of the curve C_1 , referred to the moving trihedron of the curve C , are therefore respectively proportional to*

$$(26) \quad A_1, B_1, C_1; L, M, N; P, Q, R.$$

The general theory just outlined is capable of extensive applications. It forms a powerful tool for the study of curves which are transforms of a given curve, such as involutes, evolutes, parallel curves, and so on. But limitations of space do not permit inclusion of such developments here.

3. Surfaces. First of all, some preliminary formulas in surface theory will be collected for subsequent use. Let us consider a real proper analytic surface S , not a sphere or a plane, whose parametric equations in a fixed coordinate system are

$$(27) \quad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v).$$

Let the lines of curvature be the parametric curves on the surface S , so that

$$(28) \quad F = 0, \quad D' = 0,$$

in the classical notation of Eisenhart and Bianchi. The direction cosines $\alpha^u, \beta^u, \gamma^u$ of the u -tangent at a point $P(x, y, z)$ of S are given by the formulas

$$(29) \quad \alpha^u = \frac{x_u}{E^{1/2}}, \quad \beta^u = \frac{y_u}{E^{1/2}}, \quad \gamma^u = \frac{z_u}{E^{1/2}}.$$

Similarly, the direction cosines $\alpha^v, \beta^v, \gamma^v$ of the v -tangent of S at P are given by

$$(30) \quad \alpha^v = \frac{x_v}{G^{1/2}}, \quad \beta^v = \frac{y_v}{G^{1/2}}, \quad \gamma^v = \frac{z_v}{G^{1/2}},$$

and the direction cosines a, b, c of the normal of S at P by

$$(31) \quad a = \frac{y_u z_v - y_v z_u}{(EG)^{1/2}}, \quad b = \frac{z_u x_v - z_v x_u}{(EG)^{1/2}}, \quad c = \frac{x_u y_v - x_v y_u}{(EG)^{1/2}}.$$

The curvilinear parametric equations of any curve C through the point P on the surface S are

$$(32) \quad u = u(s), \quad v = v(s),$$

the parameter s being the arc length measured from some fixed point of C . The direction cosines α, β, γ of the tangent of C at P are expressed by the formulas

$$(33) \quad \alpha = x_u u' + x_v v', \quad \beta = y_u u' + y_v v', \quad \gamma = z_u u' + z_v v' \quad (u' = du/ds, \dots).$$

Let θ be the angle from the positive half of the u -tangent to the positive half of the tangent of the curve C at the point P . Then one has

$$(34) \quad \cos \theta = E^{1/2} u', \quad \sin \theta = G^{1/2} v'.$$

The principal normal curvatures $1/R_1, 1/R_2$ of the surface S are given by the formulas

$$(35) \quad \frac{1}{R_1} = \frac{D}{E}, \quad \frac{1}{R_2} = \frac{D''}{G},$$

and the geodesic curvatures $1/r_1, 1/r_2$ of the lines of curvature by

$$(36) \quad \frac{1}{r_1} = -\frac{E_v}{2EG^{1/2}}, \quad \frac{1}{r_2} = +\frac{G_u}{2GE^{1/2}},$$

the subscript 1 in each case denoting the function associated with the u -curve, and 2 that with the v -curve, at a point P .

Formulas analogous to the Frenet formulas can be established for the local trihedron whose edges are the tangents of the lines of curvature and the normal at a point P of a surface S . These formulas express the derivatives, with respect to the arc length s of a curve C , of the direction cosines of the three edges of the local trihedron linearly in terms of these cosines themselves, the coefficients depending upon the functions $\theta, R_1, R_2, r_1, r_2$. In fact, actual calculation, the details of which will be omitted, leads to the formulas in question, namely,

$$(37) \quad \begin{aligned} (\alpha^u)' &= \left(\frac{\cos \theta}{r_1} + \frac{\sin \theta}{r_2} \right) \alpha^v + \frac{\cos \theta}{R_1} a, \\ (\alpha^v)' &= - \left(\frac{\cos \theta}{r_1} + \frac{\sin \theta}{r_2} \right) \alpha^u + \frac{\sin \theta}{R_2} a, \\ a' &= - \frac{\cos \theta}{R_1} \alpha^u - \frac{\sin \theta}{R_2} \alpha^v \quad (a' = da/ds, \dots). \end{aligned}$$

and similar formulas for the remaining derivatives. With these should be associated the easily verified result

$$(38) \quad x' = \cos \theta \alpha^u + \sin \theta \alpha^v,$$

with similar expressions for y', z' .

Let us establish a local coordinate system at a point P of a surface S , referred to its lines of curvature, with the origin at P , the ξ -axis along the u -tangent, the η -axis along the v -tangent, and the ζ -axis along the normal of S at P . The equations of transformation between the coordinates X, Y, Z of any point Q (supposed to be functions of u, v , and referred to the fixed coordinate system) and the local coordinates ξ, η, ζ of Q are

$$(39) \quad \begin{aligned} X &= x + \alpha^u \xi + \alpha^v \eta + a \zeta, \\ Y &= y + \beta^u \xi + \beta^v \eta + b \zeta, \\ Z &= z + \gamma^u \xi + \gamma^v \eta + c \zeta. \end{aligned}$$

Recursion formulas exactly analogous to those in §1 can be obtained by repeated differentiation of these equations. Differentiating once with respect to the arc length s of the curve C we find, by means of (37), (38),

$$(40) \quad X' = \alpha^u A_1 + \alpha^v B_1 + a C_1$$

and similar formulas for Y' , Z' , in which the coefficients A_1 , B_1 , C_1 are defined by

$$(41) \quad \begin{aligned} A_1 &= \cos \theta \left(1 - \frac{\eta}{r_1} - \frac{\xi}{R_1} \right) - \sin \theta \frac{\eta}{r_2} + \xi', \\ B_1 &= \cos \theta \frac{\xi}{r_1} + \sin \theta \left(1 + \frac{\xi}{r_2} - \frac{\xi}{R_2} \right) + \eta', \\ C_1 &= \cos \theta \frac{\xi}{R_1} + \sin \theta \frac{\eta}{R_2} + \xi'. \end{aligned}$$

A second differentiation, followed by appropriate reduction, gives

$$(42) \quad X'' = \alpha^u A_2 + \alpha^v B_2 + a C_2,$$

where A_2 , B_2 , C_2 are defined by

$$(43) \quad \begin{aligned} A_2 &= \cos \theta \left(-\frac{B_1}{r_1} - \frac{C_1}{R_1} \right) - \sin \theta \frac{B_1}{r_2} + A_1', \\ B_2 &= \cos \theta \frac{A_1}{r_1} + \sin \theta \left(\frac{A_1}{r_2} - \frac{C_1}{R_2} \right) + B_1', \\ C_2 &= \cos \theta \frac{A_1}{R_1} + \sin \theta \frac{B_1}{R_2} + C_1'. \end{aligned}$$

In general we find

$$(44) \quad X^{(n)} = \alpha^u A_n + \alpha^v B_n + a C_n$$

where the local components A_n , B_n , C_n of the derivative vector $X^{(n)}$, $Y^{(n)}$, $Z^{(n)}$ are given by the *recursion formulas*

$$(45) \quad \begin{aligned} A_n &= \cos \theta \left(-\frac{B_{n-1}}{r_1} - \frac{C_{n-1}}{R_1} \right) - \sin \theta \frac{B_{n-1}}{r_2} + A_{n-1}', \\ B_n &= \cos \theta \frac{A_{n-1}}{r_1} + \sin \theta \left(\frac{A_{n-1}}{r_2} - \frac{C_{n-1}}{R_2} \right) + B_{n-1}', \\ C_n &= \cos \theta \frac{A_{n-1}}{R_1} + \sin \theta \frac{B_{n-1}}{R_2} + C_{n-1}'. \end{aligned}$$

With the definitions of the functions A_n , B_n , C_n employed in this section, the formulas (12), \dots , (26) of §2 can easily be shown to be equally valid for the local trihedron of surface theory. One thus obtains a theory differing

from that of §2 only in two particulars; namely, the curve C is now supposed to lie on a given surface; and a different local trihedron is now being associated with the curve C . These considerations will not be pursued further here.

The principal interest in the theory of the moving trihedron in surface theory arises when the point P , instead of tracing a curve C on the surface S , is allowed to vary over a suitably restricted region of S . In this case the local components of the partial derivative vectors are required. These may be obtained by specializing equations (37), (38), and (40), . . . , (45), if it is kept in mind that

$$(46) \quad ds^u = E^{1/2} du, \quad ds^v = G^{1/2} dv,$$

where s^u, s^v denote arc lengths on the parametric curves. The required formulas can also be calculated directly. Either way one finds

$$(47) \quad X_u = A^u \alpha^u + B^u \alpha^v + C^u a, \quad X_v = A^v \alpha^u + B^v \alpha^v + C^v a,$$

and similar formulas for the first partial derivatives of Y, Z , where the coefficients A^u, \dots, A^v, \dots are defined by the formulas

$$(48) \quad \begin{aligned} \frac{A^u}{E^{1/2}} &= 1 - \frac{\eta}{r_1} - \frac{\xi}{R_1} + \frac{\xi_u}{E^{1/2}}, & \frac{A^v}{G^{1/2}} &= -\frac{\eta}{r_2} + \frac{\xi_v}{G^{1/2}}, \\ \frac{B^u}{E^{1/2}} &= \frac{\xi}{r_1} + \frac{\eta_u}{E^{1/2}}, & \frac{B^v}{G^{1/2}} &= 1 + \frac{\xi}{r_2} - \frac{\xi}{R_2} + \frac{\eta_v}{G^{1/2}}, \\ \frac{C^u}{E^{1/2}} &= \frac{\xi}{R_1} + \frac{\xi_u}{E^{1/2}}, & \frac{C^v}{G^{1/2}} &= \frac{\eta}{R_2} + \frac{\xi_v}{G^{1/2}}. \end{aligned}$$

Further differentiation, followed by appropriate reductions, yields

$$(49) \quad \begin{aligned} X_{uu} &= A^{uu} \alpha^u + B^{uu} \alpha^v + C^{uu} a, \\ X_{uv} &= A^{uv} \alpha^u + B^{uv} \alpha^v + C^{uv} a, \\ X_{vv} &= A^{vv} \alpha^u + B^{vv} \alpha^v + C^{vv} a, \end{aligned}$$

and similar formulas for the second partial derivatives of Y, Z , where

$$(50) \quad \begin{aligned} \frac{A^{uu}}{E^{1/2}} &= -\frac{B^u}{r_1} - \frac{C^u}{R_1} + \frac{A_u^u}{E^{1/2}}, \\ \frac{B^{uu}}{E^{1/2}} &= \frac{A^u}{r_1} + \frac{B_u^u}{E^{1/2}}, \\ \frac{C^{uu}}{E^{1/2}} &= \frac{A^u}{R_1} + \frac{C_u^u}{E^{1/2}}, \end{aligned}$$

$$\begin{aligned}
 \frac{A^{uv}}{G^{1/2}} &= -\frac{B^u}{r_2} + \frac{A_v}{G^{1/2}}, & \frac{A^{uv}}{E^{1/2}} &= -\frac{B^v}{r_1} - \frac{C^v}{R_1} + \frac{A_u^v}{E^{1/2}}, \\
 \frac{B^{uv}}{G^{1/2}} &= \frac{A^u}{r_2} - \frac{C^u}{R_2} + \frac{B_v^u}{G^{1/2}}, & \frac{B^{uv}}{E^{1/2}} &= \frac{A^v}{r_1} + \frac{B_u^v}{E^{1/2}}, \\
 \frac{C^{uv}}{G^{1/2}} &= \frac{B^u}{R_2} + \frac{C_v^u}{G^{1/2}}, & \frac{C^{uv}}{E^{1/2}} &= \frac{B^v}{R_1} + \frac{C_u^v}{E^{1/2}}, \\
 \frac{A^{vv}}{G^{1/2}} &= -\frac{B^v}{r_2} + \frac{A_v^v}{G^{1/2}}, \\
 \frac{B^{vv}}{G^{1/2}} &= \frac{A^v}{r_2} - \frac{C^v}{R_2} + \frac{B_v^v}{G^{1/2}}, \\
 \frac{C^{vv}}{G^{1/2}} &= \frac{B^v}{R_2} + \frac{C_v^v}{G^{1/2}}.
 \end{aligned}
 \tag{50}$$

The calculation of the local components of derivative vectors of higher order than the second is now purely mechanical, but none of them will be used hereinafter, and recursion formulas for the local components of the derivative vectors of the n th order need not be written.

Let us suppose for the present that the locus of the point Q , when u, v vary, is a proper surface S_1 , and let the six fundamental coefficients and other functions for this surface be indicated by subscripts 1. For the first three fundamental coefficients we find, by easy calculations from equations (47),

$$(51) \quad E_1 = \sum A^{u^2}, \quad F_1 = \sum A^u A^v, \quad G_1 = \sum A^{v^2},$$

whence

$$(52) \quad H_1^2 = E_1 G_1 - F_1^2 = \sum (B^u C^v - B^v C^u)^2.$$

The direction cosines of the u -tangent at a point of the surface S_1 , referred to the moving trihedron of the surface S at the corresponding point P , are found to be

$$(53) \quad \frac{A^u}{E_1^{1/2}}, \quad \frac{B^u}{E_1^{1/2}}, \quad \frac{C^u}{E_1^{1/2}},$$

and similarly the direction cosines of the v -tangent are

$$(54) \quad \frac{A^v}{G_1^{1/2}}, \quad \frac{B^v}{G_1^{1/2}}, \quad \frac{C^v}{G_1^{1/2}},$$

while the direction cosines of the normal of S_1 are

$$(55) \quad \frac{1}{H_1}(B^u C^v - B^v C^u), \quad \frac{1}{H_1}(C^u A^v - C^v A^u), \quad \frac{1}{H_1}(A^u B^v - A^v B^u).$$

Finally, the second fundamental coefficients for the surface S_1 are found to be given by

$$(56) \quad \begin{aligned} D_1 &= \frac{1}{H_1} \sum A^{uu}(B^u C^v - B^v C^u), \\ D_1' &= \frac{1}{H_1} \sum A^{uv}(B^u C^v - B^v C^u), \\ D_1'' &= \frac{1}{H_1} \sum A^{vv}(B^u C^v - B^v C^u). \end{aligned}$$

Since the six fundamental coefficients for the surface S_1 have been calculated, it is only a formal matter to write the expressions for the mean and total curvatures, the equation of the lines of curvature, etc., for the surface S_1 in terms of the components A^u, \dots, A^v, \dots .

4. Applications. Some applications of the theory of the moving trihedron in the theory of surfaces, as explained in the preceding section, will now engage our attention. First of all, equations (47) and the similar equations for Y, Z show that the point $Q(X, Y, Z)$ is fixed relative to the fixed coordinate system if, and only if,

$$A^u = B^u = C^u = A^v = B^v = C^v = 0.$$

Equations (48) now yield necessary and sufficient conditions that the point Q be fixed relative to the fixed coordinate system, namely,

$$(57) \quad \begin{aligned} \xi_u &= E^{1/2} \left(-1 + \frac{\eta}{r_1} + \frac{\zeta}{R_1} \right), & \xi_v &= G^{1/2} \left(\frac{\eta}{r_2} \right), \\ \eta_u &= E^{1/2} \left(-\frac{\xi}{r_1} \right), & \eta_v &= G^{1/2} \left(-1 - \frac{\xi}{r_2} + \frac{\zeta}{R_2} \right), \\ \zeta_u &= E^{1/2} \left(-\frac{\xi}{R_1} \right), & \zeta_v &= G^{1/2} \left(-\frac{\eta}{R_2} \right). \end{aligned}$$

These conditions are very useful in solving envelope problems of a type which will now be described. Let us consider a surface S referred to its lines of curvature, and a two-parameter family of surfaces such that one of them, S_1 , is associated with each point P of S . Let the equation of S_1 be

$$f(\xi, \eta, \zeta, u, v) = 0,$$

in which ξ, η, ζ are local coordinates referred to the moving trihedron of S

at P , and u, v are the curvilinear coordinates of P . It may be required to find the envelope of the surface S_1 when the point P describes a curve or a region of the surface S . The usual method of investigating the envelope entails the differentiation of the functions ξ, η, ζ with respect to u and v , and it will next be shown that the conditions (57) are precisely the needed *formulas for the differentiation of local point coordinates*. For this purpose, let us observe that the result of solving equations (39) for ξ, η, ζ is

$$\begin{aligned} \xi &= \alpha^u(X - x) + \beta^u(Y - y) + \gamma^u(Z - z), \\ \eta &= \alpha^v(X - x) + \beta^v(Y - y) + \gamma^v(Z - z), \\ \zeta &= a(X - x) + b(Y - y) + c(Z - z). \end{aligned} \quad (58)$$

Consequently the equation of the surface S_1 referred to the *fixed* coordinate system can be written in the form

$$f\left(\sum \alpha^u(X - x), \sum \alpha^v(X - x), \sum a(X - x), u, v\right) = 0,$$

the summation being for cyclical permutations. Since u, v occur explicitly and also in $\alpha^u, \alpha^v, a, x, \dots$, but not in X, Y, Z , partial differentiation yields

$$\begin{aligned} f_{\xi}\xi_u + f_{\eta}\eta_u + f_{\zeta}\zeta_u + f_u &= 0, \\ f_{\xi}\xi_v + f_{\eta}\eta_v + f_{\zeta}\zeta_v + f_v &= 0, \end{aligned} \quad (59)$$

where the partial derivatives of ξ, η, ζ are to be calculated from equations (58) by direct differentiation with X, Y, Z fixed. If use is made of equations (37), suitably specialized, to obtain the partial derivatives of $\alpha^u, \alpha^v, a, \dots$ as linear combinations of $\alpha^u, \alpha^v, a, \dots$, and if equations (58) themselves are then employed to express the derivatives of ξ, η, ζ as functions of ξ, η, ζ , the result of the differentiation can be reduced to equations (57), as was to be shown.

By way of illustration let us consider the osculating plane of the u -curve at the point P of the surface S . If the equation of this plane, referred to the fixed coordinate system, is written in the usual form, the equations of transformation (39) and the equations (29) together with the equations obtained by differentiating the latter with respect to u can be used to show that the local equation of the *osculating plane of the u -curve* is

$$\frac{\eta}{R_1} - \frac{\zeta}{r_1} = 0. \quad (60)$$

If this equation is differentiated with respect to v , the result can be reduced by means of one of the conditions of Codazzi, namely,

$$(61) \quad \left(\frac{1}{R_1}\right)_v = -\frac{G^{1/2}}{r_1} \left(\frac{1}{R_2} - \frac{1}{R_1}\right),$$

to

$$(62) \quad \frac{\xi}{r_2} - \frac{\eta}{r_1} + 1 + \left[\frac{1}{G^{1/2}} \left(\frac{1}{r_1}\right)_v - \frac{1}{R_1 R_2} \right] R_1 \zeta = 0,$$

provided that the surface S is not developable. Equations (60), (62) taken together are the equations of the characteristic of the osculating plane of the u -curve when v varies. The equations of the orthogonal projection of this characteristic onto the tangent plane are

$$(63) \quad \zeta = 0, \quad \frac{\xi}{r_2} - \frac{\eta}{r_1} + 1 + \left[\frac{1}{G^{1/2}} \left(\frac{1}{r_1}\right)_v - \frac{1}{R_1 R_2} \right] r_1 \eta = 0.$$

Since the equations of the ray of the lines of curvature, namely, the straight line joining the Laplace transformed points or ray-points $(0, r_1, 0)$ and $(-r_2, 0, 0)$, are

$$(64) \quad \zeta = 0, \quad \frac{\xi}{r_2} - \frac{\eta}{r_1} + 1 = 0,$$

it follows that the orthogonal projection of the characteristic of the osculating plane of the u -curve, when v varies, onto the tangent plane coincides with the ray if, and only if,

$$(65) \quad \frac{1}{G^{1/2}} \left(\frac{1}{r_1}\right)_v - \frac{1}{R_1 R_2} = 0.$$

Differentiation of equation (62) would enable us to find the edge of regression of the developable enveloped by the osculating plane of the u -curve when v varies.

The equation of the rectifying plane of the u -curve at the point P can easily be shown to be

$$(66) \quad \frac{\eta}{r_1} + \frac{\zeta}{R_1} = 0,$$

since this plane must contain the tangent line, $\eta = \zeta = 0$, and must be perpendicular to the osculating plane (60) of the u -curve. The equations of the characteristic of this plane when v varies can be found by the method just used for the osculating plane. The equations of the orthogonal projection of this line onto the tangent plane turn out to differ from equations (63) only in that the sign of η has been changed. Therefore the projections onto the tangent plane of the characteristics of the osculating plane and rectifying plane of the

u-curve, when *v* varies, are symmetrically placed with respect to the tangent line of the *u*-curve.

The machinery of the local trihedron can be efficiently used to investigate the focal surfaces of the congruence of normals of a surface *S*, and the Laplace transformed nets of the lines of curvature on *S*, but as the principal results are well known, this study need not be entered upon here. It may be worthy of comment, however, that it is easy to locate the centers of the osculating circle and osculating sphere of the *u*-curve. Differentiating the equation $\xi = 0$ of the normal plane of the *u*-curve with respect to *u* we obtain the equations of the polar line of the *u*-curve at the point *P*, namely,

$$(67) \quad \xi = 0, \quad \frac{\eta}{r_1} + \frac{\zeta}{R_1} = 1.$$

This line intersects the osculating plane (60) in the center of the osculating circle of the *u*-curve, whose coordinates are thus found to be

$$0, \quad \frac{\rho_1^2}{r_1}, \quad \frac{\rho_1^2}{R_1},$$

the radius of curvature ρ_1 of the *u*-curve being given by

$$(68) \quad \frac{1}{\rho_1^2} = \frac{1}{r_1^2} + \frac{1}{R_1^2}.$$

The polar line meets the surface normal, $\xi = \eta = 0$, at the center of the principal normal curvature corresponding to the *u*-curve $(0, 0, R_1)$, and meets the *v*-tangent, $\xi = \zeta = 0$, at the ray-point of the *u*-curve $(0, r_1, 0)$. A second differentiation with respect to *u* and solution of three simultaneous equations yield the coordinates of the center of the osculating sphere of the *u*-curve, namely,

$$0, \rho_1 \left(\frac{\rho_1}{r_1} + \frac{\tau_1 \rho_{1u}}{R_1 E^{1/2}} \right), \quad \rho_1 \left(\frac{\rho_1}{R_1} - \frac{\tau_1 \rho_{1u}}{r_1 E^{1/2}} \right),$$

the torsion $1/\tau_1$ of the *u*-curve being given by

$$(69) \quad E^{1/2} \frac{1}{\rho_1^2} \frac{1}{\tau_1} = \frac{1}{R_1} \left(\frac{1}{r_1} \right)_u - \frac{1}{r_1} \left(\frac{1}{R_1} \right)_u.$$

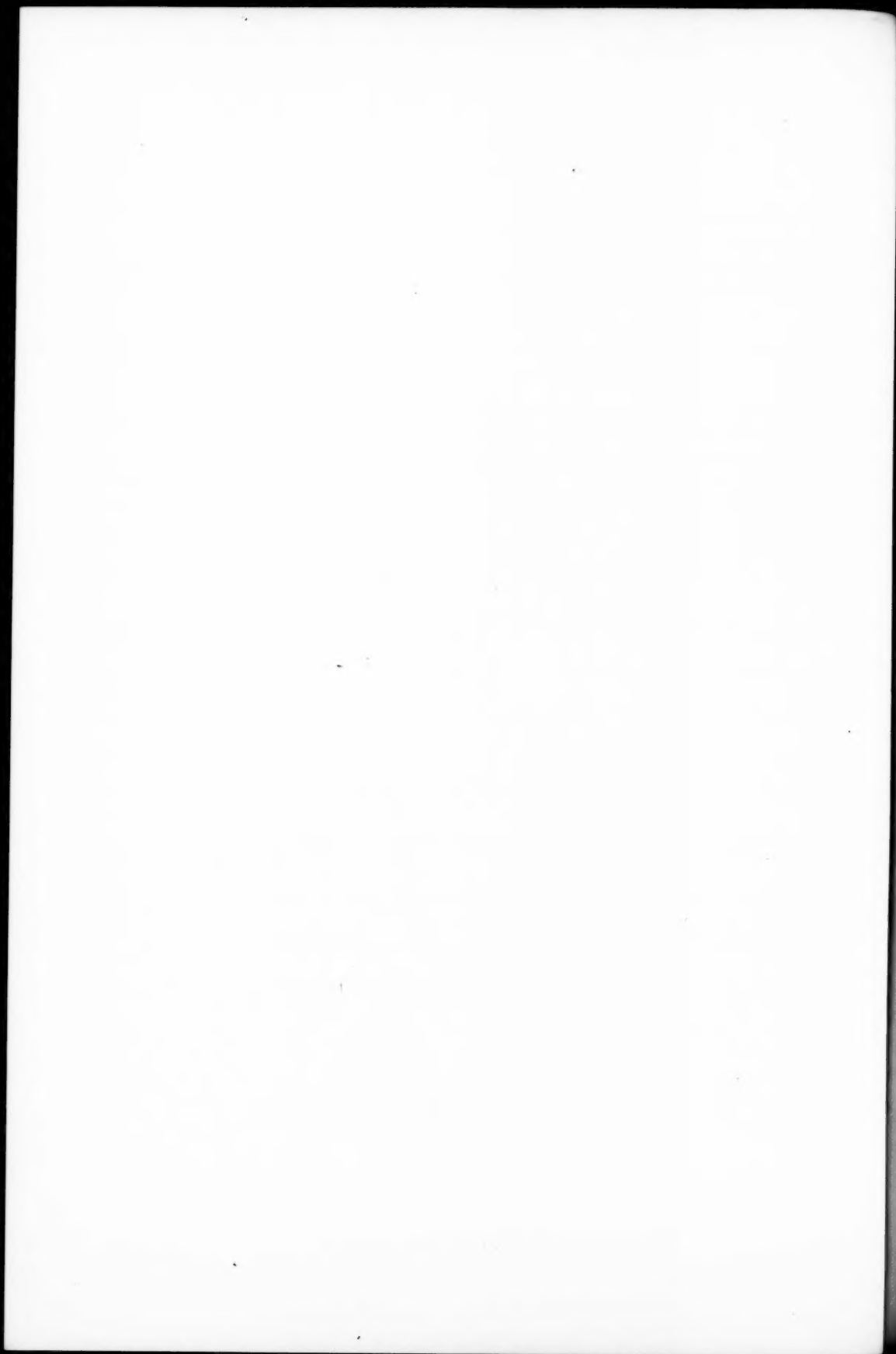
The usual formula for the radius of the osculating sphere, in terms of ρ_1 , τ_1 , and their derivatives with respect to the arc length of the *u*-curve, could easily be used to write down a condition necessary and sufficient that one family of lines of curvature, namely, the *u*-curves, on a surface be spherical. Similar results can be obtained with the *u*-curves and *v*-curves interchanged.

Necessary and sufficient conditions that the surface S_1 generated by the point Q may be obtainable from the surface S by a translation can be found in the following way. In case these surfaces differ only by a translation, the differences $X-x$, $Y-y$, $Z-z$ are constants. Differentiating equations (39) under this assumption we find the required conditions, namely,

$$(70) \quad \begin{array}{ll} A^u = E^{1/2}, & A^v = 0, \\ B^u = 0, & B^v = G^{1/2}, \\ C^u = 0, & C^v = 0. \end{array}$$

These conditions are equivalent to the conditions (57) with the modification that the terms consisting of the number -1 must be deleted from the parentheses in the formulas for ξ_u , η_v therein.

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PROPERTIES OF FUNCTIONS $f(x, y)$ OF BOUNDED VARIATION*

BY

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1. INTRODUCTION

In a recent paper† we investigated the relations between several definitions of bounded variation for functions $f(x, y)$ of two real variables.‡ These definitions are usually associated with the names of Vitali, Hardy, Arzelà, Pierpont, Fréchet, and Tonelli respectively; we proved the equivalence of the definition formulated by Pierpont and the modified form of it given by Hahn.§

Since the several definitions were assembled in CA, it is hardly necessary to repeat them here. But we shall again denote the classes of functions satisfying the respective definitions by V , H , A , P , F , and T . In addition the class of functions continuous in (x, y) will be designated by C , the class of functions belonging to the Baire classification by B , the class of functions having measurable total variation functions¶ $\phi(\bar{x})$ and $\psi(\bar{y})$ by $M_{\phi, \psi}$, and the class of functions having superficial measure by M ; and the common part of two or more classes will be indicated by the product of the corresponding letters. The domain of definition of $f(x, y)$ is generally to be understood as a rectangle with sides parallel to the axes** ($a \leq x \leq b$, $c \leq y \leq d$); the letter R , with or without a subscript, will always stand for such a rectangle.

Functions $g(x)$ of bounded variation are of great interest and usefulness because of their valuable properties, particularly with respect to additivity, decomposability into monotone functions, continuity, differentiability, meas-

* Presented to the Society, December 27, 1933; received by the editors April 30, 1934.

† Clarkson and Adams, *On definitions of bounded variation for functions of two variables*, these Transactions, vol. 35 (1933), pp. 824-854. Hereafter this paper will be referred to as CA.

‡ Since the paper CA was written, our attention has been called to two additional definitions; of these the first is due to Wiener, *Laplacians and continuous linear functionals*, Acta Szeged, vol. 3 (1927), pp. 7-16. The second is that of Nalli and Andreoli, *Sull' area di una superficie, sugli integrali multipli di Stieltjes e sugli integrali multipli delle funzioni di più variabili complesse*, Accademia dei Lincei, Rendiconti, (6), vol. 5 (1927), pp. 963-966. The fact that class $T \cdot C$ contains as a proper subclass all continuous functions satisfying the definition of Nalli and Andreoli or a modified form of it has been shown by Tonelli, *Sulla definizione di funzione di due variabili a variazione limitata*, ibid., (6), vol. 7 (1928), pp. 357-363. In this sequel to CA these additional definitions will not be further considered.

§ This will be spoken of as the P_H -form of Pierpont's definition.

¶ This must not be confused with B in CA, which stood for the class of bounded functions.

¶ $\phi(\bar{x})[\psi(\bar{y})]$ represents the total variation of $f(x, y)[f(x, y)]$ in $y[x]$; see CA.

** For brevity we shall sometimes indicate such a closed rectangle by the notation $(a, c; b, d)$.

urability, integrability, etc.; and it is largely to the possession of these properties that such functions owe their important role in the study of rectifiable curves, Fourier and other series, Stieltjes and other integrals, and the calculus of variations. Proposers of definitions of bounded variation for functions $f(x, y)$ have been actuated mainly by the desire to single out for attention a class of functions having properties analogous to some *particular* properties of a function $g(x)$ of bounded variation. It has long since become apparent that to preserve properties of one sort the definition of bounded variation for $g(x)$ should be extended to $f(x, y)$ in one way, while to preserve properties of another sort a quite different extension may be needed.

It is natural that in CA the only detailed study of properties of functions $f(x, y)$ belonging to the several classes V, H, A, P, F , and T should have had to do with the nature of the total variation functions $\phi(\bar{x})$ and $\psi(\bar{y})$, since properties of this kind seemed to bear most directly upon the problem of determining relations between the classes. Properties of functions belonging to the classes V and F (and by implication H) with respect to double Stieltjes integrals of the Riemann type have recently been examined by Clarkson.* It would seem worth while to make a systematic study of the properties of additivity, decomposability, etc., enjoyed by functions belonging to each of the six classes, and it is to this object that the present paper is mainly devoted. The determination of such properties has by no means been utterly neglected by previous writers; indeed we shall state a few results that are already well known, and certain of our theorems will constitute extensions of such results.

It will appear that the aggregate of functions in class T lacks certain desirable properties because of the necessity for $\phi(\bar{x})$ and $\psi(\bar{y})$ to be measurable. And the evidence seems to indicate that the definition of Tonelli, precisely as formulated by him, may attain its greatest usefulness when applied to functions which to a certain extent are well behaved, perhaps to the extent of belonging to the Baire classification. In order that a function $f(x, y)$ may not fail to be included in the class merely because its ϕ or ψ is non-measurable, we define the extended class T to consist of those functions f for which ϕ and ψ are respectively dominated by summable functions; this class we designate by \bar{T} . Such extension of Tonelli's class has proved desirable in recent work by Gergen† and by Morrey.‡

* Clarkson, *On double Riemann-Stieltjes integrals*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 929-936.

† Gergen, *Convergence criteria for double Fourier series*, these Transactions, vol. 35 (1933), pp. 29-63.

‡ Morrey, *A class of representations of manifolds. I*, American Journal of Mathematics, vol. 55 (1933), pp. 683-707.

Throughout this paper the difference operators Δ_{10} , Δ_{01} , and Δ_{11} , when applied to $f(x_i, y_j)$, will have the following meaning:

$$\Delta_{10}f(x_i, y_j) = f(x_{i+1}, y_j) - f(x_i, y_j),$$

$$\Delta_{01}f(x_i, y_j) = f(x_i, y_{j+1}) - f(x_i, y_j),$$

$$\Delta_{11}f(x_i, y_j) = \Delta_{10}(\Delta_{01}f(x_i, y_j)).$$

When applied to $f(x, y)$, the operators will have a similar significance, it being understood that the increments of x and y involved are greater than zero but otherwise arbitrary.

2. A PROPERTY OF CLASS P

THEOREM 1. *If $f(x, y)$ is in class P , $\phi(\bar{x})[\psi(\bar{y})]$ is dominated by a summable function.*

For each $n \geq 1$ let N_n designate the net of n^2 cells used in the P_H -form of the definition, and denote by $\phi_n^*(\bar{x})$ the sum of the oscillations of f in the cells of that column in whose base \bar{x} lies. For definiteness we may associate \bar{x} , when it is the coordinate of a line of N_n other than $x=a$, with the subinterval of (a, b) whose right-hand end point is \bar{x} . Then $\phi_n^*(\bar{x})$ is a step-function and, if B denotes a bound for the P_H -sum, we have for each n

$$(1) \quad \int_a^b \phi_n^*(x) dx = \frac{b-a}{n} \sum_{r=1}^{n^2} \omega_r' \leq B(b-a).$$

For each \bar{x} let

$$\phi^*(\bar{x}) = \liminf_{n \rightarrow \infty} \phi_n^*(\bar{x});$$

in the light of (1) it is known[†] that $\phi^*(\bar{x})$ is summable in (a, b) . Next let

$$\phi_n^{**}(\bar{x}) = \sum_{i=1}^n [\text{oscillation of } f(\bar{x}, y) \text{ in the interval } y_{i-1} \leq y \leq y_i]$$

for each n , and set

$$\phi^{**}(\bar{x}) = \liminf_{n \rightarrow \infty} \phi_n^{**}(\bar{x}).$$

For every \bar{x} in (a, b) for which $\phi(\bar{x})$ is finite we have[‡]

$$(2) \quad \phi^{**}(\bar{x}) \geq \phi(\bar{x})/2,$$

[†] See Schlesinger and Plessner, *Lebesguesche Integrale und Fouriersche Reihen*, Berlin, 1926, p. 91.

[‡] See Hobson, *Theory of Functions of a Real Variable*, 3d edition, vol. 1, Cambridge, 1927, p. 331. It is easily proved that the total variation and total fluctuation of any function $g(x)$ are equal when both are finite, and that if either is infinite the other is likewise.

and it is easily seen that when $\phi(\bar{x})$ is infinite, $\phi^{**}(\bar{x})$ is likewise. Moreover, for each \bar{x} we have

$$\phi_n^*(\bar{x}) \geq \phi_n^{**}(\bar{x})$$

for all n , whence

$$(3) \quad \phi^*(\bar{x}) \geq \phi^{**}(\bar{x})$$

except when $\phi^{**}(\bar{x})$ is infinite, in which case $\phi^*(\bar{x})$ is also infinite. The theorem for $\phi(\bar{x})$ now follows from inequalities (2) and (3); a similar proof may be given for $\psi(\bar{y})$.

COROLLARY 1. *If $f(x, y)$ is in class P and $\phi(\bar{x})$ and $\psi(\bar{y})$ are measurable,† $f(x, y)$ is also in class T .*

The common part of the overlapping classes P and T may now be specified by the relation $P \cdot T = P \cdot M_{\phi, \psi}$.

COROLLARY 2. *If $f(x, y)$ is in class P , $\phi(\bar{x})[\psi(\bar{y})]$ is finite almost everywhere.‡*

That ϕ may be infinite at an everywhere dense set§ and that $\partial f/\partial x$ and $\partial f/\partial y$ may fail to exist (finite or infinite) at a set everywhere dense in the rectangle R , when f is in P , is illustrated by the following example. Let the rational points in the interval $0 \leq x \leq 1$ be enumerated as x_1, x_2, \dots ; for $x = x_n (n = 1, 2, \dots)$ and y rational ($0 \leq y \leq 1$) let $f(x, y) = 1/2^n$; elsewhere in the unit square $I(0, 0; 1, 1)$ let $f = 0$.

From Theorem 1 we have $\bar{T} \geq P$; the relation $\bar{T} > P$ then follows from example (D) of CA. The fundamental relations of inclusiveness between the several classes are therefore

$$(4) \quad \bar{T} > P > A > H, \quad F > V > H, \quad \bar{T} > T > H;$$

and when only functions belonging to the Baire classification are admitted to consideration||,

† Montgomery, *Properties of plane sets and functions of two variables*, to appear in the American Journal of Mathematics, Theorem 17, has shown that $f \in B$ implies measurability of ϕ and ψ .

‡ Although Theorem 2 of CA was sufficient for the purposes of that paper, this corollary improves the result.

§ This first fact was illustrated by the example following the proof of Theorem 2 in CA, but the example given here is somewhat more easily shown to be in P .

|| These relations are an immediate consequence of the results of CA in conjunction with Montgomery's Theorem 17, loc. cit. From the standpoint of continuity we may remark that the inclusiveness relations are like (5) when only functions possessing one of the following properties are admitted to consideration: continuity in (x, y) [see CA], continuity in x and in y , semi-continuity in (x, y) , upper semi-continuity in one variable and lower semi-continuity in the other. A function having this last property belongs to Baire's class 1 at most; see Kempisty, *Sur les fonctions semicontinues par*

$$(5) \quad T \cdot B > P \cdot B > A \cdot B > H \cdot B, \quad F \cdot B > V \cdot B > H \cdot B.$$

These are the basis for numerous statements in the following pages.

3. CLOSURE OF THE SEVERAL CLASSES UNDER ARITHMETIC OPERATIONS

THEOREM 2. *Each of the classes V, H, A, P, F , and \bar{T} is closed under addition (and subtraction).^{*} This is not true[†] of T .*

The first part of this theorem is an immediate consequence of the definitions. For the second part we may break up example (C) of CA into monotone components as follows: E being a linearly non-measurable set of points on the downward-sloping diagonal d of the unit square $I(0, 0; 1, 1)$, set

$$f_1(x, y) = \begin{cases} 0 & \text{below } d, \\ 1 & \text{above } d, \\ 1 & \text{on } E, \\ 0 & \text{elsewhere on } d. \end{cases} \quad f_2(x, y) = \begin{cases} 0 & \text{below and on } d, \\ 1 & \text{above } d. \end{cases}$$

Each of these functions is clearly in T , although $f_1 - f_2$, which is example (C) of CA, is not.

THEOREM 3. *Each of the classes H, A , and P is closed under multiplication.[‡] This is not true of V, F, T , or \bar{T} .*

For H and A the theorem may readily be proved by aid of decomposition theorems given in §4. Since P contains only bounded functions, the proof for P flows at once from

LEMMA 1. *Let f_1 and f_2 be functions of any number of variables, defined for an arbitrary range of variation S of those variables. If f_1 and f_2 are bounded, and the least upper bound of $|f_i|$ is denoted by $B_i (i=1, 2)$, the following inequality connects the oscillations of f_1, f_2 , and $f_1 \cdot f_2$ over S :*

$$\text{Osc}(f_1 \cdot f_2) \leq B_2 \text{Osc } f_1 + B_1 \text{Osc } f_2.$$

rapport à chacune de deux variables, Fundamenta Mathematicae, vol. 14 (1929), pp. 237-241. On the other hand, when only functions upper [lower] semi-continuous in each separate variable are admitted, the relations are like (4); see example (C) of CA.

^{*} For H this fact was observed by Hardy, *On double Fourier series, and especially those which represent the double zeta-function with real and incommensurable parameters*, Quarterly Journal of Mathematics, vol. 37 (1905), pp. 53-79.

[†] It is quite clear that if $f_1(x, y)$ and $f_2(x, y)$ are both in T , $f = f_1 + f_2$ will fail to be in T when and only when at least one of its total variation functions is non-measurable. By Theorem 17 of Montgomery, loc. cit., this cannot happen if only functions belonging to the Baire classification are admitted to consideration.

[‡] For H this fact was observed by Hardy, loc. cit.

Designating by a_i and b_i respectively the greatest lower and least upper bounds of $f_i (i=1, 2)$, one may easily construct a proof of the lemma by considering seriatim all possible cases for the relationship of the intervals (a_i, b_i) to the origin.

That the product of two functions in $V \cdot C$ may not even be in F is seen at once from the following example:

$$(6) \quad \left. \begin{aligned} f_1(x, y) &= \begin{cases} x \sin(1/x) & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases} \\ f_2(x, y) &= y \end{aligned} \right\} \text{ in the unit square } I.$$

The theorem fails for T because the product of two functions in T may have a non-measurable ϕ or ψ ; viz.,

$$(7) \quad \begin{aligned} f_1(x, y) &= \begin{cases} 1 & \text{for } x \text{ in } E, y = 0, \\ 1 & \text{for } x \text{ in } C(E), y = 1, \\ 0 & \text{elsewhere,} \end{cases} \\ f_2(x, y) &= \begin{cases} 1 & \text{for } y = 0, \\ 0 & \text{for } y > 0, \end{cases} \end{aligned}$$

in I , E being a non-measurable set in the interval $(0, 1)$ and $C(E)$ its complement. The theorem also fails for T , and likewise for \bar{T} , because of the well known theorem of Lebesgue: *if $g_1(x)$ is a summable function not essentially bounded, there always exists a summable function $g_2(x)$ such that $g_1 \cdot g_2$ is not summable over the interval considered.* We may consider the interval in question as $(0, 1)$ and set in I

$$f_i(x, y) = \begin{cases} g_i(x) & \text{for } y = 0 \\ 0 & \text{for } y > 0 \end{cases} \quad (i = 1, 2).$$

Remarks. That the relations $f_1 \in V \cdot C, f_2 \in H \cdot C$ do not imply $f_1 \cdot f_2 \in F$ is shown by (6). That $f_1 \in T, f_2 \in H$ do not imply $f_1 \cdot f_2 \in T$ is apparent from (7); nevertheless one may readily show by aid of Lemma 1 and the theorem of Montgomery referred to above that if both f_1 and f_2 are in T , are bounded, and belong to the Baire classification, $f_1 \cdot f_2$ must be in T ; similarly, if f_1 and f_2 are in \bar{T} and are bounded, $f_1 \cdot f_2$ is in \bar{T} . That f_1 may be in $A \cdot C[P \cdot C \text{ or } T \cdot C]$ and f_2 in $H \cdot C$ without $f_1 \cdot f_2$ being in $H[A \text{ or } P \text{ respectively}]$ is clear from the fact that $f(x, y) \equiv 1$ is in $H \cdot C$.

THEOREM 4. *Each of the classes H , A , and P is closed under division, the denominator being assumed bounded away from zero.* This is not true of V , F , or T .*

In the light of Theorem 3 it suffices, for the first statement, to consider the case of $1/f$ for f in the class in question and $|f| \geq m > 0$. The fact has been stated for H by Hardy, loc. cit.; a proof can be constructed by aid of a little double series technique. For A one may give a proof precisely like that of the corresponding theorem for a function $g(x)$ of bounded variation.

Proof for P . Let \mathfrak{B} be a bound for the P_H -sum for f , and for each n let α_n be the number of cells, in the net of n^2 cells, in which f changes sign; then

$$(8) \quad \mathfrak{B} \geq \sum_{i=1}^{n^2} \omega'_i(f)/n \geq 2m\alpha_n/n.$$

Let us set

$$\sum_{i=1}^n \omega'_i(1/f) = \Sigma' + \Sigma'',$$

Σ' representing the sum over the cells in which f changes sign and Σ'' the sum over the remaining cells. In each cell of the first set we have

$$(9) \quad \omega'_i(1/f) \leq 2/m;$$

denoting by M_ν and m_ν , respectively the least upper and greatest lower bound of $|f|$ in the ν th cell, we have for each cell of the second set

$$(10) \quad \omega'_i(1/f) = 1/m_\nu - 1/M_\nu \leq M_\nu - m_\nu/m^2 = \omega'_i(f)/m^2.$$

From (8), (9), and (10) follows the inequality

$$\sum_{i=1}^{n^2} \omega'_i(1/f)/n \leq 2\mathfrak{B}/m^2$$

for every n , and the proof is complete.

That f may be in V and $|f| \geq m > 0$ without $1/f$ even being in F is seen from the following example. Let I be divided into subrectangles by the lines $x = 1 - 1/n$ ($n = 2, 3, \dots$). Proceeding from left to right, in the first, third, \dots rectangles let $f = 1$ except along the (closed) top and right-hand side; on the entire top and right-hand side, except at their common point where $f = 3$, let $f = 2$. At points of the even-numbered (closed) rectangles not already considered define f as 2 except along the top, where $f = 3$. For $x = 1$ and all y let $f = 1$.

If f is in T and $|f| \geq m > 0$ one readily sees that $1/f$ can fail to be in T

* The necessity for imposing this restriction is clear, since H , A , and P contain no unbounded functions.

only if its ϕ or ψ is non-measurable. This situation occurs in the case of

$$f(x, y) = \begin{cases} 3/2 & \text{for } x \text{ in } E, y = 0 \\ 1/2 & \text{for } x \text{ in } C(E), y = 1 \\ 1 & \text{otherwise} \end{cases} \text{ in } I,$$

E being a non-measurable set.

Remarks. Since V , F , and T contain unbounded functions, the restriction that the denominator be bounded away from zero in connection with these classes is perhaps more than would normally be expected. That \bar{T} is not closed under division is apparent from the following example: $f_1(x, y) = 1/x^{1/2}$ and $f_2(x, y) = x^{1/2}$ for $y=0, x>0$; and both functions equal to 1 elsewhere in I . If consideration is restricted to bounded functions in \bar{T} , it is readily seen that this subclass of \bar{T} is closed under division, the denominator being assumed bounded away from zero (see Remarks following Theorem 3).

4. RELATIONSHIPS WITH MONOTONE FUNCTIONS; DECOMPOSITION

THEOREM 5. *A necessary and sufficient condition that* $f(x, y)$ be in class V is that it be expressible as the difference between two functions, $f_1(x, y)$ and $f_2(x, y)$, satisfying the inequalities*

$$\Delta_{11}f_i(x, y) \geq 0 \quad (i = 1, 2).$$

The necessity has essentially been shown by Hobson†; the sufficiency is quite clear from Theorem 2.

THEOREM 6 (Hardy‡). *A necessary and sufficient condition that $f(x, y)$ be in class H is that it be expressible as the difference between two bounded functions, $f_1(x, y)$ and $f_2(x, y)$, satisfying the inequalities§*

$$\Delta_{10}f_i(x, y) \geq 0, \Delta_{01}f_i(x, y) \geq 0, \Delta_{11}f_i(x, y) \geq 0 \quad (i = 1, 2).$$

THEOREM 7 (Arzelà||). *A necessary and sufficient condition that $f(x, y)$ be in class A is that it be expressible as the difference between two bounded functions, $f_1(x, y)$ and $f_2(x, y)$, satisfying the inequalities*

* In order that the V -definition may always have meaning it is to be understood here that f , although perhaps unbounded, is everywhere finite. The functions f_1, f_2 are of like character.

† Hobson, loc. cit., p. 345.

‡ Hardy, loc. cit.

§ Functions satisfying these inequalities have been called "monotonely monotone" by W. H. and G. C. Young, *On the discontinuities of monotone functions of several variables*, Proceedings of the London Mathematical Society, (2), vol. 22 (1923), pp. 124-142. They belong to the class of "quasi-monotone" functions as defined by Hobson, loc. cit., p. 347.

|| Arzelà, *Sulle funzioni di due variabili a variazione limitata*, Bologna Rendiconto, (2), vol. 9 (1904-05), pp. 100-107.

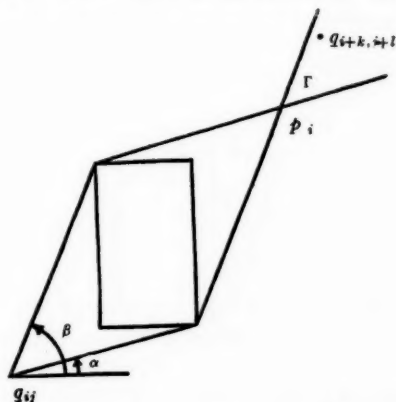
$$\Delta_{10}f_i(x, y) \geq 0, \Delta_{01}f_i(x, y) \geq 0 \quad (i = 1, 2).$$

Remarks. Although every bounded function monotone in the sense of Hobson is in class $A \cdot T$, not all such are in H . Every function quasi-monotone in the sense of Hobson is in class V and if bounded is also in H .

THEOREM 8. Every bounded *function non-decreasing in each of two directions is in class P .

Let α and β respectively ($\alpha < \beta$) be the angles made with the positive x -axis by the given directions in which f is non-decreasing. For $\alpha = 0, \beta = \pi/2$ a proof has been given by Hahn†, in establishing the relation $P \geq A$. We now prove the theorem for $0 \leq \alpha < \beta \leq \pi/2$; it will be clear that the method is applicable in all cases.

Using the P_H -form of the definition, let a net of n^2 cells be placed upon R , and let the columns of cells be numbered from left to right and the rows from bottom to top. Indices i, j may then be employed to designate the cell in the i th row and j th column. With this cell (for each pair of values i, j) we associate two points p_{ij} and q_{ij} defined as follows (see accompanying figure):



$p_{ij}[q_{ij}]$ is the point from which the cell is seen under the angle $\beta - \alpha$, the sides of the angle having the directions $\pi + \alpha, \pi + \beta$ [α, β]. Let $q_{i+k, j+l}$ be the point (or a point) of the set q_{ij} lying in the closed sector marked Γ in the figure and at a minimum distance from p_{ij} . The integers k, l are now fixed and are clearly independent of n .

* If the directions of assumed monotonicity are axial (i.e., the function is monotone in the sense of Hobson), finiteness of the function everywhere implies boundedness; otherwise this may not be so.

† Hahn, *Theorie der Reellen Funktionen*, Berlin, 1921, p. 546.

It suffices to consider $n \geq \max(10k, 10l)$. For such a value of n the sum of the oscillations of f in the first and last k rows and the first and last l columns of cells is $\leq 2B \cdot 2kn + 2B \cdot 2ln = O(n)$, B being a bound for $|f|$ in R . For each remaining cell the associated points p_{ij}, q_{ij} lie within R . These remaining cells constitute a block of $(n-2k)(n-2l)$ cells, and it will simplify matters a little to regard the row indices of these cells as running from 1 to $n-2k$ and the column indices from 1 to $n-2l$. From the above choice of k and l we clearly have

$$f(p_{ij}) \leq f(q_{i+k, j+l}) \quad (i = 1, 2, \dots, n-3k; j = 1, 2, \dots, n-3l).$$

Hence, for this remaining block of cells, the sum of the oscillations of f is

$$\begin{aligned} &\leq \sum_{i,j=1}^{n-2k, n-2l} [f(p_{ij}) - f(q_{ij})] \\ &\leq \left[\sum_{i=1}^{n-2k} \sum_{j=n-3l+1}^{n-2l} + \sum_{i=n-3k+1}^{n-2k} \sum_{j=1}^{n-3l} \right] f(p_{ij}) \\ &\quad - \left[\sum_{i=1}^k \sum_{j=1}^{n-2l} + \sum_{i=k+1}^{n-2k} \sum_{j=1}^l \right] f(q_{ij}) \\ &\leq [(n-2k)l + k(n-3l) + k(n-2l) + (n-3k)l]B = O(n). \end{aligned}$$

This completes the proof for the case considered.

It may be noted that a function non-decreasing in two directions must be non-decreasing in any third direction lying in the angle ($< \pi$) formed by the first two. Therefore, in constructing a proof for other cases, one may always reduce a case in which $\beta - \alpha$ is $> \pi/2$ to a case in which $\beta - \alpha$ is $< \pi/2$, of which that considered above is typical.

Remarks. It would be of considerable interest to determine whether a function in class P can always be decomposed into the difference between two functions each of which is bounded and monotone in two directions. If this were true it would follow at once* that every function in P has a total differential almost everywhere, settling a question left open in §6.

Lebesgue has defined a function f to be monotone if it satisfies the following condition: p being any point of the region considered and \mathcal{C} any closed curve in this region containing p in its interior, we have g.l.b. of f on $\mathcal{C} \leq f(p) \leq$ l.u.b. of f on \mathcal{C} . It is easily seen by examples that not all functions satisfying

* See Haslam-Jones, *Derivate planes and tangent planes of a measurable function*, Quarterly Journal of Mathematics, Oxford Series, vol. 3 (1932), pp. 120-132; or Saks, *Théorie de l'Intégrale*, Warsaw, 1933, p. 238.

this condition belong to any one of the classes V, H, A, P, F, T , and \bar{T} .

We conclude this Section with two quite obvious theorems concerning decomposition of a different sort, the first expressing a fact which has already been frequently observed.

THEOREM 9. *A necessary and sufficient condition that $f(x, y)$ be in class V is that $f(x, y) \equiv \bar{f}(x, y) + g(x) + h(y)$ where $\bar{f}(x, y)$ is in H .*

DEFINITION. *The subclass of F of which each function has $\phi(\bar{x})$ and $\psi(\bar{y})$ finite somewhere (and therefore finite everywhere†) will be designated by F^* .*

It should be observed that $F^* = F \cdot T$, the relationship of which to other classes was considered in CA.

THEOREM 10. *A necessary and sufficient condition that $f(x, y)$ be in class F is that $f(x, y) \equiv \bar{f}(x, y) + g(x) + h(y)$ where $\bar{f}(x, y)$ is in F^* .*

5. ADJUNCTION OR SUBDIVISION OF RECTANGLES

We state without proof two theorems.

THEOREM 11. *If a function is in any one of the several classes for each of two rectangles R_1 and R_2 whose sum is a rectangle R , it is in the same class for R .*

THEOREM 12. *If a function is in any one‡ of the classes $V, H, A, P, F, T \cdot B$, or \bar{T} for a rectangle R , it is in the same class for any subrectangle R_1 . This is not true of T .*

6. CONTINUITY, DIFFERENTIABILITY, MEASURABILITY, AND INTEGRABILITY OF FUNCTIONS BELONGING TO THE SEVERAL CLASSES

THEOREM 13. *If $f(x, y)$ is in class V and $f(x, \bar{y}) [f(\bar{x}, y)]$ for some $\bar{y}[\bar{x}]$ has only a denumerable number of discontinuities in $x[y]$, the discontinuities in $x[y]$ of $f(x, y)$ are located on a denumerable number of parallels to the y -axis [x -axis].*

Let E be the set of points at which f has a discontinuity in x and assume the existence of a non-denumerable set S of vertical lines each containing at least one point of E . Clearly only a denumerable subset of S can be made up wholly of points of E . Let the remaining lines of S constitute the subset S_1 ; then each line of S_1 contains at least one point of E and at least one point not

† See CA, Theorem 3.

‡ For H this fact was observed by W. H. Young, *On multiple Fourier series*, Proceedings of the London Mathematical Society, (2), vol. 11 (1912), pp. 133-184, especially p. 143. The failure of T to enjoy the property in question is illustrated by $f_1(x, y)$ in (7).

in E , and S_1 is non-denumerable. On each line of S_1 choose a point of E ; at this point f has a positive saltus in x . This non-denumerable set of saltuses contains a subset whose elements are the terms of a divergent series. A net can therefore be placed upon R to yield an arbitrarily large V -sum; from this contradiction flows the theorem.

THEOREM 14. *If $f(x, y)$ is in class V , the discontinuities in (x, y) which are not discontinuities in x or in y are denumerable.*

Let the oscillation at any such discontinuity (x_1, y_1) be α ; then it is clear that in every neighborhood of this point there exists a second point (x_2, y_2) such that $\Delta_{11}f$ for the cell $(x_1, y_1; x_2, y_2)$ is $> \alpha/4$. The assumption that the set of such discontinuities is non-denumerable then leads to a contradiction just as in the case of Theorem 13.

COROLLARY. *If $f(x, y)$ is in class H , the discontinuities of $f(x, y)$ are located on a denumerable number of parallels to the axes.†*

THEOREM 15. *Class V (and therefore F) contains bounded‡ functions which are everywhere discontinuous both in x and in y ; it also contains bounded non-measurable functions. Class $V \cdot C$ (and therefore $F \cdot C$) contains functions of which neither first partial derivative exists (finite or infinite) anywhere.§*

Examples. The function $f(x, y) \equiv g(x) + h(y)$, where both g and h are bounded and everywhere discontinuous, has the first property specified; if g is bounded and linearly non-measurable and h is identically zero, f has the second property; if g and h are continuous but have a derivative (finite or infinite) nowhere, f has the third property.

Of course it follows that V contains functions for which the double Lebesgue integral over R fails to exist, and that $V \cdot C$ contains functions which are nowhere totally differentiable. Nevertheless, that every function in V for which $f(x, c)$ and $f(a, y)$ have (finite) approximate derivatives almost everywhere possesses an approximate total differential almost everywhere is a consequence of Theorems 9 and 16, in conjunction with a theorem of Stepanoff.||

† This corollary is also a consequence of Theorems 2 and 6 and results obtained by W. H. and G. C. Young, loc. cit.

‡ Unbounded functions having the same property are included also.

§ It would be of considerable interest to determine whether the same is true of F^* (see §4), which from one point of view is the essential part of F and which bears to F a relationship similar to that of H to V , or whether functions in F^* possess properties of continuity, etc., more like those possessed by functions in H .

|| Stepanoff, *Sur les conditions de l'existence de la différentielle totale*, Recueil de la Société Mathématique de Moscou, vol. 32 (1925), pp. 511–526; or see Saks, loc. cit., p. 228. According to Stepanoff's theorem a necessary and sufficient condition that $f(\in M)$ have an approximate total differential almost everywhere in R is that f have (finite) approximate first partial derivatives almost everywhere in R .

THEOREM 16 (Burkill and Haslam-Jones*). *A function $f(x, y)$ in class A is totally differentiable almost everywhere.*

THEOREM 17. *A function $f(x, y)$ in class P is continuous in (x, y) almost everywhere.*

Assume the set E of points at which f has a saltus $\geq \epsilon > 0$ to have exterior measure $k > 0$. Let the area of R be denoted by S and let $[kn^2/S]$ stand for the largest integer not exceeding kn^2/S . For a net of n^2 cells under the P_H -form of the definition, we see that at least $[kn^2/S]$ cells of the net must contain points of E ; hence we have

$$\sum_{i=1}^n \omega'_i / n \geq [kn^2/S] \epsilon / n,$$

which is unbounded unless k is zero. Therefore, if f is in P , k must vanish for every $\epsilon > 0$, and by a classical argument it follows that the discontinuities of f are a set of plane measure zero.

Of course it may be inferred that the double Riemann integral over R of a function in P always exists[†]; another consequence is the relation $\bar{T} \cdot M > P$.

THEOREM 18. *If $f(x, y)$ is in class $\bar{T} \cdot M$, $\partial f / \partial x [\partial f / \partial y]$ exists (finite) almost everywhere.[‡]*

Since f is in M , the set E at which $\partial f / \partial x$ fails to exist (finite) is measurable. § Since f is in \bar{T} , E is intersected by almost every line $y = y_1$ in a set of linear measure zero. Hence, by Fubini's theorem, E is of plane measure zero.

COROLLARY 1. *A function $f(x, y)$ in class $\bar{T} \cdot M$ has an approximate total differential almost everywhere. ||*

This follows from the theorem of Stepanoff cited above.

COROLLARY 2. *If $f(x, y)$ is in class $\bar{T} \cdot M$, each first partial derivative is L -integrable[§] over R .*

It is worthy of notice that the hypothesis $f \in M$ cannot be dispensed with in Theorem 18 and its corollaries. This may easily be shown by example as fol-

* Burkill and Haslam-Jones, *Notes on the differentiability of functions of two variables*, Journal of the London Mathematical Society, vol. 7 (1932), pp. 297-305; see also Haslam-Jones, loc. cit.

† See Hobson, loc. cit., p. 477.

‡ Theorem 18 and Corollary 2 are extensions of results obtained by Morrey (loc. cit., Theorem 1, §1) on the assumption $f \in T \cdot C$. After Theorem 18 is established, his proof suffices for Corollary 2.

§ See Burkill and Haslam-Jones, loc. cit., Lemma 2.

|| Corollary 1 constitutes an extension of a similar result obtained at the expense of considerable trouble by Burkill and Haslam-Jones, loc. cit.: they assumed f to be in $T \cdot M$ and to satisfy a further measurability condition; i.e., the condition which in §7 we shall show is satisfied by all functions in H .

lows. The existence of a bounded set which is not plane measurable and of which at most two points lie on any straight line has been proved by Sierpiński*; let E be such a set entirely contained in the rectangle $(a, c; b, d)$, where $0 < a < b < 1$, $0 < c < d < 1$. Then choose any four numbers a_1, b_1, c_1, d_1 to satisfy the inequalities $b < a_1 < b_1 < 1$, $d < c_1 < d_1 < 1$ and form the set E_1 by adding to E the following points: for each $x_1 [y_1]$ in the interval $(0, 1)$, if the line $x = x_1 [y = y_1]$ contains only one point of E , add the point $(x_1, c_1) [(a_1, y_1)]$; if this line contains no point of E , add both the points $(x_1, c_1) [(a_1, y_1)]$ and $(x_1, d_1) [(b_1, y_1)]$. The characteristic function of E_1 is in T (as well as \bar{T}) but not in M , and it fails to have any of the properties specified in Theorem 18 and its corollaries.

Remarks. Example (D) of CA shows that $T \cdot M$ contains bounded functions (satisfying in addition the measurability condition considered in §7) which are everywhere discontinuous in (x, y) and hence nowhere totally differentiable. It has been proved by Saks† that there exist functions nowhere totally differentiable which are not only in $T \cdot C$ but satisfy considerably more stringent conditions. For $f \in H$, W. H. Young‡ has shown that the two cross partial derivatives of second order also exist almost everywhere.

7. A PROPERTY OF CLASS H

Let us set $V_x(x_0, y_0)$ = the total variation of $f(x, y_0)$ in x for $a \leq x \leq x_0$, $V_y(x_0, y_0)$ = the total variation of $f(x_0, y)$ in y for $c \leq y \leq y_0$; then we may formulate the

DEFINITION. A function $f(x, y)$ will be said to have the property M_v when and only when $V_x(x, y)$ and $V_y(x, y)$ are both measurable functions of (x, y) in R .

THEOREM 19. A function $f(x, y)$ in class H has the property M_v .

We give a proof for $V_y(x, y)$. Let us assume that this function is non-measurable, and in particular that α is a number such that the set $E[V_y(x, y) > \alpha]$ is non-measurable. Clearly E consists of the points on a set of inverted ordinates Ω standing on (or hanging from) the top of the rectangle R ; an ordinate may consist of a single point, or, if it contains more than one point, it may or may not have a lowest point (i.e., be closed).

* Sierpiński, *Sur un problème concernant les ensembles mesurables superficiellement*, *Fundamenta Mathematicae*, vol. 1 (1920), pp. 112–115.

† Saks, *On the surfaces without tangent planes*, *Annals of Mathematics*, (2), vol. 34 (1933), pp. 114–124.

‡ W. H. Young, *Sur la dérivation des fonctions à variation bornée*, *Comptes Rendus (Paris)*, vol. 164 (1917), pp. 622–625.

By Theorem 13 the discontinuities in y of $f(x, y)$ lie on a denumerable number of lines $y = \bar{y}$; let E_1 designate this set of values \bar{y} . The feet of the ordinates Ω form a measurable set E_2 , since E_2 is identical with the set of points x for which the measurable* function $\phi(x)$ is $> \alpha$. Let the lengths of the ordinates Ω define a function $g(x)$ over E_2 . Since $g(x)$ is $\leq d - c$, E can fail to be measurable only if the L -integral of $g(x)$ over E_2 fails to exist†, and this can occur only if $g(x)$ is a non-measurable function. Let $\beta (\geq 0)$ be a number for which $E_3 [g(x) > \beta]$ is non-measurable. There is no restriction in assuming‡, as we now do, that $d - \beta$ does not belong to E_1 . All ordinates Ω of length β will then be open.

Let E_4 be the projection of E_3 on the line $y = d - \beta$, and let $C(E_4)$ represent its complement with respect to the interval $a \leq x \leq b$ on this line. At each point of E_4 which is a limit point of $C(E_4)$, $V_v(x, d - \beta)$ is manifestly discontinuous in x ; these points constitute a set E_5 . Since $C(E_4)$ is non-measurable, E_5 must be likewise. Hence $m_e E_5$, and therefore the exterior measure of the set of points at which $V_v(x, d - \beta)$ has a discontinuity in x , is positive. On the other hand, by Theorem 12 above and Theorem 1 of CA, the discontinuities of $V_v(x, d - \beta)$ are denumerable. From this contradiction we infer the theorem.

Remarks. Example (C) of CA is a function in A which does not have the property M_v ; hence Theorem 19 fails for A , P , and T . Examples of a function (either measurable or non-measurable) in T but without the property M_v , may readily be constructed. We think it probable that Theorem 19 fails for V and F , but an example to show this does not immediately suggest itself. It should be observed that M_v is not an additive property, as is illustrated by the example following Theorem 2. Nevertheless, f being in V , V_v for f is identical with V_v for \bar{f} , where $\bar{f}(x, y) = f(x, y) - f(x, \bar{y})$ and \bar{y} is any fixed value in the interval (c, d) ; and $\tilde{f}(x, y) = \bar{f}(x, y) + f(\bar{x}, y)$, \bar{x} being any fixed value in (a, b) , can have no discontinuity in y where $\bar{f}(x, y)$ ($\subset H$) and $f(\bar{x}, y)$ are both continuous in y . Therefore the above proof of Theorem 19 can be used to establish the following assertion: if $f(x, y)$ is in V and there exists an $\bar{x}[\bar{y}]$ in (a, b) $[(c, d)]$ for which $f(\bar{x}, y)$ $[f(x, \bar{y})]$ is continuous almost everywhere, $V_v(x, y)$ $[V_v(x, y)]$ is measurable.

* See CA, Theorem 1.

† See Carathéodory, *Vorlesungen über Reelle Funktionen*, Berlin, 1918, p. 419; and Schlesinger and Plessner, loc. cit., p. 78.

‡ See Saks, *Théorie de l'Intégrale*, loc. cit., p. 37, where it is shown that measurability of the set $E[g(x) > \alpha]$ for every rational α is sufficient to insure measurability of $g(x)$. It is clear that the proof remains valid if we assume $E[g(x) > \alpha]$ measurable for any set of values α which is everywhere dense.

8. THE EFFECT OF LIPSCHITZ CONDITIONS

It is clear that the satisfaction of a Lipschitz condition,

$$|f(x + \Delta x, y + \Delta y) - f(x, y)| \leq k(\Delta x^2 + \Delta y^2)^{1/2} \quad (k = \text{constant}),$$

is sufficient to place f in class $A \cdot C$ (and therefore $P \cdot C$ and $T \cdot C$). At the same time it is insufficient to put f in H , V , or F , as the following example shows. Divide the unit rectangle I into columns by the lines $1 - 1/2^n$ ($n = 1, 2, \dots$); proceeding from left to right, divide the n th column into 2^n squares. On each square define f by the height of a regular pyramid with that square as base and with altitude equal to a side of the square, and let $f(1, y) = 0$.

Fréchet* has observed that the satisfaction of a Lipschitz condition in terms of area,

$$|\Delta_{11}f(x, y)| \leq k |\Delta x \cdot \Delta y| \quad (k = \text{constant}),$$

is sufficient to insure that f be in V (and therefore F); that it does not suffice to put f in any of the other classes may readily be seen by examples.

9. DEPENDENCE UPON AXES

It is quite clear that a function in class V , H , A , or F may fail to remain in that class when the x , y axes are rotated through a suitably chosen angle. On the other hand, definition P may easily be proved independent of the axes, and $T \cdot C$ is manifestly independent of the axes because of its geometric significance.† The question for T (or \bar{T}) is not so easily answered, and we shall construct an example to show that T (and \bar{T}) is *not* independent of the axes.‡

Let $E_x[E_y]$ be the set of numbers in the interval $(0, 1)$ which have a triadic representation free from the digit 2[1], and define $f(x, y)$ as the characteristic function of the set E of points (x, y) for which x is in E_x and y is in E_y . Since E_x and E_y are Cantor sets of measure zero, we have $f \in T$. It will be shown that f does not remain in T when the axes are rotated through the angle $\pi/4$.

The equation of the perpendicular to $y = x$ at (x_0, x_0) is $x + y = 2x_0$, and it is apparent that for any x_0 in the interval $(0, \frac{1}{2})$ this line contains at least

* Fréchet, *Extension au cas des intégrales multiples d'une définition de l'intégrale due à Stieltjes*, *Nouvelles Annales de Mathématiques*, (4), vol. 10 (1910), pp. 241-256.

† That is, a necessary and sufficient condition that a continuous surface $z = f(x, y)$ have area in the Lebesgue sense is $f(x, y) \in T$.

‡ We are indebted to Dr. W. C. Randels for suggesting this example. It is probable that our purpose would also be served by the characteristic function of some of the sets constructed by Mazurkiewicz and Saks, *Sur les projections d'un ensemble fermé*, *Fundamenta Mathematicae*, vol. 8 (1926), pp. 109-113, but the example given here seems somewhat easier to discuss.

one point of E . For, $2x_0$ being given in triadic form, corresponding digits in x_1 and y_1 such that $x_1 + y_1 = 2x_0$ can be chosen as follows: in $x_1[y_1]$ put a 0 wherever a 0 or 2[1] occurs in $2x_0$, and put a 1[2] wherever a 1[2] occurs in $2x_0$. Such a choice of digits for x_1, y_1 may be said to be "according to rule." Let $\phi_1(x)$ be the ϕ -function for the new x -axis (i.e., the line $y=x$ in the original coordinate system).

Consider first any $2x_0$ of the form $.10 \dots$, the remaining digits being arbitrary. The points (x_0, x_0) corresponding to these numbers fill an interval I_1 on $y=x$ of length $1/(3^2 2^{1/2})$. To each such number we have

$$(11) \quad \begin{cases} x_1 = .10 \dots, \\ y_1 = .00 \dots, \end{cases} \quad \begin{cases} x_1 = .01 \dots, \\ y_1 = .02 \dots, \end{cases}$$

the remaining digits in all cases being chosen according to rule. We then have

$$\int_{I_1} \phi_1 > 2/(3^2 2^{1/2}).$$

Next consider $2x_0 = .ab10 \dots$, the subsequent digits being arbitrary and a, b anything except 1, 0. The points (x_0, x_0) corresponding to these numbers fill $3^2 - 1$ intervals each of length $1/(3^4 2^{1/2})$; this set of intervals we may call I_2 . To each number $2x_0$ of the present form we may choose the third and fourth digits as the first two were chosen in (11) and choose the rest according to rule. We obtain

$$\int_{I_2} \phi_1 > 2(3^2 - 1)/(3^4 2^{1/2}).$$

Continuing in this manner we find

$$\int \phi_1 > \frac{2}{2^{1/2}} \sum_{n=0}^{3^2-1} \frac{3^2 - n}{3^4} > 1/2^{1/2}.$$

Repeating this process using blocks of $2p$ digits $1010 \dots 10$, to each of which there correspond 2^p choices instead of the two in (11), we obtain

$$\int \phi_1 > \frac{2^p}{2^{1/2}} \sum_{n=0}^{3^{2p}-1} \frac{3^{2p} - n}{3^{4p}} > 2^{p-1}/2^{1/2}.$$

Since p is arbitrary, $\int \phi_1$ does not exist and our assertion is proved.

10. FACTORABLE FUNCTIONS BELONGING TO THE SEVERAL CLASSES

For our present purposes a function $f(x, y)$ will be called *factorable* if and only if we have in R

$$f(x, y) \equiv g(x)h(y),$$

with neither g nor h identically zero.* The verification of the following equations is then immediate:

$$\Delta_{11}f(x, y) = \Delta g(x) \Delta h(y),$$

and for each net

$$\begin{aligned} \max_{i,j} \sum \epsilon_i \bar{\epsilon}_j \Delta_{11}f(x_i, y_j) &= \max \left[\sum_i \epsilon_i \Delta g(x_i) \sum_j \bar{\epsilon}_j \Delta h(y_j) \right] \\ &= \sum_i |\Delta g(x_i)| \sum_j |\Delta h(y_j)| \\ &= \sum_{i,j} |\Delta_{11}f(x_i, y_j)|. \end{aligned}$$

Conclusions may be drawn as follows.

THEOREM 20. *A necessary and sufficient condition that a factorable function be in class H is that each factor be of bounded variation. A factorable function, with one factor of unbounded variation and the other a constant, is in V and F but not in A , P , T , or \bar{T} . A factorable function, with one factor of unbounded variation and the other not a constant, is not in V , F , or A ; it is not in P , T , or \bar{T} unless the latter factor vanishes almost everywhere, and even then it may not be.*

COROLLARY. *Class A contains no factorable functions save those in H ; F contains no factorable functions save those in V ; but each of the classes V , P , and T contains factorable functions which are not in H . A factorable function in T but not in H must vanish almost everywhere in R .*

11. THE "VARIATION" OF FUNCTIONS BELONGING TO THE SEVERAL CLASSES

It is our object here to direct attention to two things: (i) the fact that a function belongs to one of the several classes conveys, in most cases, comparatively little idea of the extent to which the function fluctuates in R ; and (ii) the difficulty of associating with a function belonging to any one of the several classes, by means of the definition of that class, a number which conveys any precise notion of the amount of fluctuation of the functional values.

Let us first consider the classes V , F , and T (or \bar{T}). It has already been remarked in CA that the V -sum and the maximum F -sum for a given net N are never decreased when new horizontal or vertical lines are added to form a net N' . Therefore it might be considered natural to define the total variation of a function in V or F as the least upper bound of the respective V - or F -sum. For a function in T the quantities

* In the contrary case f is obviously in H .

$$(12) \quad \frac{1}{b-a} \int_a^b \phi(x) dx, \quad \frac{1}{d-c} \int_c^d \psi(y) dy$$

are the average total variations respectively of $f(\bar{x}, y)$ in y and $f(x, \bar{y})$ in x . One might therefore consider it desirable to define the total variation of a function in T as the larger of the numbers (12), or perhaps some linear combination of them. Under such definitions each of the classes V , F , and T contains* functions *with an arbitrarily large saltus at every point* in R whose total variation is zero! The reader inclined to be critical of our point of view may aver that it should not matter much what values a function $f(x, y)$ has on a set of plane measure zero, and it is true that a function in T whose total variation is zero according to the definition suggested above is "almost a constant"; nevertheless we are inclined to insist that when the total variation of $g(x)$ is in question, it matters a great deal what values $g(x)$ assumes on a set of linear measure zero.

If when f is in H one were to define the total variation as the least upper bound of the V -sum, very little notion of the amount of fluctuation would be conveyed; for every function $f(x, y) \equiv g(x)$, where $g(x)$ is of bounded variation, would have total variation zero as a function of two variables, independently of the value of the total variation of $g(x)$ and although in general $f(x, y)$ is not even "almost a constant".

If a function is in A , it would be natural to define its total variation as the least upper bound of the A -sum. This procedure, however, has several disadvantages, including the fact that the total variation of a function in R would not in general be the sum of its total variations in the two rectangles into which R is divided by a vertical (or horizontal) line.

It is quite clear that except when $f(x, y)$ is a constant, the total variation of a function in P , defined as the least upper bound of the P -sum, would depend upon the value of the fixed upper bound for D , the side of the square cells employed. One would naturally turn, therefore, to the P_H -form of the definition. Since the P_H -sum may decrease as n increases, it might be preferable to define the total variation, not as the least upper bound of the P_H -sum, but as the $\lim_{n \rightarrow \infty}$ of this sum. Whichever choice were made, the definition would be open to the objection that the total variation of such a function as

$$f(x, y) = \begin{cases} 1 & \text{for } x = \bar{x} \\ 0 & \text{for } x \neq \bar{x} \end{cases} \text{ in } I$$

would be different for \bar{x} rational and for \bar{x} irrational. This objection can only

* This is clearly indicated by examples given above in §6 and example (D) of CA.

be met by insisting that the oscillations ω_i' in the n^2 cells be computed for cells so defined that no two have points in common; yet if this were done, the total variation in R would not in general be equal to the sum of the total variations in the two rectangles into which R is divided by a vertical (or horizontal) line.

For the reasons described above *it would seem desirable to regard the several definitions of bounded variation for functions of two variables purely as formal generalizations of analytic conditions in common use in the theory of functions of a single variable or as conditions which single out for consideration some class of functions having one or more properties like those of a function $g(x)$ of bounded variation,* and rather completely to disassociate the term "function of bounded variation" from any notion of the amount which the function $f(x, y)$ fluctuates in the rectangle R .*

* See certain remarks in CA, pp. 826-827.

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A NEW METHOD FOR WARING THEOREMS WITH POLYNOMIAL SUMMANDS*

BY
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1. Part I of this paper is self-contained and presupposes only the rudiments of elementary theory of numbers. The method employs a pair of polynomials $p(x)$ and $q(x)$ of degree n , each uniquely determined by the other, such that there exists an identity which expresses $Iq(s)$ as a sum of m values of $p(x^2)$, where I is an integer and s is a sum of four squares. Then a Waring theorem for $q(x)$ yields one for $p(x^2)$. For, if every (large) integer is a sum of v values of $q(x)$, then every (large) multiple of I is a sum of vm values of $p(x^2)$. From the last result we readily find how many values of $p(x^2)$ suffice for all integers.

Apart from the special case in which $q(x)$ is a power of x , there is no hint in the literature of this instantaneous deduction of a Waring theorem for an even polynomial of degree $2n$ from a known Waring theorem for a polynomial of degree n . On the contrary, Maillet resorted to an extensive proof for the case $n=2$.

We feel justified in perfecting the theory of sums of four values of a quadratic function $q(x)$ in view of the resulting theorems for certain polynomials of degrees 4, 8, etc.

Since we seek Waring theorems holding for all positive integers (or with all exceptions listed), we are not content with theorems holding for all sufficiently large integers and certainly not with the asymptotic results much in vogue.

PART I. WARING THEOREMS FOR POLYNOMIALS OF DEGREES 2 AND 4

2. Using the abbreviation $s = a^2 + b^2 + c^2 + d^2$, we have the identities

$$(1) \quad 6s \equiv \sum_{12} (a \pm b)^2, \quad 6s^2 \equiv \sum_{12} (a \pm b)^4,$$

in which the summands are the powers of

$$(2) \quad a \pm b, \quad a \pm c, \quad a \pm d, \quad b \pm c, \quad b \pm d, \quad c \pm d.$$

Write

$$(3) \quad f(x) = ux^4 + vx^2, \quad q(x) = ux^2 + vx.$$

* Presented to the Society, April 7, 1934; received by the editors June 29, 1934.

Hence we have the following identity* in a, b, c, d, u, v :

$$(4) \quad 6q(s) \equiv \sum_{12} f(a \pm b).$$

First, take $u=v=1/2$. Then $q(x)=x(x+1)/2$ is called a triangular number. It is known† that every positive integer n is a sum of three triangular numbers $q(s_i)$ with $s_i \geq 0$. But each such integer s_i is a sum of four squares. Hence (4) used three times shows that $6n$ is a sum of 36 values of $f(x)=(x^4+x^2)/2$ for integers x . Every positive integer p is of the form $6n+r$, $0 \leq r \leq 5$, while $r=rf(1)$, whence p is a sum of 41 values of $f(x)$.

We can reduce 41 to 38 as follows. The numbers $f(0)=0, f(1)=1, f(2)=10, f(3)=45$ are congruent modulo 6 to 0, 1, 4, 3. Also, $2f(1)=2, f(1)+f(2) \equiv 5 \pmod{6}$. Hence if M is any integer, we can find two integers a, b , each ≥ 0 , such that

$$f(a) + f(b) \equiv M \pmod{6}, \quad f(a) + f(b) \leq 45.$$

Thus if $M \geq 45$, $M - f(a) - f(b)$ is a multiple ≥ 0 of 6 and hence is a sum of 36 values of $f(x)$, whence M is a sum of 38 such values. But $10x+y=xf(2)+yf(1)$ is a sum of $x+y$ values of $f(x)$, whence every integer <100 is a sum of fewer than 20 values.

THEOREM 1. *Every positive integer is a sum of 38 values of $(x^4+x^2)/2$.*

3. Second, take $u=1/2, v=-1/2$. Then $q(x)=(x^2-x)/2$ becomes $(y^2+y)/2$ for $y=x-1$. We may discard the negative value -1 of y corresponding to $x=0$ since $q(x)=0$ also when $x=1$, which corresponds to $y=0$. The fact that every integer $N \geq 0$ is a sum of three triangular numbers therefore implies that N is a sum‡ of three values of $q(x)$ for integers $x \geq 0$. Thus (4) implies that every positive multiple of 6 is a sum of 36 values of $f(x)=(x^4-x^2)/2$ for integers x . Conversely, any sum of values of $f(x)$ is a multiple of 6. In fact, if x is any integer, x^4-x^2 is a multiple of 4 and of 3.

THEOREM 2. *Every positive integer is a sum of 36 (always positive integral) values of $(x^4-x^2)/12$.*

4. Maillet§ investigated positive integers A which are sums of four values of

* Also when we add c to f and $2c$ to g . The modifications of the later theory are evident.

† Since $8n+3$ is a sum of three squares, each necessarily an odd square $(2x+1)^2$, where $x \geq 0$. Hence $n = \Sigma q(x)$.

‡ This $q(x)$ and $\frac{1}{2}x(x+1)$ are the only functions $q(x)=ux^2+vx$ such that every positive integer is a sum of three values of $q(x)$ and such that $q(x)$ is an integer ≥ 0 for every integer $x \geq 0$.

§ Bulletin de la Société Mathématique de France, vol. 23 (1895), pp. 40-49. He did not find the actual limit (12) for A .

$$(5) \quad q(x) = (mx^2 + nx)/2, \quad m > 0.$$

If we can find positive odd integers k and t such that

$$(6) \quad (3k - 2)^{1/2} - 1 < t < (4k)^{1/2},$$

$$(7) \quad A = mk/2 + nt/2,$$

then by Cauchy's lemma there exist four integers ≥ 0 whose sum is t and the sum of whose squares is k , whence by (5) and (7), A is a sum of four values of $q(x)$ for integers $x \geq 0$. Write

$$(8) \quad r^2 = n^2 + 2Am, \quad w^2 = 24Am + 12mn - 8m^2 + 9n^2.$$

Let $A \geq m/2$, whence $w^2 \geq (2m + 3n)^2$. Elimination of k between (6) and (7) gives

$$(9) \quad L = \frac{-2m - 3n + w}{2m} < t < G = \frac{-2n + 2r}{m}, \quad L \geq 0.$$

Let m and n be relatively prime odd integers. If $G > L + 2m$, there are m consecutive positive odd integers t_1, \dots, t_m between L and G in (9). If $2A - nt_i \equiv 2A - nt_j \pmod{m}$, then t_i and t_j are congruent modulo m and hence modulo $2m$. Since their difference is numerically $< 2m$, they are equal. Hence there is a positive odd integer t satisfying both (9) and $2A - nt \equiv 0 \pmod{m}$. Then $k = (2A - nt)/m$ is an odd integer satisfying (7). This k is positive since $2A \geq nG (> nt)$, which follows by eliminating A by (8) and using $(r - n)^2 \geq 0$.

Conversely, when k and t are positive integers, (7) and (9) imply (6) if $mt + 2n \geq 0$ (trivial if $n \geq 0$) and hence if $mL + 2n \geq 0$, viz., $w \geq 2m - n$. The latter follows from its square and hence from

$$(9') \quad 6mA \geq 3m^2 - 4mn - 2n^2 \quad (\text{if } n < 0).$$

The condition $G - L > 2m$ is equivalent to

$$(10) \quad 4r > w + T, \quad T = n - 2m + 4m^2.$$

This follows from its square: $16r^2 - w^2 - T^2 > 2wT$. The latter follows from its square (11) if its left member is ≥ 0 , which is true when

$$(10') \quad 8Am \geq T^2 - 7n^2 + 12mn - 8m^2.$$

Write $w^2 = 24Am + H$. Then

$$(11) \quad (8Am + J)^2 = (16r^2 - w^2 - T^2)^2 > (2wT)^2 = 4T^2(24Am + H),$$

$$H = 12mn - 8m^2 + 9n^2, \quad J = 8m^2 - 12mn + 7n^2 - T^2.$$

In the inequality transpose the terms with A to the left and complete the square on A . Hence

$$(12) \quad 8Am > 6T^2 - J + 2TP, \quad P^2 = H - 3J + 9T^2,$$

$$(12') \quad P^2 = 16m^2(1 + 6n - 12m + 12m^2).$$

Conversely, if $P \geq 0$, $T \geq 0$, (12) implies (11).

THEOREM 3. *If m and n are relatively prime odd integers and $m > 0$, $P \geq 0$, $T \geq 0$, every integer $A \geq \frac{1}{2}m$, large enough to satisfy (12), (9') and (10'), is a sum of four values of (5) for integers $x \geq 0$.*

5. Hence (4) used four times shows that $6A$ is a sum of 48 values of $f(x) = (mx^4 + nx^2)/2$. Let D denote the g. c. d. of 6 and $f(1)$. Let g be any given multiple of D . Then $f(1)y + 6z = g$ is solvable in integers y, z , with $0 \leq y \leq 5$. Thus $6N + g = 6(N+z) + yf(1)$. Hence every sufficiently large multiple of D is a sum of $48 + y \leq 53$ values of $f(x)$. This is equivalent to the more complicated Theorem V of Maillet, who proved it by an extensive argument based on Cauchy's lemma.

It is a new result that we may replace 53 by 50. Write

$$(13) \quad m = 2M + 1, \quad n = 2N - 1,$$

where M and N are integers. Then

$$f(0) = 0, \quad f(1) = M + N, \quad f(2) = 4(M + N), \quad f(3) = 3(M + N) \pmod{6}.$$

These with $2f(1)$ and $f(1) + f(2)$ evidently form a complete set of residues modulo 6 if $M + N$ is prime to 6. Let G be the largest of the six numbers just used. Let I be any integer $\geq G$. Hence there exist integers a, b , each ≥ 0 , such that $f(a) + f(b)$ is $\leq G$ and is $\equiv I \pmod{6}$. Thus $I - f(a) - f(b)$ is a positive multiple $6A$ of 6. If A is large enough to satisfy (12), $6A$ is a sum of 48 values of $f(x)$.

Next, if $M + N$ is even and prime to 3, we see that $f(0), f(1)$ and $2f(1)$ are congruent modulo 6 to 0, 2, 4 in some order. Thus all large even integers are sums of 50 values of $f(x)$, which is always even.

Next, if $M + N$ is an odd multiple of 3, $f(1) \equiv 3 \pmod{6}$. If I is a multiple of 3, one of $I, I - f(1)$ is a multiple of 6.

THEOREM 4. *Let m and n be relatively prime odd integers, $m > 0$. Let D denote the g.c.d. of 6 and $(m+n)/2$. Then every sufficiently large multiple I of D is a sum of 50 values of $f(x) = (mx^4 + nx^2)/2$. We may replace 50 by 49 if $D = 3$, and by 48 if $D = 6$. The theorem holds if $I \geq G + 6A$, where A is large enough to satisfy (12), (9') and (10'), while G is the largest of $2f(1), f(1) + f(2)$ and $f(3)$ if $D = 1$; $G = 2f(1)$ if $D = 2$; $G = f(1)$ if $D = 3$; $G = 0$ if $D = 6$. An equivalent statement is that every integer $\geq (G + 6A)/D$ is a sum of t values of $(mx^4 + nx^2)/(2D)$, which is always integral, where $t = 50$ if $D = 1$ or 2, $t = 49$ if $D = 3$, $t = 48$ if $D = 6$.*

6. For the case of polygonal numbers of order $m+2$, we have $n=2-m$. Then $0 < P < 4 \cdot 3^{1/2} m(2m-1)$ if $m \geq 2$, and (12) is seen to hold if $A \geq 28m^3$ (first proved by Legendre). Also (9') holds if $A \geq 5m/6$, and (10') if $A \geq 2m^3$. But (12) holds for smaller values of A . For example, if $m=3$, (12) holds if $A \geq 478$, whereas $28m^3 = 756$. By the writer's* table of sums of four polygonal numbers, the case $m=3$ shows that every integer ≤ 480 , except the six in Theorem 5, is a sum of four pentagonal numbers.

THEOREM 5. *Every integer except 9, 21, 31, 43, 55, and 89, is a sum of four pentagonal numbers $(3x^2-x)/2$. Hence every integer is a sum of five, one of which is 0 or 1.*

COROLLARY 1. *Except when N is one of those six numbers, $24N+4$ is a sum of four squares of integers $6x-1$ with $x \geq 0$.*

COROLLARY 2. *If v is a fixed one of the numbers 2-4, 6-9, every positive integer is a sum of four integers each of which is v or pentagonal.*

For, each of the six exceptions in Theorem 5 is such a sum.

7. In this section we prove two lemmas.

LEMMA 1. *Let $f(z)$ be an integer ≥ 0 for every integer $z \geq 0$. Let $g(x)$ denote the greatest integer $\leq f(x+1)/f(x)$. Then every positive integer $I < f(x+1)$ exceeds a sum of at most $g(x) + g(x-1) + \dots + g(1)$ values of $f(z)$ by an integer which is ≥ 0 and $< f(1)$.*

For, $I = C(x)f(x) + r(x)$, where $C(x) \leq g(x)$, $0 \leq r(x) < f(x)$, C and r being integers. Thus every $r(x)$ is expressible as $C(x-1)f(x-1) + r(x-1)$. Repetitions show that

$$I = C(x)f(x) + C(x-1)f(x-1) + \dots + C(1)f(1) + u,$$

where $0 \leq u < f(1)$.

LEMMA 2. *Define $f(z)$ and $g = g(2)$ as in Lemma 1. Let $f(0) = 0$, $f(1) = 1$, and $f(z+1) > f(z)$. Then $I = gf(2) - 1$ is a sum of $g-2+f(2)$, but not fewer, values of $f(z)$ for integers $z \geq 0$.*

Since $gf(2) \leq f(3)$, $I < f(3)$ and the only decompositions of I are of the form $rf(2) + s$ with $r < g$. When $r = g-1$, then $s = f(2) - 1$ and we see that I is a sum of $g-1+f(2)-1 = v$ values of $f(z)$. When $r = g-k$, $k \geq 2$, then $s = kf(2) - 1$ and the decomposition involves $v + (k-1)[f(2)-1] > v$ values of $f(z)$.

* Bulletin of the American Mathematical Society, vol. 33 (1927), p. 718. The present result was also proved directly.

It is a reasonable conjecture that every positive integer is a sum of I values of $f(z)$.

8. We now prove three theorems.

THEOREM 6. *Every positive integer is a sum of 50 values of $f(x) = (3x^4 - x^2)/2$.*

Here $f(1) = 1$, $f(2) = 22$, $f(3) = 117$, $f(4) = 376$, $f(5) = 925$. Since $m = 3$, $n = -1$, (5) is pentagonal, whence $A = 89$ by Theorem 5. In Theorem 4, $G = f(3)$, whence every integer $\geq G + 6A = 651$ is a sum of 50 values of $f(x)$. By Lemma 1 every integer $< f(4)$ is a sum of $3 + 5 + 22$ values of $f(z)$. Hence everyone $\leq 2f(4) = 752$ is a sum of 31 values.

THEOREM 7. *Every positive integer is a sum of 50 values of $f(x) = (5x^4 - 3x^2)/2$.*

Here $m = 5$, $n = -3$ and (12) holds if $A \geq 2613$ (whereas Legendre's limit is 3500), and then (9') and (10') are satisfied. The successive values of $f(x)$ are 1, 34, 189, 616, 1525, 3186, 5929, 10144, 16281; those of $g(x)$ are 34, 5, 3, 2, 2, 1, 1, 1. By Lemma 1, every integer $\leq f(9)$ is a sum of 49 values of $f(x)$. In Theorem 4, $G = f(3)$, whence every integer $\geq 189 + 6(2613) = 15867$ is a sum of 50 values.

THEOREM 8. *If every positive integer is a sum of not more than 50 values of $f(x) = (mx^4 + nx^2)/2$, then $f(x)$ is the quartic in Theorems 1, 6, or 7, or else is*

$$(14) \quad f(x) = (7x^4 - 5x^2)/2.$$

Since $f(y)$ shall represent 1 for an integer $y \geq 0$, y^2 must divide 2. Hence $y = 1$ and $(m+n)/2 = 1$. Employ (13). Then $N = 1 - M$,

$$f(x) = (\tfrac{1}{2} + M)x^4 + (\tfrac{1}{2} - M)x^2, \quad M \geq 0.$$

If $M = 0, 1$, or 2 , we have the functions in Theorems 1, 6, or 7. Hence let $M \geq 3$. Since $f(1) = 1$, $f(2) = 10 + 12M$, $f(3) = 45 + 72M$, the number $9 + 12M$ is not a sum of fewer than $9 + 12M$ values of $f(x)$. But $9 + 12M \leq 50$ only if $M \leq 3$. Thus $M = 3$ and we get (14). Then $f(1) = 1$, $f(2) = 46$, $f(3) = 267$. By Lemma 2, $229 = 4 \cdot 46 + 45$ is a sum of 49, but not fewer, values of $f(x)$. It was readily verified that 49 values suffice to $f(4) = 856$, but no further examination has been made.

One of my students is treating the many universal theorems obtained when $D = 2, 3$ or 6 .

9. Finally, we take $m = 2u$, $n = 2v$, u and v relatively prime. We shall choose A so large that there are $m/2$ consecutive positive odd integers t_k between L and G in (9). If $2A - nt_i \equiv 2A - nt_j \pmod{m}$, then $t_i \equiv t_j \pmod{m/2 = u}$.

First, let u be odd. Then $t_i \equiv t_j \pmod{m = 2u}$. But the difference between

t_i and t_j is numerically $< m$. Hence they are equal. Since the $m/2$ even integers $2A - nt_i$ are incongruent modulo m , one of them is congruent to zero. Thus

$$k = \frac{2A - nt}{m} = \frac{A - vt}{u}$$

is an integer. It is odd* if A is odd and v is even.

Second, let u be even, v and A both odd. Use only the first $m/4$ of our integers t_k . The difference between any two of them is numerically $\leq 2(m/4 - 1) < m/2$. Hence we have $m/4$ multiples $2A - nt_i$ of 4 which are incongruent modulo m . Thus one of them is $\equiv 0$. We employ the resulting k (corresponding to t) if k is odd. But if k is even, k' (corresponding to $t' = t + m/2$) is $k - n/2$, which is odd, and now we have used the last $m/4$ of our t_k .

This amplification of Maillet's argument shows that, if u and v are relatively prime, and if $u+v$ is odd, every sufficiently large odd integer A is a sum of four values of $q(x)$ in (3).

There will be $m/2$ odd integers between L and G if $G - L > m$, and hence by (9) if (10) holds with T replaced by

$$(15) \quad T_1 = n - 2m + 2m^2.$$

Then (11) and (12) hold with T and P replaced by T_1 and P_1 , where

$$(16) \quad P_1^2 = 16m^2(1 + 3n - 6m + 3m^2).$$

THEOREM 9. *Let u and v be relatively prime, and $u+v$ be odd. Let $P_1 \geq 0$, $T_1 \geq 0$. Then A is a sum of four values of $q(x)$ in (3) for integers $x \geq 0$ if $A \geq \frac{1}{2}m$, A is odd and large enough to satisfy (9'), (10') and (12) with subscripts 1 on T and P , and with the abbreviations (11), (15), (16).*

10. Then (4) used four times shows that $6A = 6(2N+1)$ is a sum of 48 values of $f(x)$ in (3). The g.c.d. of $f(1) = u+v$ and 12 is $\delta = 1$ or 3. If g is any positive multiple of δ , $f(1)y + 12z = g$ is solvable in integers y, z with $0 \leq y \leq 11$. As shown by elimination of g , $6A + g$ is a sum of $48 + y \leq 59$ values of $f(x)$. Hence every large multiple of δ is a sum of 59 values of $f(x)$.

But we may reduce Maillet's 59 to 51. First, let $u+v$ be prime to 3 (as well as to 2). We employ

$$f(0) = 0, f(1) = u + v, f(2) \equiv 4(u + v), f(3) \equiv 9(u + v) \pmod{12}.$$

Their sums by two are congruent to the products of $u+v$ by 0-2, 4-6, 8-10,

* Also if A is even and v odd. But the resulting theorem is a mere corollary to Theorem 3, with m, n replaced by u, v , as seen by doubling the numbers and function.

whence their sums by three give a complete set of residues modulo 12. Next, if $u+v$ is divisible by 3, we see that $f(0), f(1), 2f(1), 3f(1)$ are congruent modulo 12 to 0, 3, 6, 9 in some order.

THEOREM 10. *Let u, v be relatively prime, $u > 0$, and $u+v$ be odd. The g.c.d. of 12 and $u+v$ is $\delta=1$ or 3. If $\delta=1$, let G be the greatest of $3f(1), f(1)+f(2), 2f(2), f(3)+2f(1)$ and $2f(3)+f(1)$. But if $\delta=3$, let $G=3f(1)$. Let A be odd and large enough to satisfy (9'), (10') and (12) in the sense of Theorem 9. Then every integer $\geq (6A+G)/\delta$ is a sum of 51 values of $Q(x) = (ux^4+vx^2)/\delta$.*

11. We now prove

THEOREM 11. *Employ the assumptions and notations of Theorem 10. If every positive integer is a sum of 51 values of $Q(x)$, then $u+v=\delta$, $u \leq 4\delta$. Either $\delta=1$ and $u \leq 3$, or $\delta=3$ and u is one of the integers* 1, 2, 4, 5, 7, 8, 10, 11 prime to 3. Conversely, every integer is a sum of 51 values at least when $Q(x) = 2x^4 - x^2$ or $(x^4+2x^2)/3$.*

Since $1=Q(y)$ for an integer $y>0$, y^2 divides $\delta=1$ or 3. Hence $y=1$, $u+v=\delta$. Also $Q(1)=1$, $Q(2)=(12u+4\delta)/\delta$, $Q(3)>Q(2)$. Hence the only decomposition of $Q(2)-1$ into a sum of values of $Q(x)$ is that in which each $x=1$. Thus $Q(2)-1 \leq 51$, whence $u \leq 4\delta$. But if $\delta=1$, $u=4$, then $Q(2)=52$, $Q(3)=5 \times 52 + 37$, and Lemma 2 with $g(2)=5$ shows that 259 is a sum of 55, but not fewer, values.

If $u=2$, $v=-1$, we find by Theorem 9 that if A is odd and ≥ 195 , A is a sum of four values of $q(x)=2x^2-x$. The successive values of $Q(x)=2x^4-x^2$ are 1, 28, 153, 496, 1225, 2556, whence those of $g(x)$ in Lemma 1 are 28, 5, 3, 2, 2, whence 40 values suffice to $F(6)=2556$, which exceeds $6A+G=6 \times 195+307=1477$.

Let $u=1$, $v=2$. By Theorem 9, we find that every odd integer $A \geq 55$ is a sum of four values of $q(x)=x^2+2x$. The successive values of $Q(x)=(x^4+2x^2)/3$ are 1, 8, 33, 96, 225. Those of $g(x)$ in Lemma 1 are 8, 4, 2, 2, whence every positive integer $< Q(5)=225$ is a sum of 16 values of $Q(x)$. But in Theorem 10, $G=67$, whence all integers $\geq 133 \geq (67+330)/3$ are sums of 51 values.

It is readily verified that 1, 2, 4, 5, 7, 10, 13, 20, 25, 28, 37, 52 are the only integers ≤ 56 which are not sums of four values of x^2+2x . Hence every odd integer J except 1, 5, 7, 13, 25, 37 is a sum of four values with $x \geq 0$. Write $y=x+1$. Then $J+4$ is a sum of four values of y^2 , $y \geq 1$.

COROLLARY. *Every positive odd integer except 5, 9, 11, 17, 29, 41 is a sum of four squares each $\neq 0$.*

There is no such result in the literature.

* If $u=11$, Lemma 2 shows that 239 is a sum of 51, but not fewer, values.

PART II. WARING THEOREMS FOR CUBIC FUNCTIONS

12. In these Transactions, 1934, pp. 1-12, I discussed

$$(17) \quad f(x) = x + \frac{1}{6}\epsilon(x^3 - x) \quad (\epsilon \text{ an integer} > 0)$$

which is an integer ≥ 0 for every integer $x \geq 0$, while $f(x) = 1$. For ϵ prime to 3, I found positive integers C and ν such that every integer $\geq C \cdot 3^\nu$ is a sum of nine values of $f(x)$ for integers $x \geq 0$.

Call $f(x)$ universal if every positive integer is a sum of nine values of $f(x)$ for integers $x \geq 0$. Since $f(2) = 2 + \epsilon$, $f(3) = 3 + 4\epsilon$, 10 is not a sum of 9 values if $\epsilon \geq 9$. Lemma 2 shows that if $\epsilon = 8$, 29 is not a sum of fewer than 11 values; while if $\epsilon = 7$, 26 is not a sum of fewer than 10 values.

If $\epsilon = 6$, $f(x) = x^3$ is known to be universal. If $\epsilon = 2$, $f(x)$ is universal (loc. cit.). That $f(x)$ is universal if $\epsilon = 3$ was proved by Frances Baker in her Chicago dissertation (photo-printed). It has since been verified that $f(x)$ is universal* if $\epsilon = 1, 4, 5$.

THEOREM 12. Every positive integer is a sum of nine values of (17) for integers $x \geq 0$ if and only if $\epsilon = 1, \dots, 6$.

13. The result quoted at the beginning of §12 is a special case of

THEOREM 13. If ϵ and σ are relatively† prime positive integers and if ϵ is prime to 3, there exist positive integers C and ν such that every integer $\geq C \cdot 3^\nu$ is a sum of nine values of

$$(18) \quad f(x) = \sigma x + \frac{1}{6}\epsilon(x^3 - x).$$

LEMMA 3. Let s denote the least integer ≥ 0 for which $3^s \geq \sigma$. If $n \geq s+1$ and $m < \epsilon \cdot 3^n$, then $f(3m) < \gamma \cdot 3^{3n}$, where $\gamma = (9\epsilon^4 + 1)/2$. If $\epsilon = 1$ and $\sigma \leq 3^{16}/2$, it suffices to take $n \geq 8$.

The condition $f(3\epsilon \cdot 3^n) \leq \gamma \cdot 3^{3n}$ is equivalent to

$$(19) \quad 6\sigma\epsilon - \epsilon^2 \leq 3^{2n}, \quad 9\sigma^2 \leq (\epsilon - 3\sigma)^2 + 3^{2n},$$

which holds if $\sigma \leq 3^{n-1}$ and hence if $n-1 \geq s$.

The proof of Theorem 13 differs only in minor details from that for the case $\sigma = 1$ in these Transactions, 1934, pp. 3-12, a formula there numbered (j)

* If $\epsilon = 1$, there is no gap ≥ 2 in a table of sums ≤ 2000 of four values, whence all positive integers ≤ 2000 are sums of five. If $\epsilon = 4$, all positive integers $I \leq 2000$ except 17, 35, 55, 61, 73, 79, 200, 206, 213, 225 are sums of six values, whence every I is a sum of seven. If $\epsilon = 5$, the exceptions to sums by seven are 20 and 360.

† If they had a common factor $g > 1$, g would divide every number represented by $f(x)$ and hence divide any sum of its values.

being now cited as [j]. In [10] replace the term $3r$ by $3\sigma r$. In the identity below [12] replace the term $2l$ by $2\sigma l$. In T, [13], [16] and [17], replace the term 6 by 6σ . In [15] and the identity above it, replace the term 1 by σ . In [20] and S_i above it, and in [29], replace the term -6 by -6σ , and the first term ∓ 1 by $\mp \sigma$. In β_i in [26] replace the term 6 by 6σ .

When $\epsilon = 1$, the three inequalities [26] are satisfied if $b_1 = 5$, $b_2 = 7$, $b_3 = 11$, $C = 171$, $n \geq 8$, $\sigma \leq 5 \cdot 3^{15}/2$. We see that §5 holds, with the term $-6b_i$ replaced by $-6\sigma b_i$ in [30]. Hence every integer is a sum of nine values of (18) if $\epsilon = 1$, $\sigma \leq 3^{15}/2$, $\nu = 8$, $C = 171$, and also for larger values of σ when ν is increased.

The computations of b_1 , b_2 , b_3 , n , C depended only on inequalities [26], the only present change in which occurs in β_i . Hence by choice of n as a function of σ , the former values* of the b_i and C apply also here.

Near the bottom of page 8, loc. cit., we now have $\Delta \equiv 3\sigma r$, $E \equiv 3\sigma y_i \pmod{\epsilon}$. Since ϵ is now prime to 6 and σ , we can choose integers y_i so that

$$(20) \quad M'_i - 3\sigma y_i - 6\sigma - 3\sigma b_i \equiv 0 \pmod{\epsilon}, \quad 0 \leq y_i < \epsilon.$$

Since $B_i \equiv 3\sigma b_i \pmod{\epsilon}$, we see as on page 9 that [16] determines Q_i as an integer $\equiv 1 \pmod{4}$. This proves Theorem 13 when ϵ is prime to 6.

In its proof (§7) when ϵ is even, we have merely to multiply the terms $\pm 3b_i$ and -6 by σ .

14. We next prove Theorem 13 when $\epsilon = 3a$, $\sigma \not\equiv 2a \pmod{3}$. Then

$$(21) \quad D = f(z + r) - f(z) = \sigma r + \frac{1}{2}a(3z^2r + 3zr^2 + r^3 - r)$$

is divisible by 3^k if and only if r is. Hence there exists an integer m such that any given integer is congruent to $f(m)$ modulo 3^k . A slight modification of the proof of Lemma 3 yields

LEMMA 4. Let s be the least integer ≥ 0 for which $3^s \geq \sigma$. If $n \geq s-1$ and $0 \leq m < a \cdot 3^{n+1}$, then for (18) with $\epsilon = 3a$, $f(m) < \gamma \cdot 3^{3n}$, where $\gamma = 27(a^4 + 1)/2$.

We again employ the formulas in these Transactions with factors σ inserted at the places mentioned in §13, and with $f(3m)$ replaced by $f(m)$. The essential point is that Q_i is an integer. To prove this, apply the result above Lemma 4 with $k = n+1$. Hence if s_i is any given integer,

$$s_i = f(t_i) + 3^{n+1}u_i, \quad (0 \leq t_i < 3^{n+1}).$$

In D take $z = t_i$, $r = 3^{n+1}y_i$, where y_i is an arbitrary integer. Then $D = 3^{n+1}E$,

* Just as we now take $C = 171$ instead of the former $C = 168$ when $\epsilon = 1$, so also for any ϵ a lower value of n may be secured by increasing the old C somewhat.

where E is an integer, and $E \equiv \sigma y_i \pmod{a}$. Denote $u_i - E$ by q_i and $t_i + 3^{n+1}y_i$ by m_i . Then

$$f(m_i) - f(t_i) = D = 3^{n+1}E, \quad s_i = f(m_i) + 3^{n+1}q_i.$$

Thus $M_i = 3q_i$ is the number used in the general theory. Since σ is prime to a , there is a unique integer y_i such that

$$(22) \quad u_i - \sigma y_i - 2\sigma - b_i\sigma \equiv 0 \pmod{a}, \quad 0 \leq y_i < a.$$

Since $N_i \equiv M_i$, $B_i \equiv 3b_i\sigma \pmod{\epsilon}$, we see that $N_i - 6\sigma - B_i \equiv 0 \pmod{\epsilon}$, so that Q_i is an integer. As in these Transactions, vol. 36, page 10, we may take $Q_i \equiv 1 \pmod{4}$.

We find the following values of b_i , C . Those in II and III were obtained by G. C. Webber.

I. $a = 3p + 2$. $b_1 = 14p + 7$, $b_2 = 20p + 11$, $b_3 = 30p + 15$. If $p = 0$, $3076 \leq C \leq 3089$. If $p \geq 1$, C is between

$$\frac{1}{6}I_3 = 30493\frac{1}{2}p^4 + 66136\frac{1}{2}p^3 + 53480\frac{1}{4}p^2 + 19130\frac{3}{8}p + 2556\frac{7}{8}$$

and* $A + B$, $A = 36000p^4$, $B = 80262p^3 + 64827p^2 + 231155\frac{1}{2}p + 3089$.

II. $a = 3p + 1$. $b_1 = 14p + 5$, $b_2 = 20p + 7$, $b_3 = 30p + 11$. If $p = 0$, $503 \leq C \leq 514$. If $p \geq 1$, C is between

$$\frac{1}{6}I_3 = 30496\frac{1}{2}p^4 + 43699\frac{1}{2}p^3 + 23469\frac{3}{4}p^2 + 5601\frac{1}{8}p + 502\frac{1}{8},$$

$$\frac{1}{3}S_2 = 36000p^4 + 49800p^3 + 25830p^2 + 5954\frac{1}{2}p + 514\frac{5}{8}.$$

III. $a = 3p$. If $p = 1$, $b_1 = 9$, $b_2 = 13$, $b_3 = 19$, $8507 \leq C \leq 9844$. If $p \geq 2$, C is between

$$\frac{1}{6}I_3 = 30496\frac{1}{2}p^4 + 57712\frac{1}{2}p^3 + 36551\frac{1}{4}p^2 + 7718\frac{3}{8}p + 1\frac{1}{2},$$

$$\frac{1}{3}S_2 = 36000p^4 + 70200p^3 + 45630p^2 + 9886\frac{1}{2}p.$$

15. Hilbert† proved that a polynomial in x of degree d with rational coefficients has an integral value for every integer x exceeding a fixed limit if and only if it is a linear function with integral coefficients of the binomial coefficients $\binom{s}{t}$ for $s = 1, \dots, d$. Replacing x by $x + 1$, we see that every such cubic polynomial is the sum of

$$(23) \quad P(x) = A(x^3 - x)/6 + B(x^2 - x)/2 + Cx \quad (A, B, C \text{ integers})$$

and an integer, which we may take to be zero in a Waring problem. As in the footnote to Theorem 13, we may assume

$$(24) \quad A, B, C \text{ have no common factor; } A > 0.$$

* $S_1 = 37044p^4 + B$, while A is the leading term of $\frac{1}{3}S_2$.

† Mathematische Annalen, vol. 36 (1890), p. 511.

R. D. James* has proved that every integer exceeding a certain $L(A, B, C)$ is a sum of nine values of $P(x)$ if $A \not\equiv 4C \pmod{8}$. This function L was not determined, but is excessively large. Unlike our results in §§12-14, James's result is essentially only an asymptotic theorem and yields only asymptotic results for sextics (Part III).

PART III. WARING THEOREMS FOR POLYNOMIALS OF DEGREE 6

16. We list some identities of degree 4 needed later. Write

$$(25) \quad r = a^4 + b^4 + c^4 + d^4, \quad t = \sum_6 a^2 b^2, \quad r + 2t = s^2,$$

where $s = a^2 + b^2 + c^2 + d^2$. Then

$$(26) \quad \sum_{12} \sum_4 (Aa \pm Bb \pm Bc)^4 + 12A^2(4B^2 - A^2)r \equiv 24(2A^2B^2 + B^4)s^2.$$

For $A = 2B$ or $A = B$, this becomes

$$(27) \quad \sum_{48} (2a \pm b \pm c)^4 \equiv 216s^2, \quad \sum_{16} (a \pm b \pm c)^4 + 12r \equiv 24s^2.$$

For $A = 0$, (26) becomes (1₂). Next

$$(28) \quad \sum_{24} (Aa \pm Bb)^4 - 6(A^2 - B^2)^2 r \equiv 12A^2B^2s^2,$$

where the coefficient of r is ≥ 0 only when $A^2 = B^2$, and then (28) becomes (1₁). Again,

$$(29) \quad \sum_8 (a \pm b \pm c \pm d)^4 + \sum_4 (2a)^4 \equiv 24s^2,$$

$$(30) \quad \sum_{32} (2a \pm b \pm c \pm d)^4 + 88r \equiv 240s^2.$$

Except for (1₂) and (29), every such identity involves more than 32 fourth powers.

17. We employ symmetric functions of a, b, c, d of degree 6:

$$(31) \quad i = \sum_4 a^6, \quad j = \sum_{12} a^4 b^2, \quad k = \sum_4 a^2 b^2 c^2,$$

$$(32) \quad \sum_8 (a \pm b \pm c \pm d)^6 \equiv 8i + 8 \cdot 15j + 8 \cdot 6 \cdot 15k,$$

$$(33) \quad \sum_{24} (ga \pm hb)^6 \equiv 6(g^6 + h^6)i + 30wj, \quad w = g^4 h^2 + g^2 h^4.$$

* American Journal of Mathematics, vol. 56 (1934), pp. 303-315.

Every linear combination of i , (32) and (33) which is identical with a multiple of $s^3 = i + 3j + 6k$ is a multiple of

$$(34) \quad w(32) + 8(33) + M \sum (2a)^6 \equiv 120ws^3, \quad 4M = 7w - 3g^6 - 3h^6.$$

In the left member of (34) replace each exponent 6 by 4. By (28) and (29), the resulting function becomes

$$(24w + 8 \cdot 12g^2h^2)s^2 + 6 \cdot 8(g^2 - h^2)^2r + (M - w) \sum (2a)^4.$$

The sum of the last two parts will be zero if

$$4(g^2 - h^2)^2 + w - g^6 - h^6 \equiv (g^2 - h^2)^2(4 - g^2 - h^2) = 0.$$

But if the last factor is zero when g and h are integers, either $g^2=4$, $h=0$, or vice versa, whence M is negative. For a Waring problem, $M \geq 0$. Hence $g^2=h^2$ and (34) becomes the double of

$$(35) \quad g^6(32) + 8 \sum_{12} (ga \pm gb)^6 + g^6 \sum (2a)^6 \equiv 120g^6s^3,$$

while the like sum of fourth powers is equal to $(24g^6 + 48g^4)s^2$.

When $g=1$, (35) is Kempner's* identity

$$(36) \quad \sum_8 (a \pm b \pm c + d)^6 + 8 \sum_{12} (a \pm b)^6 + \sum_4 (2a)^6 \equiv 120s^3.$$

The corresponding sum† of fourth powers was seen to be $72s^2$. That of squares is $60s$. For arbitrary u, v, w , write

$$(37) \quad f(x) = ux^4 + vx^4 + wx^2, \quad q(x) = 120ux^3 + 72vx^2 + 60wx.$$

Hence

$$(38) \quad q(s) = \sum_8 f(a \pm b \pm c \pm d)^4 + 8 \sum_{12} f(a \pm b) + \sum_4 f(2a).$$

If we take $d=0$ in (38), we see that $q(a^2+b^2+c^2)$ is a sum of 107 values of $f(x)$. If we take $d=c$, and note that $f(c-d)$ becomes $f(0)=0$, we see that $q(a^2+b^2+2c^2)$ is a sum of 100 values of $f(x)$. Every positive integer not of the form $h=4^k(16m+14)$ is represented by $a^2+b^2+2c^2$. But h is not of the form $4^k(8n+7)$ of the only positive integers not represented by $a^2+b^2+c^2$. This proves

THEOREM 14. *If j is any positive integer, $q(j)$ is a sum of 107 values of $f(x)$ for integers x .*

18. We identify $q(x)$ in (37) with the product of (23) by a rational con-

* Dissertation, Göttingen, 1912. Extract in *Mathematische Annalen*, vol. 72 (1912), p. 396.

† Directly by adding (29) to the product of (12) by 8.

stant k , and insert the resulting values of u, v, w into $f(x)$. The latter now has the denominator 720. Hence we write $k = 720 N/D$, where the integers N and D are relatively prime. We get

$$(39) \quad \begin{aligned} q(x) &= 720 \frac{N}{D} P(x), & f(x) &= \frac{N}{D} F(x), \\ F(x) &= Ax^6 + 5Bx^4 + (12C - 2A - 6B)x^2. \end{aligned}$$

In a Waring theorem with summands $f(x)$, the value of $f(x)$ is assumed to be integral for every integer x . Hence D divides $F(x)$ for every integer x . We have

$$(40) \quad F(x) = A(x^6 - x^2) + 5B(x^4 - x^2) + Ex^2, \quad E = 12C - A - B,$$

$$(41) \quad F(1) = E, F(2) = 4(15A + 15B + E), F(3) = 9(80A + 40B + E).$$

Hence D divides $E, 60(A+B), 360(2A+B)$ and therefore their combinations, $720C, 360A$. Since D divides the products of A, B, C by 720 and since 1 is a linear combination of A, B, C by (24), we see that D divides 720.

By §15, every sufficiently large integer is a sum of nine values of $P(x)$. Hence by (39), every large multiple of $N(720/D)$ is a sum of nine values of $q(x)$. Then by Theorem 14, the same multiple is a sum of 9×107 values of $f(x) = NF(x)/D$. This statement is evidently equivalent to the case $N = 1$ of it and hence to Lemma 5.

LEMMA 5. *Let D divide all the values of $F(x)$ in (22) for integers x . Then D divides 720. Every sufficiently large multiple of $720/D$ is a sum of 963 values of $F(x)/D$.*

This implies

LEMMA 6. *Let L be the least positive integer such that every integer is congruent modulo $720/D$ to a sum of L values of $F(x)/D$. Then every sufficiently large integer is a sum of $L + 963$ values of $F(x)/D$.*

19. We seek the number corresponding to L when the modulus is one of the relatively prime factors 5, 9, 16 of 720. We are obliged to go into details to obtain facts which overcome the difficulty that congruent arguments need not yield congruent values of a polynomial whose coefficients are not integers (§21).

If E is prime to 5, (41) shows that the sums by two of the values of $F(0), F(1), F(2)$ are congruent modulo 5 to 0, $E, 2E, 3E, 4E$, whence every integer is congruent to a sum of two values of $F(x)$. But if E is divisible by 5, (40) and $x^5 \equiv x$ show that $F(x) \equiv 0 \pmod{5}$ for every x , and we employ $F(x)/5$.

Modulus 9. Evidently $F(9 \pm x) \equiv F(x)$. Hence all values of $F(x)$ are congruent to those with $x=0, 1, 2, 4$. But $3(A+B) \equiv -3E$. Hence

$$F(2) \equiv 7E, F(4) = 16(255A + 75B + E) \equiv 4E.$$

If E is prime to 3, every integer is congruent to a sum of three values of 0, E , $4E$, $7E$ modulo 9.

Next, let $E=3k$, where k is prime to 3. Then $x(x^2-1)$ and hence also $F(x)$ is divisible by 3 for every x . We see that $F(x)/3 \equiv 0$ or k according as x is or is not divisible by 3. Thus every integer is congruent to a sum of two values of $F(x)/3$. It follows also that, for all integers x, j ,

$$(42) \quad \frac{1}{3}F(x+3j) \equiv \frac{1}{3}F(x) \pmod{3}.$$

But if E is divisible by 9, we employ $F(x)/9$.

20. Modulus 16. Evidently $F(8 \pm x) \equiv F(x)$. Also, $4(A+B) \equiv -4E$. Hence every value of $F(x)$ is congruent to one of

$$(43) \quad F(0) = 0, F(1) = E, F(2) \equiv 8E, F(3) \equiv 9E + 8B.$$

Case B even. We may drop the term $8B$ from (43). If E is odd, every integer is congruent to a sum of 7 values of $F(x)$.

Let $E=2m$. Then $F(x)$ is always even. Also $F(x)/2 \equiv 0$ or $m \pmod{8}$ according as x is even or odd; and hence

$$(44) \quad \frac{1}{2}F(x+2j) \equiv \frac{1}{2}F(x) \pmod{8}, \text{ for all } x, j.$$

When m is odd every integer is congruent modulo 8 to a sum of 7 values of $F(x)/2$. If $E=4M$, where M is odd, we require a sum of 3 values of the residues 0, M modulo 4 of $F(x)/4$ which is always integral. If $E=8M$, M odd, every integer is congruent modulo 2 to one of the values 0, M of $F(x)/8$, which is always integral. Finally, if E is divisible by 16, we use $F(x)/16$, which is always integral.

Case B odd. If E is odd, then $F(3) \equiv E$, and we use a sum of 7 values of $F(x)$. If $E=2m$, where m is odd henceforth, the values of $F(x)/2$ are $\equiv 0, m, 5m \pmod{8}$, a sum of four of which yields every residue. Also,

$$(45) \quad \frac{1}{2}F(x+8j) \equiv \frac{1}{2}F(x) \pmod{8}, \text{ for all } x, j.$$

Let $E=4m$. Then, $F(2y)/4 \equiv 0$ and

$$\frac{1}{4}F(1) \equiv \frac{1}{4}F(7) \equiv m, \frac{1}{4}F(3) \equiv \frac{1}{4}F(5) \equiv -m \pmod{4},$$

a sum of two of which yields every residue. Here

$$(46) \quad \frac{1}{4}F(x+4) \equiv -\frac{1}{4}F(x) \pmod{4}, x \text{ odd};$$

$$(47) \quad \frac{1}{4}F(x+8j) \equiv \frac{1}{4}F(x) \pmod{4}, \text{ all } x, j.$$

Let $E=8m$. Then $F(x)/8 \equiv 0 \pmod{2}$ unless $x \equiv \pm 1 \pmod{8}$ and then $F(x)/8 \equiv 1 \pmod{2}$. Here

$$(48) \quad \frac{1}{8}F(x+4) \equiv 1 + \frac{1}{8}F(x) \pmod{2}, \quad x \text{ odd};$$

$$(49) \quad \frac{1}{8}F(x+8j) \equiv \frac{1}{8}F(x) \pmod{2}, \quad \text{all } x, j.$$

21. It remains to pass from our relatively prime moduli to the product as modulus. In case a polynomial $p(z)$ has integral coefficients, the classic method is to employ the Chinese remainder theorem and note that

$$(50) \quad z \equiv a \pmod{M} \text{ implies } p(z) \equiv p(a) \pmod{M}.$$

By (42), (44) and (45), property (50) holds also for our corresponding polynomials having denominators. There remain the cases $M=4, 2$ when B is odd. In view of (47) and (49), we have only to apply the Chinese remainder theorem when one congruence is $z \equiv a \pmod{8}$ instead of $z \equiv a \pmod{M=4}$ or 2 .

We have now proved

THEOREM 15. *Let D be the largest integer which divides all the values of $F(x)$ in (40). Then D is the g.c.d. of 720 and E . Then in Lemma 6, $L \leq 7$, so that every large integer is a sum of $L+963 \leq 970$ values of $F(x)/D$. We have $L=7$ if E is odd, or if B is even and $E=2m$, where (as below) m is odd. Next, $L=4$ if B is odd and $E=2m$. Again, $L=3$ if B is even and $E=4m$, or if E is prime to 3 and divisible by 4. Also, $L=1$ if E is divisible by 360. In all the remaining cases, $L=2$.*

22. **Examples.** By Theorem 12 every positive integer is a sum of nine values of $P(x) = (x^3+2x)/3$, viz., (17) for $\epsilon=2$. Hence $A=2, B=0, C=1$, and $F(x) = 2x^6+8x^2$. Here $E=10, D=10, L=7$. By Lemma 6, every integer is congruent modulo $720/D=72$ to a sum of seven values of $Q(x) = (x^6+4x^2)/5$. Since $0, \pm 1, \dots, \pm 35, 36$ form a complete set of residues modulo 72, every integer $\geq 7Q(36)$ is a sum of 970 values of $Q(x)$. Successive values of $Q(x)$ are 1, 16, 153, 832, 3145, 9360, 23569, 52480, $Q(9)=106353$. In Lemma 1, the successive values of $g(x)$ are 16, 9, 5, 3, 2, 2, 2, $g(8)=2$, whence every integer $< Q(9)$ is a sum of 41 values of $Q(x)$. But $Q(x+1) < 2Q(x)$ if $x \geq 9$, whence $q(x)=1$. Hence $41+27=68$ suffice to $Q(36)$, and $6+68=74$ suffice to $7Q(36)$.

THEOREM 16. *Every positive integer is a sum of 970 (integral) values of $(x^6+4x^2)/5$.*

Next, employ Theorem 12 with $\epsilon=3$. Then $A=3, B=0, C=1, E=D=9, L=7$. We see that 87 suffice to $7H(40)$, where $H(x) = (x^6+2x^2)/3$. Finally, if

$\epsilon=1, A=C=1, B=0, E=11, D=1, L=7; g(1)=9, g(2)=7, g(3)=5, g(4)=3, g(x)=2$ if $x=5-8, g(x)=1$ if $x \geq 9$.

THEOREM 17. *Every positive integer I is a sum of 970 values of $(x^6+2x^2)/3$. If $I \geq 7F(360)$, I is a sum of 970 values of $F(x)=x^6+10x^2$; if $I \leq 7F(360)$, I exceeds a sum of 389 values of $F(x)$ by an integer which is ≥ 0 and ≤ 10 .*

Various similar theorems are omitted for brevity.

PART IV. WARING THEOREMS FOR CERTAIN POLYNOMIALS OF DEGREES 8 AND 10

23. Employ the notations

$$H(x) = ux^8 + vx^4 + wx^2, \\ f(x) = 5040ux^4 + 720vx^2 + 504wx.$$

Then we have the identity* in a, b, c, d, u, v, w :

$$(51) \quad f(s) = \sum_{48} H(2a \pm b \pm c) + 6 \sum_4 H(2a) + 6 \sum_8 H(a \pm b \pm c \pm d) \\ + 60 \sum_{12} H(a \pm b).$$

When $v=w=0$, the identity is due to A. Hurwitz.† When $u=w=0$, it follows from (27₁), (29), (1₂).

Take $w=0$. Then $f(x)=720(7ux^4+vx^2)$. To apply Theorem 10, let $7u$ and v be relatively prime and $7u+v$ be odd. Let δ be the g.c.d. of the latter and 3. Hence every sufficiently large multiple of 720 is a sum of 51 values of $f(x)$ and hence by (51) (with 840 values of H) is a sum of 51×840 values of $H(x)$ or $H(x)/3$ according as $\delta=1$ or 3. A similar theorem with 51 replaced by 50 follows from Theorem 4. To apply the better Theorem 1, take $u=1/2, v=7/2$. Then $f(x)=5040(x^4+x^2)/2$. Hence every large multiple of 5040 is a sum of 38×840 values of $(x^8+7x^4)/2$. Similarly by Theorem 2, every large multiple of 30240 is a sum of 36×840 values of $(x^8-7x^4)/12$. We obtain Waring theorems as in Lemma 6.

24. J. Schur‡ expressed $22680 s^5$ as a sum of tenth powers. Replacing each exponent 10 by 4, we see that the sum becomes the sum of (27₁) and the products of (29) by 9 and (1₂) by 180, and hence is $1512 s^2$. But if we replace each exponent 10 by 6 or 8 we do not obtain a multiple of s^3 or s^4 . Hence

* It does not hold if H has a term in x^4 .

† Mathematische Annalen, vol. 65 (1908), pp. 424-7.

‡ Mathematische Annalen, vol. 66 (1909), p. 105; *History of the Theory of Numbers*, II, p. 721.

$1512(15us^5 + vs^2)$ is obtained from Schur's sum by replacing each x^{10} by $ux^{10} + vx^4$. A Waring theorem for the latter may therefore be deduced from one for* $15ux^5 + vx^2$.

* Maillet, *Journal de Mathématiques*, (5), vol. 2 (1896), pp. 363-380, proved that every large integer is a sum of a limited number of 1's and 192 values of any polynomial in x of degree 5 which is a positive integer for all integers $x \geq g$.

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DERIVED NUMBERS WITH RESPECT TO FUNCTIONS OF BOUNDED VARIATION*

BY

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1. Introduction. The present paper is a supplement to our previous paper.[†] It deals with the distribution of values of the derived numbers of a function $F(x)$ with respect to a function of bounded variation $\alpha(x)$, and the possibility of determining F from one of these derived numbers when the derived number in question is finite. That F can be determined from its derivative with respect to α when this derivative is finite has been shown by Lebesgue,[‡] and also by the present writer.[§] The method used by Lebesgue involves a transformation which reduces integration with respect to a function of bounded variation to ordinary integration. That of the present writer is direct. Lebesgue remarks^{||} concerning his method that it does not seem suitable for handling the corresponding problems which arise when the derivative with respect to α is replaced by derived numbers with respect to α . We have found that these problems will yield to the direct method of treatment, but the analysis is complicated. In the present paper we use a transformation[¶] which is different from that of Lebesgue, but which, like his, reduces the operations of differentiation and integration with respect to a function of bounded variation to the corresponding operations in the ordinary sense.

We shall be concerned with functions F which are constant on intervals throughout which α is constant, and for which $F(x-0)$ and $F(x+0)$ both exist at the discontinuities of α , and consequently at the discontinuities of ω , the variation function of α . At the points of discontinuity of ω let $\psi(x, h) = \{F(x+h) - F(x \mp 0)\} / m\omega(x, h)$, \mp holding according as $h \gtrless 0$. At the points of continuity of ω , let $\psi(x, h) = \{F(x+h) - F(x)\} / m\omega(x, h)$ when $m\omega(x, h) \neq 0$, $\psi(x, h) = 0$ when $m\omega(x, h) = 0$. Then the upper and lower limits of $\psi(x, h)$ as h

* Presented to the Society, April 15, 1933; received by the editors June 11, 1933, and, in revised form, March 19, 1934.

[†] *Non-absolutely convergent integrals with respect to functions of bounded variation*, these Transactions, vol. 34, pp. 645-675, the notation of which is carried throughout the present paper. The paper cited is referred to in what follows as T. It contains some typographical errors: p. 656, line 16, $D_\omega F$ should read $|D_\omega F|$; the numerator of the last inequality on this page should read $|F(x_{k+1}-0) - F(x_k-0)|$; the left side of the first inequality on p. 657 should read $|\sum F(x_{k+1}-0) - F(x_k-0)|$.

[‡] *Leçons sur l'Intégration*, Paris, 1928, pp. 296-307.

[§] T, p. 657, Theorem IX.

^{||} Loc. cit., p. 307.

[¶] This transformation was suggested by S. Saks. It simplified essentially our discussion.

tends to zero through positive values and through negative values respectively are the upper and lower right and left derived numbers of F with respect to ω , $D_{\omega}F^{+}$, $D_{\omega}F_{+}$, $D_{\omega}F^{-}$, $D_{\omega}F_{-}$. This set of derived numbers we designate by $\Lambda_{\omega}F$. If these derived numbers are all equal then their common value is the derivative of F with respect to ω , $D_{\omega}F$. If ω is the variation function of α then $D_{\omega}\alpha = g = \pm 1$, except for at most a set of ω -measure zero.* At points where $g = \pm 1$ we define the set $\Lambda_{\alpha}F$ by the relation $\Lambda_{\alpha}F = \Lambda_{\omega}F/g$. Where g is different from ± 1 the set $\Lambda_{\alpha}F$ is determined by considering the various limits as h tends to zero of the ratio $\{F(x+h) - F(x)\} / \{\alpha(x+h) - \alpha(x)\}$. If the limit as h tends to zero of $\psi(x, h)$ is equal to a_x for $x+h$ taking on any values except those of a set of ω -density zero† at x , then a_x is the approximate derivative of F with respect to ω , $AD_{\omega}F$, and where $g = \pm 1$, $AD_{\alpha}F = AD_{\omega}F/g$.

2. The distribution of the values of the derived numbers of F with respect to α . Let ω be the variation function of α , and e_c the points of (a, b) at which ω is continuous and which do not belong to intervals throughout which ω is constant. If $y = \omega(x)$, then according to our previous conventions‡ the set x_i of discontinuities of ω go into a countable set of open intervals $\beta_i = (b'_i, b''_i)$ on the interval $\{\omega(a), \omega(b)\} = (\mu, \nu)$, and the countable set of intervals α_j throughout which ω is constant go into a countable set y_j on (μ, ν) . Then, to each value of y on this closed interval, except the set y_j and the end points of β_i , there corresponds a single point x_y on (a, b) . For such values of y let $\phi(y) = F(x_y)$. At a point of the set y_j let $\phi(y)$ have the constant value of F on the corresponding interval of the set α_j . At the end points b'_i, b''_i of the intervals β_i let $\phi(b'_i) = F(x_i - 0)$, and $\phi(b''_i) = F(x_i + 0)$.

The function $\phi(y)$ is now defined at every point of the closed interval (μ, ν) . Let y_c be the set $\omega(e_c)$. At almost all points y of y_c the density of the set β_i is zero. At such a point y let us compare the various limits as Δy tends to zero of the ratio

$$(1) \quad \frac{\phi(y + \Delta y) - \phi(y)}{\Delta y},$$

with the corresponding limits as h tends to zero of the ratio

$$(2) \quad \frac{F(x_y + h) - F(x)}{m\omega(x, h)}.$$

* Daniell, these Transactions, vol. 19, p. 361. The result there given evidently holds under the present definition of a derivative.

† T, p. 662, where right hand ω -density of the set E is defined. ω -density is the limit as h tends to zero of the ratio $\bar{m}E(x-h, x+h)/m\omega(x-h, x+h)$.

‡ T, p. 646, 1.

If $y+\Delta y$ is a point of y_e and x_v+h is the corresponding point of e_e , then the two ratios are the same. This is also the case if $y+\Delta y$ is a point of y_j , and x_v+h is on the corresponding interval α_j . Let $y+\Delta y$ be a point of β_i . Then if $x_v+h=x_i$ we have

$$(3) \quad F(x_v+h) = \phi(y+\Delta y), \quad m\omega(x, h) = \Delta y + t_i,$$

where $|t_i| \leq m\beta_i$. Consider the ratio

$$(4) \quad \frac{m\omega(x, h)}{\Delta y} = \frac{\Delta y + t_i}{\Delta y} = 1 + \frac{t_i}{\Delta y}.$$

Now

$$(5) \quad \left| \frac{t_i}{\Delta y} \right| \leq \frac{m\beta_i}{|\Delta y|}.$$

And since at the point y the density of the set of intervals β_i is zero, it follows that

$$\frac{m\beta_i}{|\Delta y| + |t_i|} = \frac{1}{\frac{|\Delta y|}{m\beta_i} + \frac{|t_i|}{m\beta_i}}$$

tends to zero with Δy . As a result of this, and the fact that $|t_i|/m\beta_i \leq 1$, we conclude that $m\beta_i/|\Delta y|$ tends to zero with Δy . Relations (4) and (5) then show that

$$\lim_{\Delta y \rightarrow 0} \frac{m\omega(x, h)}{\Delta y} = 1.$$

It then follows from (3) and (6) that if $y+\Delta y$ is a point of β_i , and $x_v+h=x_i$, where x_i is the point of discontinuity of ω corresponding to the interval β_i , then the ratios (1) and (2) have the same limits of indetermination.

It remains to consider the case in which $y+\Delta y$ is an end point of β_i . Let $y+\Delta y=b'_i$. Then $\phi(y+\Delta y)=\phi(b'_i)=F(x_i-0)$. Let x_l be a sequence of values of x belonging to e_e or to α_j , and tending to x_i from the left. Then $F(x_l)$ tends to $F(x_i-0)=\phi(y+\Delta y)$, and if $x_v+h=x_i$ then $m\omega(x_v, h)$ tends to Δy . Thus for l sufficiently large the ratio (2) is arbitrarily near to the ratio (1). A like manner of reasoning may be used to show that the same situation prevails when $y+\Delta y=b''_i$.

We have now proved that every value that is approached by the ratio (1) as Δy tends to zero, is also approached by the ratio (2) as h tends to zero over a suitably chosen sequence of values of h . Starting with the ratio (2) and

letting h tend to zero through all possible values, it can be shown by reasoning similar to the above that for every limit approached by (2) there is a sequence of values of Δy tending to zero over which the ratio (1) approaches the same limit. It then follows that, except for a part of e_c of ω -measure zero, the distribution of the values of the set $\Lambda_\omega F$ at the points of e_c is the same as the distribution of the values of the derived numbers of $\phi(y)$ at the points of y_c . At the set x_i of discontinuities of ω , $D_\omega F$ exists and is finite. If ω is the variation function of α , then where $D_\omega \alpha = g = \pm 1$, $\Lambda_\alpha F = \Lambda_\omega F/g$. This relation holds except for a set of ω -measure zero. At a point where $g = -1$ an upper derived number with respect to ω may correspond to a lower derived number with respect to α , and conversely. But where one is finite the other is also. Furthermore, if the function F is measurable relative to α^* on (a, b) , then the function ϕ is measurable on (μ, ν) . Consequently, if we take into consideration the known facts concerning the distribution of the values of the derived numbers of measurable functions† we have the following result:

Let $\alpha(x)$ be a function of bounded variation on the interval (a, b) , and let the function $F(x)$ be finite at each point of (a, b) , measurable relative to α on (a, b) , constant on intervals throughout which α is constant, and such that at the points of discontinuity of α , $F(x-0)$ and $F(x+0)$ both exist. Then, except for at most a set of α -measure zero, the derived numbers and approximate derivatives of F with respect to α fall into one or the other of the following classes:

- (1) $AD_\alpha F$ exists and is finite.
 (2) $AD_\alpha F^+ = AD_\alpha F^- = +\infty$, $AD_\alpha F_+ = AD_\alpha F_- = -\infty$.

The points of class (1) are of four types:

- (1.1) $D_\alpha F$ exists and is finite.
 (1.2) $D_\alpha F_+ = AD_\alpha F = D_\alpha F^-$, $D_\alpha F^+ = \infty$, $D_\alpha F_- = -\infty$.
 (1.3) $D_\alpha F^+ = AD_\alpha F = D_\alpha F_-$, $D_\alpha F_+ = -\infty$, $D_\alpha F^- = \infty$.
 (1.4) $D_\alpha F^+ = D_\alpha F^- = \infty$, $D_\alpha F_+ = D_\alpha F_- = -\infty$.

3. The determination of $F(x)$ by means of the derived numbers of F with respect to α . In this section, in addition to the conditions imposed above, F is continuous where α is continuous, and, at points of discontinuity of α , $F(x)$ lies on the interval defined by $F(x-0)$ and $F(x+0)$. Furthermore, the region of definition of F is extended beyond the interval (a, b) in such a way that

* T, p. 646, §1, p. 655, §6.

† J. C. Burkill and U. S. Haslam-Jones, *The derivatives and approximate derivatives of measurable functions*, Proceedings of the London Mathematical Society, (2), vol. 32, pp. 346-355.

$F(a-0)=F(a)$, and $F(b+0)=F(b)$. Let ω be the variation function of α . On the interval $\omega(a)=\mu \leq y \leq \nu=\omega(b)$ let $\psi(y)$ be the function ϕ defined in the previous section, except for the intervals β_i , where ψ is linear, ranging from $F(x_i-0)$ to $F(x_i)$ on the left half of β_i and from $F(x_i)$ to $F(x_i+0)$ on the right half of this interval. We prove the following:

If $D_\omega F^+$ is finite at each point of (a, b) , then $D\psi^+$ is finite at each point of (μ, ν) , with the possible exception of the right hand end points of the intervals β_i .

On the intervals β_i the function ψ is linear, and consequently $D\psi^+$ is finite at each point y for which $b'_i \leq y < b''_i$. For y a point of y_e and $y+\Delta y$ a point of y_e or y_j , the limits of indetermination of the ratios

$$\frac{\psi(y+\Delta y) - \psi(y)}{\Delta y} \quad \text{and} \quad \frac{F(x_\nu + h) - F(x_\nu)}{m\omega(x_\nu, h)}$$

are the same provided $x_\nu + h$ is so chosen that $y+\Delta y = \omega(x_\nu + h)$. The same statement holds if y is a point of y_j , provided x_ν is the right hand end point of α_j , and $F(x_\nu)$ is replaced by $F(x_\nu-0)$. Hence, if $D_\omega F^+$ is finite, either the upper limit of the first ratio is finite or this ratio becomes positively infinite as Δy tends to zero with $y+\Delta y$ on intervals of the set β_i . Let $y+\Delta y$ be on $\beta_i = (b'_i, b''_i)$; let $b'_i - y = \Delta'y$, $\Delta y = \Delta'y + t'_i$, and let $m\beta_i = t_i$. There are then two cases to consider: (i) $\Delta'y/t_i$ bounded from zero; (ii) $\Delta'y/t_i$ tending to zero as $\Delta'y$ tends to zero. In case (i) the ratio $\Delta\psi/\Delta y$ lies between the two ratios

$$\frac{F(x_i+0) - F(x_\nu)}{\Delta'y + t'_i} \quad \text{and} \quad \frac{F(x_i-0) - F(x_\nu)}{\Delta'y + t'_i},$$

i.e., between

$$\frac{F(x_i+0) - F(x_\nu)}{\Delta'y + t_i} \bigg/ \frac{\Delta'y + t'_i}{\Delta'y + t_i} \quad \text{and} \quad \frac{F(x_i-0) - F(x_\nu)}{\Delta'y} \bigg/ \frac{\Delta'y + t'_i}{\Delta'y}.$$

Since $\Delta'y + t_i = m\omega(x_\nu, x_i)$, $\Delta'y = m\omega(x_\nu, x_i-0)$, and since $D_\omega F^+$ is finite, it follows that the numerators of these last two expressions are bounded above, and since $\Delta'y/t_i$ is bounded from zero, it follows that their denominators are bounded from zero. Thus it has been shown that $D\psi^+ < \infty$. It remains to be shown that $D\psi^+ > -\infty$. If the ratio $\Delta\psi/\Delta y$ becomes negatively infinite for every sequence of values of Δy , then the ratio $\Delta F/\Delta\omega$ becomes negatively infinite for $x_\nu + h$ points of e_e or α_j . Hence, since $D_\omega F^+$ is finite, we must have $\Delta F/\Delta\omega$ tending to a finite limit for $x_\nu + h$ points of the set x_i of discontinuities of ω . In this case,

$$\frac{\Delta F}{m\omega(x_\nu, h)} = \frac{F(x_i) - F(x_\nu)}{m\omega(x_\nu, x_i)} = \frac{\psi(y+\Delta y) - \psi(y)}{m\omega(x_\nu, x_i)},$$

where $\Delta y = \Delta' y + t_i/2$ and $m\omega(x_y, x_i) = \Delta' y + t_i$. But this makes $\Delta y/m\omega(x_y, x_i) \geq \frac{1}{2}$. Hence, if $\Delta\psi/\Delta y$ becomes negatively infinite, so does $\Delta\psi/\Delta\omega$ and its equivalent ratio $\Delta F/m\omega(x_y, x_i)$, from which it follows that $D_\omega F^+ = -\infty$. But this is a contradiction. We can, therefore, conclude that when $\Delta' y/t_i$ is bounded from zero, $D\psi^+$ is finite.

Remark. In case (i) if the function ψ is defined as above, except that it is a single linear function on β_i ranging from $F(x_i-0)$ to $F(x_i+0)$, the same conclusions hold in regard to $D\psi^+$. It was this definition of ψ that was suggested by Saks. But if ψ is defined in this manner there exist functions F and ω such that in case (ii) $D_\omega F^+$ is finite and $D\psi^+ = -\infty$. We exhibit such an example. It throws light on the whole situation.

On the interval $(0, e)$ let

$$x_n = \sum_{i=n}^{\infty} \frac{1}{i!}, \text{ and let } \omega(x) = \sum_{i=n}^{\infty} \frac{1}{i!} \text{ on } x_n \leq x < x_{n-1}.$$

It is easily verified that if $\beta_n = \omega(x_n-0) < y < \omega(x_n+0)$ then for $y=0$, $\Delta' y/t_i$ tends to zero as $\Delta' y$ tends to zero, which is the condition of case (ii). Let $F(0)=0$, and on $x_n < x \leq x_{n-1}$ let

$$F(x) = - \sum_{i=n-1}^{\infty} \frac{1}{i!}; \text{ then } \frac{F(x_{n-1}) - F(0)}{m\omega(0, x_{n-1})} = -1,$$

which shows that $D_\omega F^+$ is finite. Also

$$\frac{F(x_{n-1} + h) - F(0)}{m\omega(0, x_{n-1} + h)} = \frac{- \sum_{i=n-2}^{\infty} \frac{1}{i!}}{\sum_{i=n-1}^{\infty} \frac{1}{i!}} = - (n-1) \frac{1 + \frac{1}{n-1} + \frac{1}{n(n-1)} + \cdots}{1 + \frac{1}{n} + \frac{1}{n(n+1)} + \cdots},$$

which becomes negatively infinite as n increases. Hence $D_\omega F_+ = -\infty$. Now let ψ be linear on β_n and range from $F(x_n-0)$ to $F(x_n+0)$. Then if $y=0$, and $y+\Delta y$ is on β_n ,

$$\frac{\Delta\psi}{\Delta y} = \frac{\sum_{i=n-1}^{\infty} \frac{1}{i!} - t_i''}{\sum_{i=n}^{\infty} \frac{1}{i!} + t_i'} = -n \frac{\theta_1(n) + n(n-1)!t_i'}{\theta_2(n) + n!t_i'},$$

since $t_i''/t_i' = n$. The functions $\theta_1(n)$ and $\theta_2(n)$ tend to unity as n increases. From this it follows that the last member of the foregoing equality, and consequently $\Delta\psi/\Delta y$, becomes negatively infinite as n increases. Hence $D\psi^+ = -\infty$.

We now show that for the function ψ defined at the beginning of this section we have in case (ii), just as in case (i), that $D_\omega F^+$ finite implies $D\psi^+$ finite. If $D_\omega F^+$ is finite and $D\psi^+ = +\infty$, then $\Delta\psi/\Delta y$ must become infinite for $y+\Delta y$ on β_i . We then have

$$\frac{\Delta\psi}{\Delta y} = \frac{\psi(y+\Delta y) - \psi(y+\Delta'y)}{t'_i} \bigg/ \frac{\Delta'y + t'_i}{t'_i} + \frac{\psi(y+\Delta'y) - \psi(y)}{\Delta'y} \bigg/ \frac{\Delta'y}{\Delta'y}.$$

The denominators of both ratios on the right are greater than or equal to unity, and since $D_\omega F^+$ is finite the numerator of the second ratio is finite. Hence if $\Delta\psi/\Delta y$ becomes positively infinite so does the numerator of the first ratio. If we set $\psi(y+\Delta y) - \psi(y+\Delta'y) = t''_i$, then t''_i/t'_i becomes infinite, and

$$\frac{\Delta\psi}{\Delta y} = \frac{F(x_i) - F(x_y)}{\Delta'y + t'_i} + \frac{t''_i}{\Delta'y + t'_i} = A + B.$$

Let us compare A and B with

$$A' = \frac{F(x_i) - F(x_y)}{\Delta'y + \tau'_i} \text{ and } B' = \frac{\tau''_i}{\Delta'y + \tau'_i},$$

where $t''_i \leq \tau''_i \leq F(x_i)$, and $t'_i \leq \tau'_i \leq t_i/2$. If A is negative then $A' \geq A$, and if A is positive, $A' \geq 0$. Since $t''_i/t'_i = \tau''_i/\tau'_i$ it follows that $B' \geq B$. Since $\Delta'y/t_i$ tends to zero, and since τ'_i/τ'_i becomes positively infinite, it follows that B' becomes infinite as $\Delta'y$ tends to zero and τ'_i tends to $t_i/2$. Hence, if $\Delta\psi/\Delta y$ becomes positively infinite, so does $A' + B'$ for $\tau'_i = t_i/2$. But this means that the ratio $\{F(x_i) - F(x_y)\}/t_i$ tends to $+\infty$. Then, since $\Delta'y/t_i$ tends to zero, it follows that $\{F(x_i) - F(x_y)\}/m\omega(x_y, x_i)$ tends to $+\infty$. But this means that, at the point x_y , $D_\omega F^+ = +\infty$, which is a contradiction. We conclude, therefore, that $D\psi^+ < \infty$. The proof that $D\psi^+ > -\infty$ is the same as in case (i).

We now know that if, at each point of (a, b) , $D_\omega F^+$ is finite, then, at each point of (μ, ν) , $D\psi^+$ is finite, except possibly the right hand end points of the intervals β_i , at which points the left hand derivative of ψ exists and is finite. Hence if, at each point of (a, b) , $D_\omega F^+$ is finite, then at each point of (μ, ν) one of the derived numbers of ψ is finite. At the points of y_e , except at most a null set, this finite derived number can be taken as $D\psi^+$, and will be equal to $D_\omega F^+$ at the corresponding points of e_e , which is all of e_e except at most a set of ω -measure zero. Furthermore, on an interval of the set β_i ,

$$\int_{b_i'}^{b_i''} D\psi^+ dy = \psi(b_i'') - \psi(b_i') = F(x_i + 0) - F(x_i - 0) = \int_{x_i} D_\omega F d\omega.$$

Let $a' \leq x \leq a''$ be an interval on (a, b) with (a', a'') points of continuity of ω , and $b' = \omega(a') \leq y \leq b'' = \omega(a'')$, the corresponding interval on (μ, ν) . Let $f(y) = D\psi^+$ where $D\psi^+$ is finite, and otherwise let $f(y)$ be the left hand derivative of ψ . Then

$$\int_{b'}^{b''} f(y) dy = \psi(b'') - \psi(b') = F(a'') - F(a'),$$

where the integration is in the sense of Denjoy.* Since at almost all of y_e the function $f(y) = D_\omega F^+$ at the corresponding points of e_e , and since the integral of $f(y)$ over β_i is equal to the integral of $D_\omega F$ over x_i , it follows that $D_\omega F^+$ is Denjoy integrable with respect to ω^\dagger on (a', a'') , and that

$$\int_{a'}^{a''} D_\omega F^+ d\omega = \int_{b'}^{b''} f(y) dy = F(a'') - F(a').$$

Now let (l, m) be any interval on (a, b) , (a_n', a_n'') a sequence of intervals for which $l < \dots < a_2' < a_1' < a_1'' < a_2'' < \dots < m$, where a_n' and a_n'' are points of continuity of ω , a_n' tending to l and a_n'' tending to m . Then

$$F(m - 0) - F(l + 0) = \lim_{n \rightarrow \infty} \int_{a_n'}^{a_n''} D_\omega F^+ d\omega = \int_{l < x < m} D_\omega F^+ d\omega,$$

where the integration with respect to ω is in the sense of Denjoy. Also

$$F(l + 0) - F(l - 0) = \int_l D_\omega F^+ d\omega,$$

$$F(m + 0) - F(m - 0) = \int_m D_\omega F^+ d\omega.$$

By putting $l = a$ and $x = m$ these results then permit us to state the following theorem:

If $f(x)$ is finite at each point of $a \leq x \leq b$, and is the upper right derivative with respect to a non-decreasing function ω , where F satisfies the conditions laid down at the beginning of this section, then if x is any point on (a, b) ,

* Lebesgue, loc. cit., p. 150.

† T, p. 665, §10.

$$F(x-0) - F(a) = \int_{a \leq t < x} f(t) d\omega, \quad F(x+0) - F(a) = \int_{a \leq t \leq x} f(t) d\omega,$$

where the integration with respect to ω is in the sense of Denjoy.

In a similar manner the same result can be established for any of the other derived numbers of F with respect to ω . If it is known that f is equal to one of the derived numbers of F with respect to ω , not necessarily the same derived number at each point, and if further for the intervals α_i throughout which ω is constant f is a right hand derived number at the upper end, or a left hand derived number at the lower end, then it follows by reasoning similar to the above that one of the derived numbers of ψ is finite at each point of (μ, ν) , and this in turn leads to the truth of the foregoing theorem in the present case.

If α is a function of bounded variation on (a, b) and ω the variation function of α , then where $g = \pm 1 = D_\omega \alpha$, $\Lambda_\alpha F = \Lambda_\omega F/g$. Hence, if at such points one of the set $\Lambda_\alpha F$ is finite, then one of the set $\Lambda_\omega F$ is finite. Where g is different from ± 1 , the set $\Lambda_\alpha F$ is determined from the ratio $\Delta F/\Delta \alpha$. It is not difficult to construct functions F and α for which $D_\alpha F^+$ is finite, $D_\omega F^+ = +\infty$, $D_\alpha F_- = -\infty$. If, however, at this exceptional set for all $|h|$ sufficiently small, $\Delta \alpha = \alpha(x+h) - \alpha(x)$ does not change sign unless h changes sign, it is easily shown that one of the set $\Lambda_\alpha F$ finite implies one of the set $\Lambda_\omega F$ finite. When the function α satisfies this condition, we have

If the function $f(x)$ is finite at each point of the interval $a \leq x \leq b$ and equal to one of the derived numbers of F with respect to α , for the intervals of the set α_i throughout which α is constant either a right hand derived number at the upper end or a left hand derived number at the lower end, then

$$F(x-0) - F(a) = \int_{a \leq t < x} f(t) d\alpha, \quad F(x+0) - F(a) = \int_{a \leq t \leq x} f(t) d\alpha,$$

where the integration with respect to α is in the sense of Denjoy.*

At the points for which $g = \pm 1$, $f = \theta/g$ where θ is one of the derived numbers of F with respect to ω . If we set $\theta_1(x) = \theta(x)$ where $g = \pm 1$, and θ_1 equal to a finite derived number of F with respect to ω at the remaining points of (a, b) , then on the intervals $a \leq t < x$ and $a \leq t \leq x$, the function θ_1 is integrable in the sense of Denjoy with respect to ω to the values $F(x-0) - F(a)$ and

* The integral of $f d\alpha$ is defined as the integral of $f g d\omega$, in whatever sense the latter integral exists, T, p. 655, §6. It is stated, T, p. 675, §13, that if the integral of $f d\omega$ exists in the sense of Denjoy then the integral of $f g d\omega$ exists in the same sense. Obviously, this is only necessarily true when f is summable with respect to ω , since g may be negative where f is negative and positive where f is positive.

$F(x+0)-F(a)$ respectively. Hence for the interval (a, x) we have, since $\theta = \theta_1$ except for at most a set of ω -measure zero,

$$\int f d\alpha = \int f g d\omega = \int \theta d\omega = F(x \mp 0) - F(a),$$

\mp holding according as the integration is taken over the interval $a \leq t < x$, or $a \leq t \leq x$. This establishes the theorem.

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PROBABILITY AND STATISTICS*

BY

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The theory of probability has made much progress recently in the direction of completely mathematical formulations of its methods and results.‡ The purpose of this paper is to make a further contribution in this direction. In order to analyze the results of repeated trials of an experiment, a certain space of infinitely many dimensions is the proper tool. This space is discussed in the first section of the paper. In the second section, the results of the first are applied to obtain for the first time a complete proof of the validity of the method of maximum likelihood of R. A. Fisher, which is used in statistics to estimate the true probability distribution when the results of a repeated experiment are known.

1. THE SPACE $\Omega(F)$

It will be seen that the space $\Omega(F)$ described below provides the natural basis for the analysis of experiments with repeated trials. The preliminary facts, which are not new, will be stated in the form of a theorem.

THEOREM 1. *Let $F(x)$ be a monotone non-decreasing function, defined for $-\infty < x < \infty$, and satisfying*

$$(1) \quad F(x-0) = F(x), \quad \lim_{x \rightarrow \infty} F(x) = 1, \quad \lim_{x \rightarrow -\infty} F(x) = 0.$$

There is a σ -field§ of point sets on the x -axis, including all Borel measurable sets, and a completely additive non-negative set function $p_F(A)$ defined on this σ -field, such that if I is any interval $a \leq x < b$, $p_F(I) = F(b) - F(a)$.||

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‡ Cf. the treatment of A. Kolmogoroff, *Ergebnisse der Mathematik*, vol. 2, No. 3: *Grundbegriffe der Wahrscheinlichkeitsrechnung*.

§ A field is a collection of point sets with the property that if A and B are sets in the collection, $A+B$, $A-B$, $A \cdot B$, $A \cdot \bar{B}$ are also. A field is a σ -field if whenever A_1, A_2, \dots is a sequence of sets in the field, $\sum_{i=1}^{\infty} A_i$ is also in the field. It will then follow that $\prod_{i=1}^{\infty} A_i$ is in the field. A set function $p(A)$ defined on the sets of a σ -field is completely additive if when A_1, A_2, \dots is a sequence of disjunct sets in the field, $p(\sum_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} p(A_i)$.

|| The sets in the field of definition of p_F will be called measurable with respect to $F(x)$. If A is measurable with respect to $F(x)$, $p_F(A)$ is the variation of $F(x)$ over A . The definitions of functions measurable with respect to $F(x)$ and of their integration are formulated in the usual way, giving the Lebesgue-Stieltjes integral.

Let $\Omega(F)$ be the space whose points are the sequences $(\dots, x_{-1}, x_0, x_1, \dots)$, where x_j is any real number. There is a σ -field of point sets of $\Omega(F)$, including all sets determined by conditions of the form

$$(2) \quad x_j \in E_j \quad (j = 0, \pm 1, \dots),$$

where the point sets E_1, E_2, \dots are measurable with respect to $F(x)$ and a completely additive non-negative set function $P_F(\Lambda)$ defined on this field, such that if Λ is of the type (2),

$$(3) \quad P_F(\Lambda) = \prod_{j=-\infty}^{j=\infty} p_F(E_j).$$

The sets in the field of definition of P_F will be called measurable with respect to $F(x)$; the measurability (with respect to $F(x)$) and integration of functions defined on $\Omega(F)$ are then defined in the usual way. This space was first discussed by Daniell.*

It should be noted that if $\phi(\omega)$ is a measurable function on $\Omega(F)$, and if $\phi(\omega)$ depends only on x_1 : $\phi(\omega) = f(x_1)$, $f(x)$ is measurable with respect to $F(x)$, and

$$(4) \quad \int_{\Omega(F)} \phi(\omega) d\omega = \int_{-\infty}^{\infty} f(x) dF(x)$$

where the existence of either integral implies that of the other.

This space $\Omega(F)$ is introduced as a tool in the rigorous analysis of certain ideas in the theory of probability. Let $F(x)$ determine a probability distribution, i.e. we suppose that there is a chance variable x such that the probability that $x < x$ is $F(x)$. Then (1) is satisfied. If a single trial is made, $p_F(A)$ is the probability that the value of x obtained will be in the set A . If a finite succession of trials is made, obtaining values ξ_1, \dots, ξ_n , and if Λ is a point set of $\Omega(F)$ on which P_F is defined, $P_F(\Lambda)$ is the probability that there is a point $\omega: (\dots, x_0, \dots)$ in Λ such that $x_j = \xi_j, j = 1, \dots, n$. The usual interpretation if Λ is a set of the form (2) is obvious. The advantage of this point of view† is that the set-up is independent of the number of trials. Chance varia-

* *Annals of Mathematics*, (2), vol. 20 (1919), pp. 281-288. Daniell actually only considered the space whose points are sequences of the form (x_1, x_2, \dots) , but the treatment of $\Omega(F)$ could be carried through in the same way. These considerations concerning the space $\Omega(F)$ can be considered as a particular case of a general treatment given by Kolmogoroff, loc. cit., pp. 24-30.

† A similar point of view was taken by A. Khintchine, *Zeitschrift für angewandte Mathematik und Mechanik*, vol. 13 (1933), pp. 101-103, who treated the case of a chance variable which only takes on the values 1 or 0 (making less restrictions on $P_F(\Lambda)$ however). This space was used for the same purpose by E. Hopf, *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, vol. 13 (1934), pp. 51-102. The place of these methods in the theory of stochastic processes was discussed by the writer in the *Proceedings of the National Academy of Sciences*, vol. 20 (1934), pp. 376-379.

bles become measurable functions on $\Omega(F)$, and their integrals on $\Omega(F)$ are their expectations. The law of large numbers will be seen to correspond to the ergodic theorem of Birkhoff.* The convergence of a sequence of chance variables in probability† is simply convergence in measure on $\Omega(F)$.‡

THEOREM 2. *The transformation T of $\Omega(F)$ into itself,*

$$T: \quad x'_j = x_{j+1} \quad (j = 0, \pm 1, \dots),$$

is a one-to-one measure-preserving transformation. If Λ is a measurable set invariant under T , $P_F(\Lambda) = 0$, or $P_F(\Lambda) = 1$.§ If $\phi(\omega)$ is any measurable function on $\Omega(F)$ such that $\int_{\Omega(F)} |\phi(\omega)|^2 d\omega$ exists and such that $\phi(T\omega) = e^{i\lambda} \phi(\omega)$ for some real number λ ,

$$\phi(\omega) = \int_{\Omega(F)} \phi(\omega) d\omega$$

almost everywhere on $\Omega(F)$.

The second part of the theorem includes the first part if $\lambda = 0$, and if $\phi(\omega)$ is considered as the characteristic function of a point set, so only the second part of the theorem need be considered. The proof will be given in several steps.

(i) Let $F(x)$ be 0 for $x < 0$, x for $0 \leq x \leq 1$ and 1 for $x > 1$, and let $p_F(A)$ and $P_F(\Lambda)$ for this $F(x)$ be denoted by $p_0(A)$, $P_0(\Lambda)$, respectively. Let Ω_0 be the subset of $\Omega(F)$ consisting of the points $(\dots, x_{-1}, x_0, x_1, \dots)$ whose coordinates satisfy the inequalities $0 \leq x_j \leq 1$, $j = 0, \pm 1, \dots$. It will be shown that the general set functions $p_F(A)$ and $P_F(\Lambda)$ can be derived from $p_0(A)$ and $P_0(\Lambda)$. In fact, let $y = F(x)$ transform the points of the x -axis into points of the interval $0 \leq y \leq 1$, where if $F(x)$ has a jump at x_0 , the point x_0 will be made to correspond to the interval $F(x_0) \leq y \leq F(x_0 + 0)$. Then $p_F(A)$ is defined for those and only those sets whose images on the y -axis are Lebesgue measurable, and for such sets $p_F(A)$ is defined as the Lebesgue measure of the image of A . In the same way the set Λ_x on $\Omega(F)$ measurable with respect to $F(x)$ goes over into a set Λ_y on Ω_0 on which $P_0(\Lambda)$ is defined, and $P_F(\Lambda_x)$

* Cf. A. Khintchine, loc. cit., and E. Hopf, loc. cit., p. 95.

† For the definition of convergence in probability, see for instance Kolmogoroff, loc. cit., p. 31.

‡ Convergence in measure was defined and discussed by F. Riesz, Paris Comptes Rendus, vol. 148 (1909), pp. 1303-1305.

§ If $F(x)$ does not increase, except for equal jumps at $x = 0, \dots, 9$, the set function $P_F(\Lambda)$ has a simple interpretation as ordinary two-dimensional Lebesgue measure, and this property (metrical transitivity) was proved by W. Seidel, Proceedings of the National Academy of Sciences, vol. 19 (1933), pp. 453-456. Hopf obtained this result from the second part of the corollary to this theorem (see below) by a different method.

$= P_0(\Lambda_v)$. Then it is sufficient to prove Theorem 2 for the space Ω_0 and the set function $P_0(\Lambda)$.

(ii) The set of all complex-valued functions $\phi(\omega)$ on Ω_0 whose real and imaginary parts are measurable on Ω_0 and such that

$$\int_{\Omega_0} |\phi(\omega)|^2 d\omega$$

exists can be considered as the set of elements of a Hilbert space* \mathfrak{H} if the inner product of $\phi_1(\omega)$, $\phi_2(\omega)$ is defined in the usual way as

$$\int_{\Omega_0} \phi_1(\omega) \overline{\phi_2(\omega)} d\omega. \dagger$$

It is easily seen that the set of functions of the form

$$\exp \left\{ 2\pi i \sum_{j=1}^n n_j x_j(\omega) \right\}$$

where $x_j(\omega)$ is the value of x_j for the point $\omega: (\dots, x_{-1}, x_0, x_1, \dots)$ and where n_j , n are arbitrary integers, form a complete orthonormal set of functions in \mathfrak{H} .[‡] If these functions, arranged in some order, are $\phi_0(\omega)$, $\phi_1(\omega)$, \dots where $\phi_0(\omega) \equiv 1$, to every function $\phi(\omega)$ in \mathfrak{H} corresponds a series $\sum_{j=0}^{\infty} a_j \phi_j(\omega)$, where the coefficient a_j is determined by

$$(5) \quad a_j = \int_{\Omega_0} \phi(\omega) \overline{\phi_j(\omega)} d\omega,$$

such that

$$(6) \quad \int_{\Omega_0} |\phi(\omega)|^2 d\omega = \sum_{j=0}^{\infty} |a_j|^2.$$

(iii) Now suppose that $\phi(T\omega) = e^{i\lambda} \phi(\omega)$. Then if b_0, b_1, \dots are the coefficients corresponding to $\phi(T\omega)$,

$$(7) \quad b_j = e^{i\lambda} a_j,$$

and, from the simple form of the transformation T , if $j > 0$,

$$(8) \quad a_j = b_{r(j)} = e^{i\lambda} a_{r(j)}$$

* For a general reference to Hilbert space see, for instance, M. H. Stone, *Linear Transformations in Hilbert Space*, American Mathematical Society Colloquium Publications, vol. 15 (especially chapter I). The properties of Ω_0 which are needed here (separability, etc., if distance is properly defined), are given by Daniell, loc. cit., p. 281. Using these properties the proof that the functions $\{\phi(\omega)\}$ form a Hilbert space follows the lines of a similar theorem in Stone, pp. 23-29.

† If ξ is a complex number, $\bar{\xi}$ will denote its conjugate.

‡ This concept is discussed by Stone, loc. cit., pp. 7-14, where the facts stated below are proved.

where $\tau(j) \neq j$. Repeating this we find a sequence of coefficients a_{m_1}, a_{m_2}, \dots , where $m_1 = j$, $m_i = \tau(m_{i-1})$ if $j > 1$, whose absolute values are all equal. Evidently $m_i \neq m_j$ if $i \neq j$. This contradicts (6) unless $a_j = 0$. Then $a_j = 0$ if $j > 0$, and $\phi(\omega) = a_0$, as was to be proved.*

COROLLARY. (i) If $\phi(\omega)$ is any integrable function on $\Omega(F)$,

$$(9) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \phi(T^i \omega) = \int_{\Omega(F)} \phi(\omega) d\omega$$

almost everywhere on $\Omega(F)$.

(ii) If $\phi_1(\omega), \phi_2(\omega)$ are measurable functions the squares of whose absolute values are integrable on $\Delta(F)$,

$$(10) \quad \lim_{n \rightarrow \infty} \int_{\Omega(F)} \phi_1(T^n \omega) \phi_2(\omega) d\omega = \left\{ \int_{\Omega(F)} \phi_1(\omega) d\omega \right\} \left\{ \int_{\Omega(F)} \phi_2(\omega) d\omega \right\}.$$

(i) This part of the corollary is simply the ergodic theorem in this case.†

(ii) This part of the corollary corresponds to the extension of the ergodic theorem given by E. Hopf, B. O. Koopman and J. von Neumann, to the particular case where there are no "angle variables."‡ It is obvious when $\phi_1(\omega)$ and $\phi_2(\omega)$ each depend only on a finite number of the coordinates of $\omega: (\dots, x_0, \dots)$ since in that case the terms in (10) are equal to the limit prescribed for sufficiently large values of n . Since any measurable function can be approximated by functions depending only on a finite number of coordinates,§ the general theorem can be reduced to this case.

The following lemma is needed for the proof of the next theorem.

LEMMA. Let $F(x)$ be defined as in Theorem 1. Define measure on the x -axis by the set function p_F . Let $f(x)$ be a function defined for almost all values of x and measurable (with respect to $F(x)$). Then if

$$(11) \quad \limsup_{n \rightarrow \infty} |f(x_n)|/n < \infty$$

on a set of points $\omega: (\dots, x_0, \dots)$ of $\Omega(F)$ of positive measure, $\int_{-\infty}^{\infty} f(x) dF(x)$ exists (as a Stieltjes-Lebesgue integral).||

* Stone, loc. cit., p. 10.

† For a simple proof of the ergodic theorem, following the lines of the first proof, given by Birkhoff, cf. A. Khintchine, *Mathematische Annalen*, vol. 107 (1933), pp. 485-488. In this proof the function $\phi(x, r)$ corresponds to the function $\sum_{i=1}^r \phi(T^i \omega)$ used here.

‡ E. Hopf, *Proceedings of the National Academy of Sciences*, vol. 18 (1932), pp. 204-209; B. O. Koopman and J. von Neumann, *ibid.*, pp. 255-263. In these treatments a continuous set of transformations is considered, instead of the set of iterates of a single transformation as here, but the treatment needs no essential change to make it applicable to this case.

§ Cf. Daniell, loc. cit., p. 283.

|| The Stieltjes-Lebesgue integral is defined in the same way as the ordinary Lebesgue integral except that p_F -measure is used instead of ordinary Lebesgue measure.

By hypothesis there is a positive number M such that

$$(12) \quad \limsup_{n \rightarrow \infty} \frac{|f(x_n)|}{n} \leq M$$

on a set of points Λ of $\Omega(F)$, $P_F(\Lambda) > 0$. Let Λ_N be the point set on $\Omega(F)$ at which

$$\text{L. U. B. } \left\{ \frac{f(x_n)}{n} \right\}_{n \geq N} > M.*$$

Then $\Lambda_N \supset \Lambda_{N+1}$ and

$$\lim_{N \rightarrow \infty} P_F(\Lambda_N) = 1 - P_F(\Lambda) < 1.$$

Let E_n be the set of values of x at which $f(x) > nM$. Then a point $\omega: (\dots, x_0, \dots)$ belongs to the complement of Λ_N if and only if x_n is in the complement of E_n for $n \geq N$. Then the complement of Λ_N is of the form (2), so from (3),

$$(13) \quad P_F(\Lambda_N) = 1 - \sum_{n=N}^{\infty} [1 - p_F(E_n)].$$

Since $\lim_{N \rightarrow \infty} P_F(\Lambda_N) < 1$, the infinite product is convergent. Then $\sum_{n=0}^{\infty} p_F(E_n)$ must be convergent,[†] and it is easily shown from the definition of the Lebesgue-Stieltjes integral that this implies that $f(x)$ is integrable (with respect to $F(x)$) over the set E_0 . Substituting $-f(x)$ for $f(x)$, the proof shows that $f(x)$ is also integrable (with respect to $F(x)$) over the set where it is negative. Then $\int_{-\infty}^{\infty} f(x) dF(x)$ exists, as was to be proved.

The following theorem will be put in the phraseology of the theory of probability. Like the lemma, it is simply a theorem on integration on $\Omega(F)$.

THEOREM 3. Let x_1, x_2, \dots be a sequence of independent chance variables with the same distributions.

(i) If the expectation E of x_i exists, then

$$(14) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n x_j = E$$

with probability 1.

(ii) If there is a sequence of real numbers c_1, c_2, \dots such that the probability is positive that

* Throughout this paper, if a_1, a_2, \dots is a sequence of real numbers, L.U.B. $\{a_n\}$ will denote its least upper bound.

† W. F. Osgood, *Lehrbuch der Funktionentheorie*, vol. 1, 4th edition, p. 528.

$$(15) \quad \limsup_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{j=1}^n x_j - c_n \right| < \infty,$$

it follows that the expectation of x_j exists, and we have Case (i) again.*

Let $F(x)$ be the probability that $x_j < x$. If the expectation of x_j exists, it is, by (4),

$$\int_{\Omega(F)} x_j(\omega) d\omega = \int_{-\infty}^{\infty} x dF(x)$$

where ω is the point (\dots, x_0, \dots) .

(i) The first part of the theorem is simply the Corollary of Theorem 2 applied to the function $\phi(\omega) = x_0(\omega)$.

(ii) We can suppose in (ii) that there is a point set Λ on $\Omega(F)$ of positive P_F -measure, a positive number M and an integer N such that

$$(16) \quad \left| \frac{1}{n} \sum_{j=1}^n x_j(\omega) - c_n \right| < M$$

on Λ if $n \geq N$. On replacing n by $n-1$ and multiplying by $(n-1)/n$,

$$(17) \quad \left| \frac{1}{n} \sum_{j=1}^{n-1} x_j(\omega) - \frac{n-1}{n} c_{n-1} \right| < M$$

on Λ if $n \geq N+1$. Subtracting (17) from (16),

$$(18) \quad \left| \frac{x_n(\omega)}{n} - \left(c_n - \frac{n-1}{n} c_{n-1} \right) \right| < 2M$$

on Λ if $n \geq N+1$. By (1), $x_n(\omega)/n$ approaches 0 in measure as n becomes infinite. Then there is an integer $N_1 \geq N+1$ such that on a subset Λ_n of Λ of positive P_F -measure

$$|x_n(\omega)/n| < M \text{ if } n \geq N_1.$$

Hence

$$(19) \quad |c_n - c_{n-1}(n-1)/n| < 3M.$$

From (18) and (19),

$$|x_n(\omega)/n| < 5M$$

on Λ . The lemma can now be applied, and it shows that $\int_{-\infty}^{\infty} x dF(x)$ exists as a Stieltjes-Lebesgue integral. This integral is the expectation of the chance variable x_j .

* A. Kolmogoroff, *Ergebnisse der Mathematik*, vol. 2, No. 3: *Grundbegriffe der Wahrscheinlichkeitsrechnung*, p. 59, announced the first part of this theorem, and also the second part, under the assumption that the probability is 1 that the upper limit in (15) is 0.

The following theorem will be needed in the application of the results of this section. Its proof is simple and will be omitted.

THEOREM 4. *If $F(x)$ is defined as in Theorem 1, and if $F(x)$ has an integrable derivative $f(x)$:*

$$F(x) = \int_{-\infty}^x f(x)dx,$$

there is a point set $\Lambda(F)$ on $\Omega(F)$, $P_F[\Lambda(F)] = 1$, with the following property. If $g(x)$ is any function defined and continuous almost everywhere (in the sense of p_F -measure) on the infinite interval $-\infty < x < \infty$, and such that $\int_{-\infty}^{\infty} g(x)f(x)dx$ exists, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n g(x_j) = \int_{-\infty}^{\infty} g(x)f(x)dx$$

*at every point $\omega: (\dots, x_0, \dots)$ of $\Lambda(F)$.**

2. THE METHOD OF MAXIMUM LIKELIHOOD

For each value of p in some point set E let $f(x, p)$ be a probability density over the interval $-\infty < x < \infty$.† Assume that the chance variable x has a probability distribution whose density is $f(x, p)$ for some (unknown) value of p in E . Then an important problem in statistics is that of estimating the true value of p by means of large samples of values of x , obtained independently. This is done by the method of maximum likelihood of R. A. Fisher‡, which has supplanted the use of Bayes' theorem. If x_1, \dots, x_n is a sample of values of x , and if $f(x, p)$ is the probability density of the distribution of values of x , the probability of obtaining a sample of values x'_1, \dots, x'_n where x'_j is in a small interval with midpoint x_j , is, in the limit, proportional to $\prod_{j=1}^n f(x_j, p)$. The method of maximum likelihood takes as an approximation to p_0 , the true value of p , the value p_n of p (or one of them if there are several) which makes this product a maximum. If p_n approaches p_0 in probability as the samples become larger, p_n is called a consistent estimate of p . A

* The theorem will be needed as here stated. It can be stated in terms of Riemann-Stieltjes integration, making unnecessary any restrictions on $F(x)$.

† This means that $f(x, p) \geq 0$, that $f(x, p)$ is defined for almost all values of x , is measurable and integrable over the x -axis, and that $\int_{-\infty}^{\infty} f(x)dx = 1$. It is supposed that there is a chance variable $x(p)$ whose values are distributed in such a way that the probability of $x(p)$ being in any measurable point set A is $\int_A f(x)dx$.

‡ Philosophical Transactions of the Royal Society of London, (A), vol. 222, pp. 309-368, especially pp. 309-330. The proofs given by Fisher and by H. Hotelling, these Transactions, vol. 32 (1930), pp. 847-859, of the validity of the method of maximum likelihood (in the sense that theorems similar to the ones to be proved in this section hold) are not rigorous.

rigorous proof will be given in this section that, under certain hypotheses, the method of maximum likelihood furnishes consistent estimates.

THEOREM 5. For each value of p in a point set E let $f(x, p)$ be a probability density on the infinite interval $-\infty < x < \infty$. Let x be a chance variable whose distribution is determined by the probability density $f(x)$, and suppose that for each set of numbers $x_1, \dots, x_n, n=1, 2, \dots$, it is possible to find a value of p in $E: p = p_n(x_1, \dots, x_n)$ such that

$$(20) \quad \prod_{i=1}^n f(x_i, p_n) \geq \prod_{i=1}^n f(x_i).$$

Then if

$$F(x) = \int_{-\infty}^x f(x) dx,$$

there is a set of points Λ of $\Omega(F)$ of total probability 1: $P_F(\Lambda) = 1$, with the following properties. Let $\omega: (\dots, x_0, \dots)$ be a point of Λ and let $\{p_{a_n}(x_1, \dots, x_n)\}$ be any subsequence of $\{p_n(x_1, \dots, x_n)\}$ for $x_j = x_j(\omega), j=1, 2, \dots$. Set

$$(21) \quad f_n(x) = \text{L.U.B.}_{n \geq n} \{f(x, p_{a_n})\}.$$

Suppose that $f_n(x)/f(x)$ is continuous, except possibly for a set of values of x of zero probability*, and that

$$(22) \quad \int_{-\infty}^{\infty} f(x) \log^+ \left[\frac{f_n(x)}{f(x)} \right] dx \dagger$$

exists. It follows

(i) that the integral

$$(23) \quad \int_{-\infty}^{\infty} f(x) \log \left[\frac{\limsup_{n \rightarrow \infty} f(x, p_{a_n})}{f(x)} \right] dx$$

exists and is not negative;

(ii) that if $\limsup_{n \rightarrow \infty} f(x, p_{a_n})$ is integrable, and if

$$(24) \quad \int_{-\infty}^{\infty} \limsup_{n \rightarrow \infty} f(x, p_{a_n}) dx \leq 1,$$

then $\limsup_{n \rightarrow \infty} f(x, p_{a_n}) = f(x)$ except possibly on a set of zero probability;

(iii) that if the sequence $\{f(x, p_{a_n})\}$ converges (except possibly on a set of 0 probability), the limit function is $f(x)$ (except possibly on a set of 0 probability).

* This means that the integral of $f(x)$ over the exceptional set is 0, i.e., that $f(x)=0$ almost everywhere (in the sense of Lebesgue measure) on the set. In the following integrals, in which ratios with $f(x)$ in the denominator appear, we define the ratios as 1 when $f(x)=0$.

† If $\xi \geq 0$, $\log^+ \xi$ is defined as $\log \xi$ when $\xi > 1$, and 0 otherwise.

In the application to statistical problems, it is part (iii) which would be customarily used. Thus, consider the problem of estimating the mean of a normal distribution, where the density is

$$(25) \quad f(x, p) = \frac{1}{(2\pi)^{1/2}} e^{-(x-p)^2/2},$$

the true value of p being p_0 . In this case if p_{a_n} approaches any finite value, it is seen at once that (22) exists. Since $f(x, p)$ is continuous in p , (iii) shows that $f(x, p_{a_n})$ approaches $f(x, p_0)$, so that p_{a_n} converges to p_0 , the true value. On the other hand, suppose that p_{a_n} converges to either $+\infty$ or $-\infty$. Then the integral (22) exists. By (iii), $f(x, p_{a_n})$, which converges to 0 (since $|p_{a_n}| \rightarrow \infty$), approaches $f(x, p_0)$. This is impossible, so $\lim_{n \rightarrow \infty} p_n = p_0$, with probability 1. It is usual to take for the approximation p_n the average $(1/n) \sum_{j=1}^n x_j$.

It is evident that if Λ_0 is the set of points $\omega: (\dots, x_0, \dots)$ of $\Omega(F)$ such that at least one coordinate x_j is in the set of values of x at which $f(x) = 0$, $P_F(\Lambda_0) = 0$. It will be shown that the set Λ of this theorem can be taken as the set $\Lambda(F) - \Lambda_0 \cap \Lambda(F)$, where $\Lambda(F)$ was described in Theorem 4. Suppose then that $\omega: (\dots, x_0, \dots)$ is in this set.

(i) From (20) and (21), if $L_t(y)$ is defined for every positive number t as $\log y$ if $y \geq t$, and as $\log t$ if $y < t$,

$$(26) \quad \frac{1}{a_n} \sum_{j=1}^{a_n} L_t \left[\frac{f_N(x_j)}{f(x_j)} \right] \geq \frac{1}{a_n} \sum_{j=1}^{a_n} \log \left[\frac{f_N(x_j)}{f(x_j)} \right] \geq \frac{1}{a_n} \sum_{j=1}^{a_n} \log \left[\frac{f(x_j, p_{a_n})}{f(x_j)} \right] \geq 0,$$

if $n \geq N$. Now since $f \log^+ (f_N/f)$ is integrable, $f L_t(f_N/f)$ is integrable (over the entire x -axis). Then letting n become infinite in (26), we have, from Theorem 4,

$$(27) \quad \int_{-\infty}^{\infty} f(x) L_t \left[\frac{f_N(x)}{f(x)} \right] dx \geq 0.$$

As N increases, $L_t(f_N/f)$ does not increase, and

$$\lim_{n \rightarrow \infty} L_t(f_N/f) = L_t(\hat{f}/f),$$

where

$$\hat{f}(x) = \limsup_{n \rightarrow \infty} f(x, p_{a_n}).$$

Then we can go to the limit under the integral sign in (27)*, obtaining

$$(28) \quad \int_{-\infty}^{\infty} f(x) L_t \left[\frac{\hat{f}(x)}{f(x)} \right] dx \geq 0.$$

Let E_t be the set of values of x at which $f(x) \leq t$. The integral (28) can be separated into integrals over E_t and its complement, CE_t . Doing this, we find that

$$(29) \quad 0 \leq p_F(E_t) \log \frac{1}{t} \leq \int_{CE_t} f(x) L_t \left[\frac{\hat{f}(x)}{f(x)} \right] dx \text{ if } t \leq 1.$$

Letting t approach 0, (29) shows that $p_F(E_0) = 0$ and that furthermore the integral (23) exists and is not negative.

(ii) From (i),

$$(30) \quad 0 \leq \int_{-\infty}^{\infty} f(x) \log \left[\frac{\hat{f}(x)}{f(x)} \right] dx.$$

Now by a well known inequality†, and using (24),

$$(31) \quad \int_{-\infty}^{\infty} f(x) \log \left[\frac{\hat{f}(x)}{f(x)} \right] dx \leq \log \int_{-\infty}^{\infty} \hat{f}(x) dx \leq 0.$$

There is equality in (31) only when $\hat{f}(x) = f(x)$ for almost all x (in the sense of p_F -measure), and there is necessarily equality, by (30), so (ii) is proved.

(iii) To prove (iii) it is only necessary to reduce it to (ii), by showing that, if

$$\hat{f}(x) = \lim_{n \rightarrow \infty} f(x, p_{a_n}),$$

$\int_{-\infty}^{\infty} \hat{f}(x) dx$ exists and is not greater than 1. We have

$$\int_{-\infty}^{\infty} f(x, p_{a_n}) dx = 1,$$

so by Fatou's lemma‡, $f(x)$ is integrable over $-\infty < x < \infty$ and $\int_{-\infty}^{\infty} f(x) dx \leq 1$.

* The situation is visualized more readily when the integral is written as

$$\int_{-\infty}^{\infty} L_t \left[\frac{f_N(x)}{f(x)} \right] dF(x).$$

The integrand is bounded uniformly above by the integrable function $L_t(f_1/f)$ and below by $\log t$, so we can integrate term by term.

† Making the substitution $y = F(x)$, the inequality needed becomes

$$\int_0^1 \log g(y) dy \leq \log \int_0^1 g(y) dy,$$

where $g(y) = \hat{f}(x)/f(x)$.

‡ P. Fatou, Acta Mathematica, vol. 30 (1906), pp. 375-376.

The treatment of the principle of maximum likelihood given above was for continuous distributions. The most general statement of the other extreme is as follows. To each integer $n \geq 1$ is assigned a probability $a(n, p)$ depending on p which varies on some point set. The intrinsic conditions are

$$a(n, p) \geq 0, \quad \sum_{n=1}^{\infty} a(n, p) = 1.$$

For each sample of integers r_1, \dots, r_n there is a value p_n of p such that

$$\prod_{j=1}^n a(r_j, p_n) \geq \prod_{j=1}^n a(r_j, p_0),$$

where p_0 is the true value of p . The problem is to show (under suitable restrictions on $a(n, p)$), that p_n approaches p_0 in probability. This problem can be treated in a similar manner to the one just treated.

The method of maximum likelihood, when analyzed more carefully, yields further information. Reverting to continuous distributions, suppose that for each value of p in a neighborhood of p_0 , $f(x, p)$ is the density of a probability distribution. The function $p_n(x_1, \dots, x_n)$ will be called an n th approximation of maximum likelihood to p_0 if it is defined on $\Omega(F)$ (where $F(x) = \int_{-\infty}^x f(x, p_0) dx$) on a set of P_F -measure 1, if

$$\prod_{j=1}^n f(x_j, p_n) \geq \prod_{j=1}^n f(x_j, p_0)$$

and if $\prod_{j=1}^n f(x_j, p)$ for fixed x_1, \dots, x_n has a relative maximum at $p = p_n$. It is no restriction to assume that $p_0 = 0$.

THEOREM 6. *For each value of p in some neighborhood $|p| \leq a_1$, $a_1 > 0$, of $p = 0$, let $f(x, p)$ be a probability density in the infinite interval $-\infty < x < \infty$. Let the true distribution of x be determined by the probability density $f(x, 0)$. Suppose*

(i) *that $\log f(x, p)$ can be expressed in the form*

$$(32) \quad \log f(x, p) = \log f(x, 0) + p\alpha(x) + \frac{p^2}{2}\beta(x) + \gamma(x, p),^*$$

where $\alpha(x)f(x, 0)$, $\alpha(x)^2f(x, 0)$, $\beta(x)f(x, 0)$ are Lebesgue measurable and integrable over $-\infty < x < \infty$ and where

* We shall assume in the discussion of this theorem that x does not take on any value at which $f(x, 0) = 0$. This means leaving out sets of total probability 0 on the x -axis and on $\Omega(F)$, where

$$F(x) = \int_{-\infty}^x f(x, 0) dx.$$

$$\frac{\partial}{\partial p} \gamma(x, p) = \gamma_p(x, p)$$

exists for $|p| \leq a_2 \leq a_1, a_2 > 0$, and is continuous at $p=0$;

(ii) that if

$$(33) \quad \phi(x) = \text{L. U. B.}_{0 < |p| \leq a_2} \left\{ \frac{|\gamma_p(x, p)|}{p^2} \right\}$$

then $\phi(x)f(x, 0)$ is integrable over $-\infty < x < \infty$ *;

(iii) that if $\delta(x, p)$ is defined by

$$(34) \quad f(x, p) = f(x, 0) \left\{ 1 + p\alpha(x) + \frac{p^2}{2} [\beta(x) + \alpha(x)^2] + \delta(x, p) \right\},$$

$$(35) \quad \lim_{p \rightarrow 0} \frac{1}{p^2} \int_{-\infty}^{\infty} \delta(x, p) f(x, 0) dx = 0. \dagger$$

Then

$$(36) \quad \int_{-\infty}^{\infty} \alpha(x)^2 f(x, 0) dx + \int_{-\infty}^{\infty} \beta(x) f(x, 0) dx = 0.$$

Suppose that

$$\sigma^2 = \int_{-\infty}^{\infty} \alpha(x)^2 f(x, 0) dx > 0.$$

Then if $p_n(x_1, \dots, x_n)$ is an n th approximation of maximum likelihood to $p=0$, and if p_n approaches 0 in probability:

$$(37) \quad \lim_{n \rightarrow \infty} \bar{P}_F(|p_n| > \epsilon) = 0 \ddagger$$

for every $\epsilon > 0$,

$$(38) \quad \lim_{n \rightarrow \infty} \bar{P}_F(\sigma n^{1/2} p_n < \lambda) = \lim_{n \rightarrow \infty} \underline{P}_F(\sigma n^{1/2} p_n < \lambda) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\lambda} e^{-z^2/2} dz,$$

for every constant λ , uniformly in λ .

* We take this to mean that $\int_{-\infty}^{\infty} \phi(x) dF(x)$ exists so that $\phi(x)$ can be $+\infty$ on a set of zero p_F -measure.

† Since $f(x, p)$ is integrable over $-\infty < x < \infty$, it follows from (i) that $\delta(x, p)f(x, 0)$ is also.

‡ Such expressions will be taken to mean the probability that $|p_n| > \epsilon$ (i.e. the P_F -measure of the set of those points on $\Omega(F)$ where $|p_n| > \epsilon$), etc. In (37) we use \bar{P}_F , the outer measure on $\Omega(F)$, instead of P_F , since we have not assumed that $p_n(x_1, \dots, x_n)$ is measurable with respect to $F(x)$. Similarly, \underline{P}_F will denote the inner measure on $\Omega(F)$.

The theorem states simply that, under suitable restrictions on the character of $f(x, p)$ in p , p_n will be normal for large n , with variance $1/(\sigma^2 n)$.*

Since

$$(39) \quad \int_{-\infty}^{\infty} f(x, p) dx = 1$$

for all p in the neighborhood considered,

$$(40) \quad p \int_{-\infty}^{\infty} \alpha(x) f(x, 0) dx + \frac{p^2}{2} \int_{-\infty}^{\infty} [\beta(x) + \alpha(x)^2] f(x, 0) dx \\ + \int_{-\infty}^{\infty} \delta(x, p) f(x, 0) dx = 0.$$

Dividing through by p and letting p approach 0, we find that, in view of (35),

$$(41) \quad \int_{-\infty}^{\infty} \alpha(x) f(x, 0) dx = 0.$$

Dividing (40) through by p^2 and letting p approach 0, we find in view of (35), that (36) is true.

The logarithm of the likelihood of a value of p , obtained from n trials, is defined as

$$(42) \quad L_n(p) = \sum_{j=1}^n \log f(x_j, p) = \sum_{j=1}^n \log f(x_j, 0) \\ + p \sum_{j=1}^n \alpha(x_j) + \frac{p^2}{2} \sum_{j=1}^n \beta(x_j) + \sum_{j=1}^n \gamma(x_j).$$

Since $L_n(p)$ has a relative maximum at p_n ,

$$(43) \quad L'_n(p_n) = \sum_{j=1}^n \alpha(x_j) + p_n \sum_{j=1}^n \beta(x_j) + \sum_{j=1}^n \gamma_p(x_j, p_n) = 0,$$

if we suppose that $|p_n| < a_2$.

(A) If $p_n = 0$, $\sum_{j=1}^n \alpha(x_j) = 0$ also, excluding possibly a set of zero probability on $\Omega(F)$. For if $p_n = 0$, (43) becomes $\sum_{j=1}^n \alpha(x_j) = 0$ (if a set of zero probability on $\Omega(F)$ is ignored), since the hypotheses of the theorem imply that $\gamma_p(x, 0) = 0$ on a set of p_F -measure 1 on the x -axis.

(B) Let m be defined by

$$(44) \quad m = \int_{-\infty}^{\infty} \phi(x) f(x, 0) dx.$$

* R. A. Fisher, loc. cit., p. 359.

H. Hotelling, loc. cit., pp. 836-858. Through an oversight, this theorem is stated, on p. 850, with the variance of p_n as $\sigma^2 n$.

Then

$$(45) \quad P_F \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \phi(x_j) = m \right\} = 1$$

by the Corollary to Theorem 2, or by Theorem 3, so that

$$(46) \quad \lim_{n \rightarrow \infty} \bar{P}_F \left\{ \frac{|\hat{p}_n|}{n} \sum_{j=1}^n \phi(x_j) \geq \epsilon \right\} = 0$$

for any $\epsilon > 0$.*

$$(C) \quad P_F \left\{ \lim_{n \rightarrow \infty} \frac{-1}{\sigma^2 n} \sum_{j=1}^n \beta(x_j) = 1 \right\} = 1,$$

by the Corollary to Theorem 2 or by Theorem 3.

Now from (43), if $0 < |\hat{p}_n| \leq a_2$ and if the denominator does not vanish,

$$(47) \quad n^{1/2} \sigma \hat{p}_n = \frac{1}{\sigma n^{1/2}} \sum_{j=1}^n \alpha(x_j) + R_n,$$

where

$$(48) \quad R_n = \frac{\frac{1}{\sigma n^{1/2}} \sum_{j=1}^n \alpha(x_j) \left\{ 1 + \frac{1}{\sigma^2 n} \sum_{j=1}^n \beta(x_j) + \frac{1}{\sigma^2 n \hat{p}_n} \sum_{j=1}^n \gamma_p(x_j, \hat{p}_n) \right\}}{-\frac{1}{\sigma^2 n} \sum_{j=1}^n \beta(x_j) - \frac{1}{\sigma^2 n \hat{p}_n} \sum_{j=1}^n \gamma_p(x_j, \hat{p}_n)}.$$

We define R_n as 0 if $\hat{p}_n = 0$.

Using (A), (B), (C), we shall show that

$$\lim_{n \rightarrow \infty} \bar{P}_F(|R_n| > \epsilon) = 0$$

for every $\epsilon > 0$. Since

$$\left| \frac{1}{n \hat{p}_n} \sum_{j=1}^n \gamma_p(x_j, \hat{p}_n) \right| \leq \frac{|\hat{p}_n|}{n} \sum_{j=1}^n \phi(x_j),$$

and since by the Laplace-Liapounoff theorem†

* Equation (45) expresses the fact that a certain sequence $\{h_n\}$ of functions on $\Omega(F)$ converges to m almost everywhere on $\Omega(F)$. Then since the sequence $\{|\hat{p}_n|\}$ converges in measure to 0 on $\Omega(F)$, by hypothesis, the sequence $\{|\hat{p}_n| h_n\}$ converges in measure to 0 on $\Omega(F)$, which fact is expressed by (46).

† A. Khintchine, *Ergebnisse der Mathematik*, vol. 2, No. 4: *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, pp. 1-8.

$$(49) \quad \lim_{n \rightarrow \infty} P_F \left\{ \frac{1}{\sigma n^{1/2}} \sum_{j=1}^n \alpha(x_j) \leq \lambda \right\} = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\lambda} e^{-x^2/2} dx,$$

the numerator of R_n converges in measure to 0 and the denominator to 1 as n becomes infinite. Then R_n converges in measure to 0 on $\Omega(F)$ as n becomes infinite:

$$(50) \quad \lim_{n \rightarrow \infty} \bar{P} \left\{ \left| \sigma n^{1/2} p_n - \frac{1}{\sigma n^{1/2}} \sum_{j=1}^n \alpha(x_j) \right| \geq \epsilon \right\} = 0$$

for every $\epsilon > 0$. Now suppose that $\sigma n^{1/2} p_n < \lambda$ on the set E_n on $\Omega(F)$. Fix $\epsilon > 0$ and suppose that the difference in (50) is less than ϵ on the set

$$F_n: \quad \lim_{n \rightarrow \infty} \underline{P}_F(F_n) = 1.$$

Then the points of $\Omega(F)$ where $\sigma n^{1/2} p_n < \lambda$ for any constant λ are included in the points of the complement of F_n or in the points common to F_n and the set on which

$$\frac{1}{\sigma n^{1/2}} \sum_{j=1}^n \alpha(x_j) < \lambda + \epsilon.$$

The points of $\Omega(F)$ where $\sigma n^{1/2} p_n < \lambda$ include the points where

$$\frac{1}{\sigma n^{1/2}} \sum_{j=1}^n \alpha(x_j) < \lambda - \epsilon$$

which also belong to F_n . These considerations show that (38) is true, since (49) is uniform in λ .

Theorem 6 requires a slight modification if the parameter p is replaced by several parameters, $p^{(1)}, \dots, p^{(r)}$. Theorem 5 evidently needs no essential change in this case. In Theorem 6 we replace (32) by

$$(32') \quad \begin{aligned} \log f(x, p^{(1)}, \dots, p^{(r)}) &= \log f(x, p) \\ &= \log f(x, 0) + \sum_{i=1}^r p^{(i)} \alpha_i(x) + \frac{1}{2} \sum_{i,k=1}^r p^{(i)} p^{(k)} \beta_{ik}(x) + \gamma(x, p), \\ \beta_{ik}(x) &= \beta_{ki}(x), \end{aligned}$$

where we take the true set of parameters as $(0, \dots, 0)$, and where we suppose that the first partial derivatives of $\gamma(x, p^{(1)}, \dots, p^{(r)})$ exist in a neighborhood of the origin in the r -dimensional p -space, and are continuous at the origin. Conditions (ii) and (iii) are modified in an obvious way, and (36) becomes

$$(36') \quad \int_{-\infty}^{\infty} \alpha_i(x) \alpha_k(x) f(x, 0) dx + \int_{-\infty}^{\infty} \beta_{ik}(x) f(x, 0) dx = 0,$$

proved as before. If we set

$$\sigma_{ik} = \int_{-\infty}^{\infty} \alpha_i(x) \alpha_k(x) f(x, 0) dx,$$

the theorem states that the joint distribution of $p_n^{(1)}, \dots, p_n^{(r)}$, the n th approximation of maximum likelihood, approaches normality, where the matrix of the variances and covariances of the $p_n^{(i)}$ becomes $1/n$ times the inverse matrix of $\|\sigma_{ik}\|$, which we assume non-singular. The proof will be sketched briefly. The theorem is stated in a way invariant under non-singular linear transformations of $p^{(1)}, \dots, p^{(r)}$. We can assume that a linear transformation has been performed already, if necessary, reducing the positive definite quadratic form

$$(51) \quad - \sum_{i,j=1}^r p^{(i)} p^{(j)} \int_{-\infty}^{\infty} \beta_{ij}(x) f(x, 0) dx = \int_{-\infty}^{\infty} \left[\sum_{i=1}^r p^{(i)} \alpha_i(x) \right]^2 f(x, 0) dx$$

to canonical form, so that

$$(52) \quad - \int_{-\infty}^{\infty} \beta_{ij}(x) f(x, 0) dx = \delta_{ij}$$

where δ_{ij} is the usual Kronecker delta. Equation (43) becomes

$$(43') \quad \left. \frac{\partial L_n}{\partial p^{(k)}} \right|_{p_n^{(k)}} = \sum_{j=1}^n \alpha_k(x_j) + \sum_{i=1}^r \sum_{j=1}^n p^{(i)} \beta_{ki}(x_j) + \gamma_{p^{(k)}}(x, p_n) = 0,$$

and (47) becomes

$$(47') \quad (\sigma_{ii} n)^{1/2} p_n^{(i)} = \frac{1}{(\sigma_{ii} n)^{1/2}} \sum_{j=1}^n \alpha_i(x_j) + R_n^{(i)}.$$

It is shown as before that R_n approaches 0 in probability as n becomes infinite. For large n , the estimates $p_n^{(i)}$ are then distributed nearly normally, with variances and covariances obtained from $1/n$ times the inverse of the matrix $\|\sigma_{ik}\|$.

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METABELIAN GROUPS OF ORDER p^{n+m} WITH COMMUTATOR SUBGROUPS OF ORDER p^{m*}

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INTRODUCTION

We consider a metabelian group G obtained by extending an abelian group H of order p^n and type $1, 1, \dots$ by means of m operators U_1, \dots, U_m of order p from a Sylow subgroup of its group of isomorphisms. Every operator of $U = \{U_1, \dots, U_m\}$ determines a partition of n and the fact that G is metabelian is equivalent to the requirement that no operator of U determine a partition of n in which the greatest term is greater than 2.† We require further that H be a maximal invariant abelian subgroup of G ; this implies that no operator, except identity, in U determines a partition of n with greatest term smaller than 2. Throughout the first four sections we shall require that every operator of U , except identity, determine the partition $n = 2 + 2 + 1 + \dots + 1$. Such groups for $m = 3$ as well as the groups such that every operator of U , except identity, determines the partition $n = 2 + 1 + \dots + 1$ have been classified.‡ In the case $m = 3$ we found that there is but one group satisfying the conditions which we impose here, but the considerations necessary to show it indicated that extremely interesting results were to be found for larger values of m .

In §1 we suppose generators of G to satisfy a set of relations of a special type and are able to show that the problem of the classification of the resulting groups is exactly the problem of the classification of polynomials of degree m in a single variable x with coefficients in the modular field, mod p , under the group of projective transformations on x with coefficients also in the modular field. This is applied in §2 to the groups for $m = 4$, where some obvious properties of the groups suggest a further analysis of the relation between polynomial and group. In particular it becomes apparent that the polynomial is in most cases independent of the special form of the generating relations used in §1; also there appears a group which belongs in the class but has no set of generators satisfying these special relations. In §3 it is shown that the classification of the groups with central of order p^{n-2} is equiva-

* Presented to the Society, April 6, 1934; received by the editors March 10, 1934.

† *On isomorphisms of abelian groups of type 1, 1, \dots*, American Journal of Mathematics, vol. 56 (1934), p. 53.

‡ *On metabelian groups*, offered to the American Journal of Mathematics.

lent to the classification of matrices $M + xN$, where M and N are m -rowed square matrices with elements in the modular field, under "rational" projective transformations on x and "rational" elementary transformations on M and N simultaneously. This is extended in §5 to show the equivalence of the theory of groups with centrals of order p^{n-k} with the theory of matrices $x_1M_1 + \dots + x_kM_k$ under the same set of transformations. In §4 the invariant factors of $M + xN$ are used to discover some of the properties of the groups.

1. A SPECIAL CASE

Let the generators of H be s_1, s_2, \dots, s_n . Let the generators of $G = \{H, U\}$ satisfy the following relations and no others except such as are consequences of these:

$$(1) \quad \begin{aligned} U_1^{-1} s_1 U_1 &= s_1 s_4, & U_2^{-1} s_1 U_2 &= s_1 s_5, & U_{m-1}^{-1} s_1 U_{m-1} &= s_1 s_{m+2}, \\ U_1^{-1} s_2 U_1 &= s_2 s_3, & U_2^{-1} s_2 U_2 &= s_2 s_4, & \dots, & U_{m-1}^{-1} s_2 U_{m-1} &= s_2 s_{m+1}, \\ U_m^{-1} s_1 U_m &= s_1 s_3 s_4 \dots s_{m+2}, \\ U_m^{-1} s_2 U_m &= s_2 s_{m+2}. \end{aligned}$$

The central of G is obviously of order p^{n-2} , being generated by s_3, s_4, \dots, s_n , and the commutator subgroup is of order p^m , being generated by s_3, s_4, \dots, s_{m+2} . If U contained an operator of type I, i.e. one which determines the partition $n = 2 + 1 + \dots + 1$, then $\{s_1, s_2\}$ would contain an operator $s_1 s_2^x$ permutable with some operator of U .^{*} Any operator of U may be written $U' = U_1^{k_1} U_2^{k_2} \dots U_m^{k_m}$. Transforming $s_1 s_2^x$ by U' we have

$$U'^{-1} s_1 s_2^x U' = s_1 s_2^x (s_4 s_3)^{x k_1} (s_5 s_4)^{x k_2} \dots (s_{m+2} s_{m+1})^{x k_{m-1}} (s_3 s_4 \dots s_{m+2})^{a_m x k_m}.$$

The commutator in the above must be identity and we thereby obtain the following system of congruences linear and homogeneous in the k 's,

$$\begin{aligned} x k_1 &+ a_1 k_m \equiv 0, \\ k_1 + x k_2 &+ a_2 k_m \equiv 0, \\ k_2 + x k_3 &+ a_3 k_m \equiv 0, \\ \dots &\dots \dots \\ k_{m-2} + x k_{m-1} + a_{m-1} k_m &\equiv 0, \\ k_{m-1} + (x + a_m) k_m &\equiv 0. \end{aligned}$$

^{*} Cf. the last reference where the question is considered for $m = 3$.

The condition that the system have a solution is that the determinant of the matrix of coefficients be zero, which is

$$(2) \quad x^m + a_m x^{m-1} - a_{m-1} x^{m-2} + a_{m-2} x^{m-3} - \dots + (-1)^{m-1} a_1 \equiv 0, \text{ mod } p.$$

This condition will not be satisfied by any x if the polynomial in (2) contains no linear factor in the modular field. Since there exist irreducible congruences of any degree it follows that a_1, a_2, \dots, a_m in (1) may be chosen so that U contains no operator of type I.

A different choice of generators s_1, s_2, \dots, s_n and U_1, U_2, \dots, U_m in the group G would be expected to result in a different congruence (2), it being understood that the new generators satisfy relations similar to (1) in that the commutator of U_i and s_2 is the same as that of U_{i-1} and $s_1, i=2, 3, \dots, m$. We undertake to show first that if s_1 and s_2 are left unchanged and U_1, U_2, \dots, U_m are changed to any set which satisfy a set of relations similar to (1) then the congruence (2) remains unchanged.

It is evident that, since U is of order p^m and the commutator subgroups arising from transformation of s_1 and s_2 by U are both of order p^m , the choice of U_1 determines all the U 's thereafter. We shall prove our statement by proving that (2) is unchanged

(a) by replacing U_1 by U_1^k ,

(b) by replacing U_1 by U_2 ,

(c) by replacing U_1 by the product of U_i and U_j each of which gives the original congruence (2) when used for U_1 .

If we replace U_1 by $U_1' = U_1^k$ the operators U_2', U_3', \dots, U_m' may be determined successively and it is obvious that they are $U_i' = U_i^k$. Consequently the commutator of U_m' and s_1 is

$$s_3^{a_1 k} s_4^{a_2 k} \dots s_{m+2}^{a_m k}.$$

This operator expressed in terms of the preceding commutators is

$$s_3^{k a_1} s_4^{k a_2} \dots s_{m+2}^{k a_m}.$$

This proves the statement for the case (a).

If we replace U_1 by $U_1' = U_2$, we have $U_i' = U_{i+1}, i=1, 2, \dots, m-1$. The commutator of $U_{m-1}' = U_m$ and s_1 is

$$(3) \quad s_3^{a_1} s_4^{a_2} \dots s_{m+2}^{a_m}$$

and therefore we must have $U_m' = U_1^{a_1} U_2^{a_2} \dots U_m^{a_m}$. Then the commutator of U_m' and s_1 is

$$(4) \quad s_3^{a_1} s_4^{a_2} \dots s_{m+2}^{a_m} (s_3^{a_1} s_4^{a_2} \dots s_{m+2}^{a_m})^{a_m} = s_3^{a_1 a_m} s_4^{a_2 a_m} s_5^{a_3 a_m} \dots s_{m+2}^{a_{m-1} a_m + a_m^2}.$$

This may be expressed in terms of the preceding commutators $s'_{i-1} = s_i$, $i \leq m+2$ and s'_{m+2} as given by (3). Or, if we evaluate

$$s_3^{a_1} s_4^{a_2} \cdots s_{m+2}^{a_m}$$

in terms of s_3, s_4, \dots, s_{m+2} , we have (4). This proves our statement for (b).

From the above facts it follows that (2) remains unchanged if U_1 is replaced by $U'_1 = U_i^{k_i}$. Since the commutators are all permutable among themselves it follows that if U_i and U_j are such that each leaves (2) unchanged when used for U_1 in (1), then the product $U'_1 = U_i U_j$ used for U_1 will determine a set U'_2, U'_3, \dots, U'_m and a set of commutators $s'_3, s'_4, \dots, s'_{m+2}$ such that the commutator of U'_m and s_1 will be expressible as

$$s_3^{a'_1} s_4^{a'_2} \cdots s_{m+2}^{a'_m}.$$

This last relation holds regardless of whether or not the operators U'_1, U'_2, \dots, U'_m are independent, though we are interested only in the case where they are independent, as otherwise the U'' 's would not serve with H to generate G . This restriction is contained in (c). As a result of these considerations we have

(5) *The congruence (2) determined by a set of generators $s_1, s_2, \dots, s_n, U_1, U_2, \dots, U_m$ of G is independent of the choice of the U 's provided they are chosen from $\{U_1, U_2, \dots, U_m\}$ and the commutators satisfy the relation*

$$U_{i-1}^{-1} s_1 U_{i-1} s_1^{-1} = U_i^{-1} s_2 U_i s_2^{-1}, \quad i = 2, 3, \dots, m.$$

We consider next the effect of a new choice of generators of H . An essential of the relations (1) is that but two of the generators of H are outside the central of G , and since the congruence (2) depends on the form (1) it is obvious that a change in generators of H must be a change to a set of which but two are outside the central of G . If s_1 and s_2 are left fixed and s'_3, s'_4, \dots, s'_n are chosen from $\{s_3, s_4, \dots, s_n\}$ the change amounts to a renaming of operators in the central and the commutator subgroup and does not affect the relations connecting operators. It is on these relations that (2) depends. Moreover, a choice

$$s'_1 = s_1 \prod_{i=3}^n s_i^{k_i}$$

and

$$s'_2 = s_2 \prod_{j=3}^n s_j^{k_j}$$

has no effect on (2), since the commutator of U_i and s_j' is the same as that of U_i and s_j . Consequently we need consider only the effect of a choice of s_1' and s_2' from the group $\{s_1, s_2\}$. We proceed to prove the following theorem:

(6) If $s_1' = s_1^a s_2^b$ and $s_2' = s_1^c s_2^d$ where s_1, \dots, U_m satisfy (1) and determine the congruence (2), then $s_1', s_2', s_3, \dots, U_m$ satisfy a set of relations similar to (1) and determine a congruence which is obtained from (2) by subjecting x to the transformation $x = (ax' + b)/(cx' + d)$.

To prove the theorem we need consider only the special cases

- (a) $s_1' = s_1^a, \quad s_2' = s_2, \text{ and } x = ax',$
- (b) $s_1' = s_1 s_2, \quad s_2' = s_2, \text{ and } x = x' + 1,$
- (c) $s_1' = s_2, \quad s_2' = s_1, \text{ and } x = 1/x'.$

It is not necessary to record the details of the computation here, for the operations are all rational. The two transformations, one on the generators of G and the other on the variable x , determine in each case the same transformation on the congruence (2). Any transformation on the generators of $\{s_1, s_2\}$ is a product of transformations of the above types; corresponding to it will be a product of transformations of the three types above on x . The matrices of the two products will be identical.

It results from the above considerations that the problem of the classification of groups whose generators satisfy (1) is exactly the problem of the classification of congruences (2), which have no roots in the modular field, under the group of projective transformations on the variable with coefficients in the modular field.

2. THE GROUPS G FOR $m=4$

The indicated classification of the congruences (2) has been carried out for $m=3$ and $m=4$.^{*} These two cases present striking differences and the latter points the way to the results to be expected for a general m .

When $m=3$ and the left-hand side of (2), which we shall denote hereafter by $f(x)$, contains no linear factor in the modular field, then $f(x)$ is irreducible. This is not always true when $m=4$. All irreducible cubics are conjugate under the linear homogeneous group with coefficients in the modular field, and consequently any two groups G with a given H and $m=3$ are simply isomorphic. It is obvious that not all quartics with no linear factor are conjugate under that group, and further that not all irreducible quartics are conjugate under

^{*} On cubic congruences, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 962-969; and Irreducible quartic congruences, also offered to the same Bulletin.

it, for a necessary condition for conjugacy is that their absolute invariants be the same. The identity of the absolute invariants is sufficient for conjugacy under the general projective group but not for conjugacy under its subgroup whose coefficients are in the modular field. If a given quartic is irreducible and a second quartic is conjugate to it under the general projective group but not under its "rational" subgroup, the second quartic is the product of two quadratic factors, one of which is irreducible.

When $m=4$ there are $p+1$ distinct groups G whose generators satisfy (1). They correspond to $p+1$ quartics none of which has a linear factor. We are not interested here in making the count of the polynomials, but rather in comparing groups corresponding to different types of polynomial and interpreting properties of the polynomials in terms of properties of the groups.

Let us denote the polynomial by $f(x)$. Let us consider an $f(x)$ of degree 4 which is the product of two irreducible quadratics. There exists a group G whose generators satisfy (1) where the a 's are the coefficients of $f(x)$. For the sake of simplicity let us suppose that

$$f(x) = (x^2 - \lambda_1)(x^2 - \lambda_2)$$

where λ_1 and λ_2 are not squares. There is a group G' determined by H and four U 's which satisfy the relations

$$(7) \quad \begin{aligned} U_1^{-1} s_1 U_1 &= s_1 s_4, & U_2^{-1} s_1 U_2 &= s_1 s_3^{\lambda_1}, & U_3^{-1} s_1 U_3 &= s_1 s_6, & U_4^{-1} s_1 U_4 &= s_1 s_6^{\lambda_2}, \\ U_1^{-1} s_2 U_1 &= s_2 s_3, & U_2^{-1} s_2 U_2 &= s_2 s_4, & U_3^{-1} s_2 U_3 &= s_2 s_5, & U_4^{-1} s_2 U_4 &= s_2 s_6. \end{aligned}$$

The generators which were selected for G' do not satisfy (1), nevertheless the condition that $\{U_1, \dots, U_4\}$ contain no operator of type I is readily seen to be that $f(x)$ have no linear factor. Moreover, it can be shown that if λ_1 and λ_2 are distinct, generators of G' can be chosen which do satisfy (1). Such a set is obtained by taking $U'_1 = U_1 U_3$, in which case the resulting polynomial is $f(x)$. Therefore the two groups G and G' are simply isomorphic. From relations (7) it is easy to see that G' , and consequently G , contains two subgroups $\{H, U_1, U_2\}$ and $\{H, U_3, U_4\}$ of order p^{n+2} each with commutator subgroup of order p^2 . Looking at generators of G it is obvious that G , and consequently G' , contains subgroups of order p^{n+2} with commutator subgroups of order p^2 . These facts are dependent on the condition that $f(x)$ is the product of two distinct irreducible quadratics. If this condition is not satisfied there are two possibilities: (a) $f(x)$ is irreducible and then G contains no subgroup of order p^{n+2} with commutator subgroup of order p^2 ; and (b) $f(x)$ is the square of an irreducible quadratic, and G does not contain two subgroups of order p^{n+2} with commutator subgroups of order p^2 and with commutator subgroups distinct except for the identity. This last fact may be seen readily by consider-

ing a set of generators of G' which satisfy (7) where $\lambda_1 = \lambda_2$. Such a group exists and determines the polynomial $f(x) = (x^2 - \lambda_1)^2$. Every subgroup $\{H, U', U''\}$ has a commutator subgroup of order p^2 or of order p^4 . G' cannot then be simply isomorphic with G whose generators satisfy (1) and whose polynomial is $f(x)$.

Granting that there are $p+1$ conjugate sets of quartics which have no linear factors, we are able to distinguish $p+2$ types of group G for $m=4$ which satisfy the conditions of the introduction. Of those there are $p+1$ which come under the special case of §1. We are able to separate these $p+2$ groups into three types by a consideration of their subgroups. Those with no subgroups of order p^{n+2} with commutator subgroup of order p^2 correspond to irreducible quartics. Those with such subgroups but also with subgroups of order p^{n+2} with commutator subgroup of order p^3 correspond to reducible quartics. The one with no subgroup of order p^{n+2} with commutator subgroup of order p^3 corresponds (not in the sense of §1) to the square of a quadratic.

An interesting question is that of the existence of some subgroup or set of subgroups by means of which we may distinguish among the $(p+1)/2$ groups whose quartics are irreducible and the $(p-1)/2$ groups whose quartics are products of two distinct quadratics. The question looks sufficiently interesting to warrant our posing it in detail for the simple case where $p=7$. The four conjugate sets of irreducible quartics are represented by

$$\begin{aligned} (a) \quad & x^4 + 4x^2 + 5x + 2 \equiv 0, \\ (b) \quad & x^4 + 6x^2 + 4x + 2 \equiv 0, \\ (c) \quad & x^4 + 2x + 3 \equiv 0, \\ (d) \quad & x^4 + 4x + 4 \equiv 0. \end{aligned}$$

Groups of order 7^{n+4} corresponding to these quartics are generated by operators satisfying (1) where

$$U_4^{-1}s_1U_4 = s_1s_k,$$

$$U_4^{-1}s_2U_4 = s_2s_6,$$

and s_k takes the respective forms

$$(a) \quad s_k = s_3^5s_4^5s_6^3, \quad (b) \quad s_k = s_3^5s_4^4s_6, \quad (c) \quad s_k = s_3^4s_4^2, \quad (d) \quad s_k = s_3^3s_4^4.$$

All of these groups are identical with respect to the following:

- the order is 7^{n+4} ;
- they are metabelian;
- the central is of order 7^{n-2} ;
- the commutator subgroup is of order 7^4 ;

G/H is abelian, of order 7^4 and type 1, 1, 1, 1;
 every subgroup of order 7^{n+1} has a commutator subgroup of order 7^2 ;
 no subgroup of order 7^{n+2} has a commutator subgroup of order 7^2 ;
 every subgroup of order 7^{n+3} has a commutator subgroup of order 7^4 .

It would seem that there is a possibility of difference in the numbers of subgroups of order 7^{n+2} with commutator subgroups of orders 7^3 and 7^4 . However it seems extremely unlikely that such differences could correspond to the distinction implied by the differences in value of the absolute invariant of the corresponding quartics, especially since the absolute invariant does not distinguish between an irreducible and a reducible quartic.

While it is true that two groups which differ in the number of subgroups having a given property cannot be simply isomorphic, it is not to be assumed that the converse is true. There is no compelling reason to expect the non-isomorphism of two groups to be reflected in properties of their subgroups. Nevertheless, the author is not aware of any prior example where the non-isomorphism of two groups is not easily deducible from a consideration of their subgroups. The number of occasions where a number-theoretic argument is indispensable in the theory of groups is small, although the occasions themselves are crucial as is evidenced by the amount of the theory that depends on the existence of primitive roots in a Galois field.

3. TWO GENERAL THEOREMS

The statements of the last section for the case $m=4$ can all be established easily for that special case. The relation between properties of the polynomial $f(x)$ and properties of the group and the obvious direction in which a generalization of the results should proceed suggest a closer scrutiny of $f(x)$. This polynomial was obtained as the condition that U contain no operator of type I but it is obviously more important than that would imply. Also it seemed to be connected with a certain selection of the generators of U , but the last section has shown that it appears when the generators do not satisfy the conditions (1).

In this section we shall generalize the situation in §1 and we shall furnish incidentally the necessary proofs for the statements of §2. We require now only that G be metabelian, that the central and commutator subgroups be of orders p^{n-2} and p^m . We may assume in that case that a set of independent generators of H is chosen which contains m independent operators of the commutator subgroup and $n-2$ independent operators of the central. We also still require the U 's to be of order p and permutable. The generalization consists in not requiring the generators of G to satisfy the relations (1). Under these conditions an independent set of relations on generators of G is com-

pletely described by a pair of m -rowed square matrices. We associate one matrix M with the operator s_1 and the other matrix N with s_2 . We let the i th row of each matrix correspond to the generator U_i , and we let the j th column of each matrix correspond to s_{j+2} , where s_3, s_4, \dots, s_{m+2} are independent generators of the commutator subgroup of G . The elements in the i th row and the j th column of M and N are the exponents of s_{j+2} in the commutator of U_i with s_1 and s_2 respectively. For example, M and N for relations (1) are

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_m \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

In the case where the U 's can be separated into sets each set satisfying (1), of which relations (7) describe the simplest case, the matrix M takes the form

$$M = \begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & \cdot & \cdot & \cdot \\ & & & M_k \end{pmatrix},$$

where M_i is of the above form and all other elements are zeros. In this case also N is the identity matrix.

The congruence (2) is immediately recognizable as the condition for the vanishing of the determinant $|M + xN|$. The condition that the U 's be all of type II is that $|M + xN|$ have no linear factor in the modular field. The general case requires no further argument in these respects.

A change of generators of the commutator subgroup amounts to replacing s_i , $i=3, 4, \dots, m+2$, by

$$s'_i = s_3^{c_{i1}} s_4^{c_{i2}} \cdots s_{m+2}^{c_{im}},$$

where C , the matrix of the c_{ij} 's, is non-singular. The effect on M and N is to replace them by MC^{-1} and NC^{-1} respectively. Likewise a change in generators of U is equivalent to replacing U_i by

$$U'_i = U_1^{d_{i1}} U_2^{d_{i2}} \cdots U_m^{d_{im}},$$

where D is also non-singular. The effect of this is to replace M and N by DM and DN respectively. The theory of the groups described in the introduction is therefore equivalent to the theory of pairs of matrices with elements in a

modular field.* Applying the theory of pairs of matrices we have the following theorem:

(8) *Two groups satisfying the conditions of the introduction and determined by M, N and M', N' are simply isomorphic if and only if $M + xN$ and $M' + xN'$ have invariant factors which are conjugate under some operator of the projective group of transformations on x with coefficients in the modular field.*

We proceed to our second theorem. We consider a group G whose generators satisfy (1) and determine the congruence $f(x) \equiv 0$. We suppose that G contains a subgroup $G' = \{H, V_1, \dots, V_m\}$ with commutator subgroup of order $p^{m'}$. Let the congruence determined by G' be $f'(x) \equiv 0$. We wish to prove that $f'(x)$ is a factor of $f(x)$. Let x_1 be a root of $f'(x) \equiv 0$.† Then by means of a set of linear homogeneous congruences similar to that preceding (2) x_1 determines a set of numbers $l_1, l_2, \dots, l_{m'}$ such that $V_1^{l_1} V_2^{l_2} \dots V_{m'}^{l_{m'}}$ is permutable with $s_1 s_2^{-1}$. The V 's are expressible in terms of the U 's and consequently the l 's determine a set of numbers k_1, k_2, \dots, k_m such that $U_1^{k_1} U_2^{k_2} \dots U_m^{k_m}$ is permutable with $s_1 s_2^{-1}$. Consequently, x_1 is a root of $f(x) \equiv 0$. Hence,

(9) *If G determines the congruence $f(x) \equiv 0$, if G contains a subgroup G' of order $p^{n+m'}$ with commutator subgroup of order $p^{m'}$, and if G' determines the congruence $f'(x) \equiv 0$ where the s_1 and s_2 used to determine $f'(x)$ are those used to determine $f(x)$, then $f'(x)$ is a factor of $f(x)$.*

These two theorems establish and generalize all the unsubstantiated statements of §2. The first determines a canonical form for the generating relations of G and the second interprets the irreducible factors of $f(x)$ in terms of subgroups of G .

4. CLASSIFICATION OF THE GROUPS G AND DISCUSSION OF PROPERTIES

It has been shown elsewhere‡ that m cannot be greater than $2n-4$, if U contains an operator of type II and no operator except those of types I and II. Since here we require every operator of U to be of type II it is obvious that m is limited by the order of the commutator subgroup. We must have $m \leq n-2$. To find all the groups for a given m we may first determine all the conjugate sets of polynomials $f(x)$ of degree m under the "rational" linear fractional group. There exists at least one group for each conjugate set. If

* For the theory of pairs of matrices, cf. Dickson, *Modern Algebraic Theories*, 1930, p. 112.

† x_1 is not in the modular field. We beg to be excused from interpreting $s_1 s_2^{-1}$ and $V_1^{l_1} V_2^{l_2} \dots V_{m'}^{l_{m'}}$. This, however, does not affect the argument.

‡ On metabelian groups, loc. cit.

$f(x)$ is irreducible, or if the irreducible factors of $f(x)$ are relatively prime, then there exists but one group for the conjugate set to which $f(x)$ belongs, for the invariant factors of $M+xN$ are $f(x), 1, 1, \dots$. Let us suppose that $f(x)$ has one irreducible factor $f_1(x)$ which is repeated, so that $f(x) = [f_1(x)]^r f_2(x)$ where the factors of $f_2(x)$ are relatively prime and prime to $f_1(x)$. Then the invariant factors of $M+xN$, having the property that each is divisible by all those which follow it and being such that their product is $f(x)$, may be selected in as many ways as there are partitions of r . The number of such groups is therefore $\theta(r)$. In general,

(10) *If $f(x)$ has the distinct irreducible factors $f_1(x), \dots, f_k(x)$ and they appear to the powers r_1, \dots, r_k , then the number of distinct groups G determined by $f(x)$ is equal to the product $\Pi_{i=1}^k \theta(r_i)$ of the numbers of partitions of the r_i 's.*

Let us consider a group G and its corresponding polynomial $f(x)$ where the irreducible factors of $f(x)$ are relatively prime. The invariant factors of $M+xN$ are $f(x), 1, 1, \dots$. Generators of U and of the commutator subgroup may be chosen so that the transformed determinant $M'+xN'$ is in canonical form. If this is done the generators of G satisfy relations (1) where the a 's are the coefficients of $f(x)$. Looking at the present case from another point of view let us suppose the irreducible factors of $f(x)$ to be $f_1(x), \dots, f_k(x)$ of degrees m_1, \dots, m_k . Let us consider k sets of U 's, U_{11}, \dots, U_{im_i} , each set satisfying relations (1), the resulting commutator subgroups being distinct, all the U 's being permutable, and the a 's in (1) for the i th set being the coefficients of $f_i(x)$. The group generated by $H, U_{11}, \dots, U_{km_k}$ obviously determines the congruence $f(x) \equiv 0$, and since its irreducible factors are relatively prime, the invariant factors of $M+xN$ are necessarily $f(x), 1, 1, \dots$. This group is therefore the same as the group described at the beginning of the paragraph. Consequently, when the irreducible factors of $f(x)$ are relatively prime, G contains a set of k subgroups of orders $p^{n+m_i}, i=1, \dots, k$, with commutator subgroups of orders p^{m_i} , and G contains no other subgroup of order $p^{n+\alpha}$ with commutator subgroup of order p^α except such as are obtained by combining these. In fact these subgroups are characteristic.

The essential condition on G in order that it be possible to write the two sets of generating relations made use of above is that the invariant factors be all unity except one. This may still hold if some or all of the irreducible factors are repeated. In that case, however, there will exist subgroups of order $p^{n+\alpha}$ with commutator subgroups of order p^α where α is not one of the m_i 's, or a combination of them. Let us suppose that $f(x) = [f_1(x)]^r$, where $f_1(x)$ is irreducible and of degree m_1 , and let us suppose further that the invariant factors of $M+xN$ are $f(x), 1, 1, \dots$. Now let us consider two sets of $(r-1)m_1$

and m_1 U 's respectively. Let the first set satisfy relations (1) where the a 's are the coefficients of $[f_1(x)]^{r-1}$. Let the second set determine with H a group whose commutator subgroup is of order p^{m_1+1} , and let the commutator subgroup determined by the two sets be of order p^m . This can obviously be done by selecting the commutator of U_m and s_1 from the group generated by the rm_1 commutators which precede it and not in either of the groups generated by the first $(r-1)m_1$ or the last m_1 commutators. Clearly this commutator of U_m and s_1 can be chosen so that the invariant factors of $M+xN$ are $f(x)$, 1, 1, \dots , since it is simply a question of requiring the canonical form of $M+xN$ to have certain coefficients and there is so much freedom in the choice of the commutator. From this it follows that the group G contains at least one subgroup of order $p^{n+\alpha}$ with commutator subgroup of order p^α where $\alpha = km_1$ and k is any number from 1 to r .

If in the above case we selected the commutator of U_m and s_1 in the group generated by the preceding m_1 commutators and selected it so that the congruence determined by H and the set of m_1 U 's was $f_1(x)$, we should have the canonical form for $M+xN$ with invariant factors $[f_1(x)]^{r-1}, f_1(x), 1, 1, \dots$. In this case G contains two groups of order p^{n+m_1} with commutator subgroups of order p^{m_1} . The two sets of U 's which determine these groups give with H a group of order p^{n+2m_1} with commutator subgroup of order p^{2m_1} none of whose subgroups of order p^{n+m_1} has a commutator subgroup of order p^{m_1+1} .

The effects on G of an increase in the number of repeated factors of $f(x)$ or of the invariant factors of $M+xN$ different from unity can be determined by considerations similar to the above. Rather than pursue this further we shall give a brief description of the groups G for $m=6$. We omit $m=5$ because in that case $f(x)$ could have no repeated factors.

When $m=6$, $f(x)$ may be (1) irreducible, (2) the product of an irreducible quartic and a quadratic, (3) the product of two irreducible cubics, or (4) the product of three quadratics.

Case (1). There are as many groups as there are conjugate sets of irreducible sextics under the "rational" projective group, a number as yet undetermined. None of these groups has a subgroup of order $p^{n+\alpha}$ with commutator subgroup of order p^α , $\alpha < 6$.

Case (2). The quartic may be transformed into one of $(p+1)/2$ depending on the value of the absolute invariant. The operator of order $2i$, $i=1, 2$, which transforms the quartic into itself transforms its roots into their p^{2i} th powers and consequently transforms every element in the Galois field determined by it into its p^{2i} th power. The roots of the quadratic are in that $GF(p^4)$ and the quadratic is also transformed into itself. Therefore for each of the $(p+1)/2$ quartics there are as many groups as there are irreducible quadratics belong-

ing to the modular field. This number is $p(p-1)/2$. The number of groups is $p(p^2-1)/4$. Each of the groups has generators which satisfy relations (1). Each has subgroups of orders p^{n+2} and p^{n+4} with commutator subgroups of orders p^2 and p^4 respectively, and no other subgroups of order p^{n+a} with commutator subgroup of order p^a .

Case (3). One of the cubics can be transformed into a given irreducible cubic and no further specialization may be made. The other cubic may then be any one of $p(p^2-1)/3$, one of which is the first one. There are therefore $(p^3-p+3)/3$ groups of this kind. All but one have generators which satisfy relations (1). The odd group contains no subgroups of order p^{n+3} with commutator subgroup of order p^4 . One other contains one subgroup of order p^{n+3} with commutator subgroup of order p^3 , and all the others contain two.

Case (4). One of the quadratics can be transformed into $(x^2-\lambda)$ and a second into one of $(p+1)/2$ quadratics which are taken one from each of the conjugate sets of quadratics under the group which leaves $(x^2-\lambda)$ fixed. Having decided which of the three quadratics are first and second and having transformed them to the desired form, no other simplification is possible. Hence, the third quadratic may be any one of $p(p-1)/2$. If the three quadratics are the same, then we may suppose $f(x)$ to be $(x^2-\lambda)^3$ and there are three groups: one, corresponding to the invariant factors $f(x), 1, 1, \dots$, which contains subgroups of orders p^{n+2} and p^{n+4} with commutator subgroups of orders p^3 and p^5 respectively; one, corresponding to invariant factors $(x^2-\lambda)^2, (x^2-\lambda), 1, 1, \dots$, which contains subgroups of the first type but none of the second; and one, corresponding to invariant factors $(x^2-\lambda), (x^2-\lambda), (x^2-\lambda), 1, 1, \dots$, which contains no subgroups of either type. If two of the quadratics are the same and the third is distinct from this, we may transform the repeated one into $(x^2-\lambda)$ and then the third may be any one of $(p-1)/2$. For each of these there are two groups, according as one or two of the invariant factors of $M+xN$ are different from one. The two groups corresponding to the same $f(x)$ are again distinguished by their subgroups of order p^{n+2} . There are $p-1$ of these groups. If the three quadratics are distinct, $f(x)$ can be reduced to one of $(p-1)(p^2-p-4)/4$ forms, but the reduction can be made in more than one way since it involved the selection of a first and a second quadratic. A different selection of the first and second quadratics may or may not change $f(x)$, depending on the relations of the three quadratics. We shall not make the count of the number of groups, but shall note that for each such $f(x)$ there is but one group, and that each such group contains three and only three subgroups of order p^{n+2} with commutator subgroup of order p^2 and that it contains subgroups of orders p^{n+2} and p^{n+4} with commutator subgroups of orders p^3 and p^5 respectively.

5. A GENERALIZATION

There are two obvious directions in which the results so far obtained may be generalized. The theory of pairs of matrices as expounded by Dickson (cf. the reference above) does not require the matrices to be non-singular, whereas the condition that U contain no operator of type I does require that M and N be non-singular. If U contains an operator of type I we have seen that $M + xN$ contains a linear factor. Obviously, there exists in that case a transformation on x which transforms one of the roots of $f(x) \equiv 0$ to zero, and such a transformation replaces M and N by M' and N' where M' is singular. The new generator s_1' is therefore permutable with one of the operators of U . If $f(x)$ has a linear factor which is repeated, then after the above transformation more than one of the U 's will be permutable with s_1' . If $f(x)$ has two distinct linear factors, then a transformation on x will put one of the roots of $f(x) \equiv 0$ into zero and another into infinity, in which case both M' and N' will be singular.* We shall not pursue this question at this time but shall consider another extension.

Let us remove the restriction that the operators of U be of type II, assuming that U contains an operator of type K where the type K is distinguished by the fact that the operator determines the partition

$$n = 2 + 2 + \cdots + 2 + 1 + 1 + \cdots + 1$$

in which there are k 2's. The central of G will then be of order at most p^{n-k} . On the other hand if the central of G is of order p^{n-k} and U contains no operator corresponding to a partition of n with a greatest term greater than 2, then U can contain no operator of type J where $j > k$. We shall require further that the commutator subgroup of G and that U be of order p^m . The considerations of the first paragraph of this section indicate that groups satisfying these restrictions are of fundamental importance.

The U 's are assumed to be permutable as before. Now generators of H can be selected so that $n - k$ of them are in the central and m of those are in the commutator subgroup. The relations among generators of G are then completely described by k matrices M_1, M_2, \dots, M_k , one corresponding to each of the non-invariant generators s_1, s_2, \dots, s_k , and defined exactly as M and N in §3. The condition that all the operators of U be of type K is obtained by considering the commutator of $s_1^{x_1}s_2^{x_2} \cdots s_k^{x_k}$ and $U_1^{k_1}U_2^{k_2} \cdots U_m^{k_m}$. This leads to a system of linear homogeneous congruences in k_1, k_2, \dots, k_m , the condition for whose solution is $|x_1M_1 + x_2M_2 + \cdots$

* Compare with the classification of groups for $m=3$, *On metabelian groups*, loc. cit.

$+x_k M_k \equiv 0, \text{ mod } p$. There will be no operator of a lower type in U if this polynomial $f(x_1, x_2, \dots, x_k)$ contains no linear factor.

Exactly as before we may restrict our attention to changes in $f(x_1, \dots, x_k)$ due to changes in the generators which satisfy the following conditions: the U 's are selected from the group U ; m of the independent generators of H are in the commutator subgroup; and k of the independent generators of H are in the group $\{s_1, s_2, \dots, s_k\}$. The first condition makes certain that the U 's are permutable and disregards all transformations that leave the M 's simultaneously invariant; the second is necessary if the M 's are to remain square matrices; and the third is necessary if the number of M 's is to remain unchanged. A selection of new sets of generators of U and the commutator subgroup subject to these conditions results in the transformation $M'_i = CM_i D$, where C and D are non-singular. This does not change the polynomial $f(x_1, \dots, x_k)$, and does not change the invariant factors* of the matrix $x_1 M_1 + \dots + x_k M_k$.

For the effect of changes in the generators of $\{s_1, \dots, s_k\}$ let us consider the transformation

$$s'_i = s_1^{a_{i1}} s_2^{a_{i2}} \dots s_k^{a_{ik}} \quad (i = 1, 2, \dots, k).$$

The matrix M'_i is obtained by considering the commutators of s'_i with U_1, \dots, U_m and is obviously $a_{i1} M_1 + a_{i2} M_2 + \dots + a_{ik} M_k$. The operator $s_1^{x'_1} s_2^{x'_2} \dots s_k^{x'_k}$ expressed in terms of the s 's is $s_1^{x'_1} s_2^{x'_2} \dots s_k^{x'_k}$, where the x 's are obtained from the x 's by a linear transformation whose matrix is the matrix A of exponents a_{ij} above. After this transformation we have the matrix $x'_1 M'_1 + x'_2 M'_2 + \dots + x'_k M'_k$ in place of $M = x_1 M_1 + \dots + x_k M_k$. If now instead of carrying out the transformation on the generators we subject the x 's to the transformation just described, the matrix M becomes, when the terms in x'_i are collected,

$$x'_1 (a_{11} M_1 + a_{12} M_2 + \dots + a_{1k} M_k) + x'_2 (a_{21} M_1 + a_{22} M_2 + \dots + a_{2k} M_k) \\ + \dots + x'_k (a_{k1} M_1 + a_{k2} M_2 + \dots + a_{kk} M_k)$$

which is $x'_1 M'_1 + x'_2 M'_2 + \dots + x'_k M'_k$. Therefore,

(11) *Two metabelian groups $G = \{H, U\}$ and $G' = \{H, U'\}$ in which both the U 's and the U' 's are permutable and of order p , and such that generating relations are defined by sets of matrices M_1, \dots, M_k and M'_1, \dots, M'_k respec-*

* The term "invariant factor" is used here in a sense which is merely an extension to k matrices of the definition given by Dickson, loc. cit., p. 104, for two matrices M and N . It is clear that the polynomials are homogeneous and are left unchanged when the matrix $M = x_1 M_1 + \dots + x_k M_k$ is replaced by CMD .

tively, are simply isomorphic if and only if the invariant factors of $x_1M_1 + x_2M_2 + \dots + x_kM_k$ and $x'_1M'_1 + x'_2M'_2 + \dots + x'_kM'_k$ are conjugate under some operator of the linear homogeneous group on the variables x_1, \dots, x_k .

We return again briefly to the ideas of the first paragraph of this section. In the situation we have been considering we have assumed that all the operators of U were of type K and this implies that each of the M_i 's is non-singular and also that $f(x_1, \dots, x_k)$ contains no linear factor. If $f(x_1, \dots, x_k)$ contains a linear factor, then U contains an operator of type J where $j < k$. It is obvious that under those circumstances there are several possibilities. If the operator in question is of type I it would be possible to select the generators s'_1, \dots, s'_k and U'_1, \dots, U'_m so that all but one of the matrices M'_i would be singular. If however the operator were of type $(K-1)$, generators of G could not be selected to make more than one of the M'_i 's singular. Thus the condition that $f(x_1, \dots, x_k)$ have a single linear factor in the modular field seems to permit the possibility of many distinct groups. This seems to lead to a large subject in the theory of forms on which not much is to be found in the literature. Dickson has considered* the types of forms that can be written as determinants with linear elements. The classification of these forms involves the theory of modular invariants. The classification of the groups involves considerations beyond the modular invariants since the latter do not take account of the invariant factors of M .

6. CONCLUDING REMARKS

In conclusion we wish to point out the relation of our investigations to the practically impossible problem of the classification of groups of order p^a . One important sub-class of these groups, which from some points of view may be considered as the most elementary, is made up of those groups whose operators are all of order p , in other words, those groups which are conformal with the abelian group of type 1, 1, \dots . It is with groups of this class that we have been concerned. Every such group contains a maximal invariant abelian subgroup of type 1, 1, \dots . Of these groups the most elementary are the metabelian groups. Our plan of classification is to determine all of those such that U contains at least one operator of type K and no operator of type greater than K . In the cases where k is 1 or 2 this plan of classification never allows a given group to appear in more than one class, but for $k > 2$ it is necessary to look out for repetitions. For example, if $k=3$ and $m=2$, G contains a maximal invariant abelian subgroup $\{U_1, U_2, s_4, s_5, \dots, s_n\}$ of order p^{n-1} and the operators s_1, s_2, s_3 which now serve as the U 's are all of type I or

* These Transactions, vol. 22 (1921), p. 167.

type II. However those repetitions and all others are avoided by insisting that m be at least as great as k .

But we are not yet willing to consider all metabelian groups which are conformal with the abelian group of type $1, 1, \dots$, for a restriction which has been important throughout is that the U 's be permutable. It is obvious that U 's can be chosen, in the groups which we have considered, which are not permutable, for example, $U'_1 = s_1 U_1$ is not permutable with U_2 . Consequently, the removal of that restriction will not only increase our difficulties in managing the groups but will increase greatly the number of repetitions. It appears quite likely that the best method of procedure when the restriction in question is removed is to arrange the groups according to the differences in orders of the commutator subgroup of G and the commutator subgroup arising from transformation of H by U . In all the groups we have considered these orders are equal; the question as to whether or not the converse is true does not seem to have an obvious answer.

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SUFFICIENT CONDITIONS FOR THE PROBLEM OF BOLZA IN THE CALCULUS OF VARIATIONS*

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1. Introduction. Let the end points of the arcs

$$(1:1) \quad y_i = y_i(x) \quad (x^1 \leq x \leq x^2; i = 1, \dots, n)$$

be denoted by the symbols $(x^1, y_1^1, \dots, y_n^1)$ and $(x^2, y_1^2, \dots, y_n^2)$. The problem to be considered is that of finding in a class of arcs (1:1) and sets $(\alpha) = (\alpha_1, \dots, \alpha_r)$ satisfying the differential equations and end conditions

$$(1:2) \quad \phi_\beta(x, y, y') = 0 \quad (\beta = 1, \dots, m < n),$$

$$(1:3) \quad x^s = x^s(\alpha), \quad y_i^s = y_i^s(\alpha) \quad (s = 1, 2)$$

one which minimizes a functional of the form

$$J = \theta(\alpha) + \int_{x^1}^{x^2} f(x, y, y') dx.$$

This problem was first formulated by Bolza (II, p. 431)‡ and will be called the *problem of Bolza*. The formulation here given is due to Morse and Myers (VI, p. 236). Of special importance is the case in which the end conditions (1:3) are of the form

$$(1:4) \quad \begin{aligned} x^1 &= x^1(\alpha_1, \dots, \alpha_p), & y_i^1 &= y_i^1(\alpha_1, \dots, \alpha_p), \\ x^2 &= x^2(\alpha_{p+1}, \dots, \alpha_r), & y_i^2 &= y_i^2(\alpha_{p+1}, \dots, \alpha_r) \end{aligned}$$

and the function $\theta(\alpha)$ is of the form

$$\theta(\alpha) = \theta^1(\alpha_1, \dots, \alpha_p) - \theta^2(\alpha_{p+1}, \dots, \alpha_r).$$

The latter problem will be called the *problem of Bolza with separated end conditions*. In the proof of Theorem 9:2 below it will be shown that the two problems are equivalent not only in the sense that each can be transformed into one of the other type but also in the sense that the theory of the one can be deduced from that of the other.

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† National Research Fellow. A considerable portion of the results here given were obtained while the author was a Research Assistant to Professor Bliss at the University of Chicago.

‡ Roman numerals in parentheses refer to the list of references at the end of this paper.

Sufficient conditions for a minimum in the problems of Bolza were first given by Morse (VIII) and later by Bliss (IX) and Hu (XIX). However, the normality assumptions, which they make, prevent these conditions from being applicable without further modification to the problem of Mayer (III; XI), to the case in which the functions ϕ_β contain no derivatives, and to a number of other problems. Sufficient conditions for the problem of Mayer have been deduced by Bliss and Hestenes (XVII; XVIII) who make similar restrictive normality assumptions. In §9 below we give for the first time sets of sufficient conditions for the problem of Bolza containing no normality assumptions whatsoever. We merely assume the existence of a set of multipliers of the form $\lambda_0=1$, $\lambda_\beta(x)$ with which the arc g under consideration satisfies suitable analogues of the usual sufficiency conditions. It is clear that the results of the present paper are applicable at once to the problem of Mayer and thus unify the problems of Bolza and Mayer so that they are equivalent not only in the sense that each can be transformed into one of the other type but also in the sense that the theory for the one can be deduced from that of the other without further modification. The results of this paper also show that the classical problem of Mayer can be considered as a problem of Lagrange with one variable end point (cf. I, p. 224). Moreover by the use of a device given by Bliss (V, p. 703) the results here given can be applied to the case in which the functions ϕ_β contain no derivatives. One obtains thereby an extension of the results given by Bower (XX).

In order to obtain the sufficient conditions here given we derive in §4 a new analogue of the necessary condition of Mayer for the problem of Bolza with separated end conditions. A similar condition has been given by Currier (XII, p. 699) for parametric problems without differential side conditions and with special end conditions. The methods of Currier, however, do not seem to be readily extensible to the problem of Bolza without making stringent normality assumptions. A very special case of this necessary condition has been given by Bliss for variable end point problems in the plane (IV, pp. 324-6).

The sufficiency proof given in §§ 6 and 9 below is new and is simpler than those given hitherto for the problems of Bolza. It is a direct extension of the classical method used for fixed end point problems and does not make use of the famous theorem of Hahn (IX, p. 267).

The author has made extensive use of the papers of Bliss and Morse listed at the end of this paper.

2. First necessary conditions. Let us suppose that we have given an open region \mathcal{R} of points (x, y, y') in which the functions f, ϕ_β have continuous derivatives of the first three orders. A set (x, y, y') is said to be *admissible* if it is in \mathcal{R} and satisfies the equations $\phi_\beta=0$. A *differentiably admissible arc* is

a continuous arc having a continuously turning tangent except possibly at a finite number of points on it and having all of its elements (x, y, y') admissible. A differentially admissible arc (1:1) and a set of constants $(\alpha) = (\alpha_1, \dots, \alpha_r)$ satisfying the end conditions (1:3) are said to form an *admissible arc*.

We center our attention on a particular admissible arc g and propose to find under what conditions g will surely furnish a minimum to J relative to neighboring admissible arcs. We assume that the matrix $\|\phi_{\beta y_i'}\|$ has rank m on g and that the set (α) belonging to g is the set $(\alpha) = (0)$. The functions $\theta(\alpha)$, $x^s(\alpha)$, $y_i^s(\alpha)$ ($s=1, 2$) are assumed to have continuous first and second partial derivatives near $(\alpha) = (0)$.

The tensor analysis summation convention will be used throughout.

The following necessary condition is well known and has been established by Morse and Myers (VI, p. 245) and by Bliss and Schoenberg (X, pp. 681-3) and by others.

THEOREM 2:1. *If g affords a minimum to J then there exist for it constants c_1, \dots, c_n and a function $F = \lambda_0 f + \lambda_\beta(x) \phi_\beta$ ($\beta=1, \dots, m$) such that the equations*

$$(2:1) \quad F_{y_i'} = \int_x^2 F_{y_i'} dx + c_i, \quad \phi_\beta = 0 \quad (i=1, \dots, n)$$

hold at every point of g . Moreover on g the equation

$$(2:2) \quad [(F - y_i' F_{y_i'}) dx + F_{y_i'} dy_i]_1^2 + \lambda_0 d\theta = 0^*$$

is an identity in $d\alpha_h$ when the differentials $dx^1, dy_i^1, dx^2, dy_i^2, d\theta$ are expressed in terms of the differentials $d\alpha_h$. The multiplier λ_0 is a constant. The multipliers $\lambda_\beta(x)$ are continuous except possibly at values of x defining corners of g . The elements of the set $\lambda_0, \lambda_\beta(x)$ do not vanish simultaneously at any point on g .

By the order q of anormality of g on an interval $x'x''$ relative to the conditions (2:1) is meant the number q of linearly independent sets of multipliers of the form $\lambda_0=0, \lambda_\beta(x)$ with which g satisfies the conditions (2:1) on $x'x''$. The order q of g on $x'x''$ cannot exceed the number m of differential equations $\phi_\beta=0$. This follows because for every $m+1$ sets of multipliers of the form $\lambda_0=0, \lambda_\beta(x)$ there exists at least one linear combination of these sets having constant coefficients not all zero and vanishing at x' and hence vanishing for all values of x on $x'x''$. The case $q=0$ on every sub-interval of x^1x^2 has been

* The symbol $[]_1^2$ denotes the value of $[]$ at the final end point 2 on g minus its value at the initial end point 1 on g .

treated by Morse and Bliss. In this case g is said to be normal on every sub-interval.

Carathéodory (XV, XVI) has shown that in the analytic case the order q of anormality of g is the same on every sub-interval of x^1x^2 . In the non-analytic case this is not necessarily true, as will be seen in the example given at the end of §9.

By the order p of anormality of g relative to the conditions (2:1) and (2:2) is meant the number p of linearly independent sets of multipliers of the form $\lambda_0=0, \lambda_\beta(x)$ with which g satisfies the conditions (2:1) and (2:2). Clearly the order p of g cannot exceed the order q of g on the interval x^1x^2 defined by its end points. If $p=0$ then g is said to be normal. In the normal case there exists an infinity of admissible arcs in every neighborhood of g . In the anormal case this is not necessarily true. Moreover for a normal minimizing arc g there exists a unique set of multipliers of the form $\lambda_0=1, \lambda_\beta(x)$ satisfying the conditions of Theorem 2:1 (V, pp. 693-5).

We have the following analogue of the necessary condition of Weierstrass which has been established by Graves (XIII, p. 751).

THEOREM 2:2. *If g is a normal minimizing arc then at each element (x, y, y', λ) on g the inequality*

$$E(x, y, y', \lambda, Y') \geq 0$$

must hold for every admissible set $(x, y, Y') \neq (x, y, y')$ whose matrix $\|\phi_{\beta y_i'}(x, y, Y')\|$ has rank m , where

$$(2:3) \quad E(x, y, y', \lambda, Y') = F(x, y, Y', \lambda) - F(x, y, y', \lambda) - (Y'_i - y'_i)F_{y_i'}(x, y, y', \lambda).$$

The analogue of the necessary condition of Clebsch given in Theorem 4:5 below can also be obtained from Theorem 2:2 by the arguments given by Bliss (V, pp. 718-9).

An *extremal* arc is defined to be a differentially admissible arc and a set of multipliers

$$y_i = y_i(x), \quad \lambda_\beta = \lambda_\beta(x) \quad (x^1 \leq x \leq x^2)$$

having continuous derivatives $y_i', y_i'', \lambda_\beta'$ and satisfying with $\lambda_0=1$ the *Euler-Lagrange equations*

$$(2:4) \quad (d/dx)F_{y_i'} - F_{y_i} = 0, \quad \phi_\beta = 0.$$

An extremal is said to be *non-singular* if the determinant

$$\begin{vmatrix} F_{y_i' y_k'} & \phi_{\beta y_i'} \\ \phi_{\beta y_k'} & 0 \end{vmatrix}$$

is different from zero at each element (x, y, y', λ) on it. A study of the extremal family has been made by Bliss (V, p. 687).

In the sequel it will be understood that the admissible arc g under consideration is an extremal arc satisfying the conditions (2:1) and (2:2) of Theorem 2:1 unless otherwise expressly stated.

3. The second variation and the accessory minimum problem. In this section we are concerned with the functional

$$J_2(\eta, w) = b_{hl}w_hw_l + \int_{x^1}^{x^2} 2\omega(x, \eta, \eta')dx \quad (h, l = 1, \dots, r)$$

evaluated along the extremal g , where (s not summed; $s = 1, 2$)

$$\begin{aligned} b_{hl} &= \theta_{hl} + [(F_x - y'_i F_{y_i})x_h^s x_l^s + (F - y'_i F_{y_i'})x_h^s x_l^s \\ &\quad + F_{y_i'}(x_h^s y_{il}^s + x_l^s y_{ih}^s) + F_{y_i'} y_{ih}^s]^2, \\ 2\omega &= F_{y_i y_k} \eta_i \eta_k + 2F_{y_i y_k'} \eta_i \eta_k' + F_{y_i' y_k'} \eta_i' \eta_k' \quad (i, k = 1, \dots, n). \end{aligned}$$

Here the symbols x^s, y_i^s denote the functions $x^s(\alpha), y_i^s(\alpha)$ and the subscripts h and l denote differentiation with respect to α_h and α_l respectively at $(\alpha) = (0)$. The matrix $\|b_{hl}\|$ is symmetric. The functions $\eta_i(x)$ are assumed to possess continuous derivatives except possibly at a finite number of values of x on the interval $x^1 x^2$ and to satisfy with the constants w_h the equations

$$\begin{aligned} \Phi_{\beta}(x, \eta, \eta') &= \phi_{\beta y_i} \eta_i + \phi_{\beta y_i'} \eta_i' = 0, \\ \eta_i^s &= c_{ih}^s w_h \quad (s = 1, 2; h = 1, \dots, r) \end{aligned}$$

evaluated along g , where $c_{ih}^s = y_{ih}^s(0) - y_i'(x^s)x_h^s(0)$ (s not summed). Such a set η_i, w_h is called a set of *admissible variations* for g . The functional $J_2(\eta, w)$ is called the *second variation* of the functional J along g (cf. VIII, pp. 520-1).

THEOREM 3:1. *If g is a normal minimizing extremal arc, then along g the second variation J_2 of J must satisfy the condition $J_2(\eta, w) \geq 0$ for every set of admissible variations η_i, w_h having continuous second derivatives except possibly at a finite number of values of x on the interval $x^1 x^2$ defined by the end points of g .*

The theorem follows readily from the derivation of the second variation given by Morse (VIII, pp. 520-1) provided that we show that for every set of admissible variations η_i, w_h having the continuity properties described in the theorem there exists a one-parameter family of admissible arcs

$$y_i = y_i(x, e), \quad \alpha_h = \alpha_h(e) \quad [x^1(\alpha) \leq x \leq x^2(\alpha)]$$

containing g for $e=0$, having η_i, w_h as its variations along g , and having the following continuity properties. The functions $y_i(x, e), \alpha_h(e)$ have continuous

first and second derivatives with respect to e near $e=0$. The derivatives $y_{ix}, y_{ixxx}, y_{ixx}$ exist and are continuous for values (x, e) near those belonging to g except possibly at a finite number of values of x on x^1x^2 . The existence of such a family is readily established by the methods of Bliss (V, p. 695: cf. VI, p. 249) with suitable modifications in order to obtain the necessary derivatives.

Theorem 3:1 leads us to the study of the *accessory minimum problem*, namely, the problem of minimizing the functional $J_2(\eta, w)$ in the class of admissible variations η_i, w_h . This problem is a problem of Bolza of the type described in §1. From Theorem 2:1 we obtain the following equations which a minimizing arc without corners must satisfy:

$$(3:1) \quad (d/dx)\Omega_{\eta_i'} - \Omega_{\eta_i} = 0, \quad \Phi_\beta = 0 \quad (\beta = 1, \dots, m),$$

$$(3:2) \quad \eta_i^s - c_{ih}^s w_h = 0 \quad (i = 1, \dots, n; s = 1, 2),$$

$$(3:3) \quad \xi_i^2 c_{ih}^2 - \xi_i^1 c_{ih}^1 + \mu_0 b_{hl} w_l = 0 \quad (h, l = 1, \dots, r),$$

where $\Omega = \mu_0 \omega + \mu_\beta(x) \Phi_\beta$, $\xi_i = \Omega_{\eta_i'}$. The equations (3:1) are known as the *accessory equations*, the equations (3:2) as the *secondary end conditions*, the equations (3:3) as the *secondary transversality conditions*. The extremals for this problem will be called *secondary extremals*. The secondary end conditions are said to be *regular* in case the $2n \times r$ -dimensional matrix $\|c_{ih}^s\|$ has rank r on g .

If g is non-singular the equations

$$\xi_i = \Omega_{\eta_i'}(x, \eta, \eta', \mu), \quad \Phi_\beta(x, \eta, \eta') = 0$$

with $\mu_0=1$ can be solved for the variables η_i', μ_β . The accessory equations with $\mu_0=1$ are then found to be equivalent to equations of the form

$$(3:4) \quad d\eta_i/dx = G_i(x, \eta, \xi), \quad d\xi_i/dx = H_i(x, \eta, \xi),$$

where G_i, H_i are linear in the variables η_i, ξ_i (V, p. 727). For every pair of solutions η_i, ξ_i and u_i, v_i of these equations the expression $\xi_i u_i - \eta_i v_i$ is a constant (V, p. 738). If this constant is zero the solutions are said to be *conjugate solutions*. A set of n mutually conjugate linearly independent solutions is said to form a *conjugate system*.

In the separated end point case the quadratic form $b_{hl} w_h w_l$ is of the form

$$b_{\mu\nu}^1 w_\mu w_\nu - b_{\sigma\tau}^2 w_\sigma w_\tau \quad (\mu, \nu = 1, \dots, \rho; \sigma, \tau = \rho + 1, \dots, r),$$

where

$$(3:5) \quad b_{\mu\nu}^1 = \theta_{\mu\nu}^1 - (F_x - y_i' F_{y_i}) x_\mu^1 x_\nu^1 - (F - y_i' F_{y_i'}) x_\mu^1 x_\nu^1 \\ - F_{y_i}(x_\mu^1 y_{i\nu}^1 + x_\nu^1 y_{i\mu}^1) - F_{y_i'} y_{i\mu\nu}^1$$

evaluated at the initial point 1 on g and $b_{\sigma\tau}^2$ is a similar expression in $\theta_{\sigma\tau}^2$, x_{σ}^2 , $y_{i\sigma}^2$, $x_{\sigma\tau}^2$, $y_{i\sigma\tau}^2$ evaluated at the final end point 2 on g . The matrices $\|b_{\mu\nu}^1\|$ and $\|b_{\sigma\tau}^2\|$ are symmetric. Moreover the equations (3:2) and (3:3) with $\mu_0=1$ can be written in the form

$$(3:6) \quad \eta_i^1 = c_{i\mu}^1 w_{\mu}, \quad \zeta_{i\mu}^1 = b_{\mu\nu}^1 w_{\nu}, \quad (\mu, \nu = 1, \dots, \rho),$$

$$(3:7) \quad \eta_i^2 = c_{i\sigma}^2 w_{\sigma}, \quad \zeta_{i\sigma}^2 = b_{\sigma\tau}^2 w_{\tau}, \quad (\sigma, \tau = \rho+1, \dots, r).$$

If the matrix $\|c_{i\mu}^1\|$ has rank ρ then there are n and at most n linearly independent solutions

$$(3:8) \quad \eta_{ik}(x), \zeta_{ik}(x), w_{\mu k} \quad (k = 1, \dots, n)$$

of equations (3:4) and (3:6), as one readily verifies. Moreover the secondary extremals η_{ik}, ζ_{ik} in (3:8) form a conjugate system since at $x=x^1$ we have

$$\begin{aligned} \zeta_{ik}\eta_{ij} - \zeta_{ij}\eta_{ik} &= \zeta_{ik}c_{i\mu}^1 w_{\mu j} - \zeta_{ij}c_{i\mu}^1 w_{\mu k} \\ &= b_{\mu\nu}^1 w_{\mu j} w_{\nu k} - b_{\mu\nu}^1 w_{\mu k} w_{\nu j} = 0 \end{aligned}$$

and since these secondary extremals are linearly independent, as follows readily from the fact that the matrix $\|c_{i\mu}^1\|$ has rank ρ . Similarly if the matrix $\|c_{i\sigma}^2\|$ has rank $r-\rho$, then there are n and at most n linearly independent solutions

$$(3:9) \quad u_{ik}(x), v_{ik}(x), w_{\sigma k} \quad (k = 1, \dots, n)$$

of equations (3:4) and (3:7). It is clear that the secondary extremals u_{ik}, v_{ik} also form a conjugate system.

The following lemma will be useful:

LEMMA 3:1. *The order p of anormality of g is equal to the number of linearly independent secondary extremals η_i, μ_{β} having $(\eta) \equiv (0)$ on x^1x^2 and satisfying the equations (3:3) with the set $(w) = (0)$.*

This result follows because the first n equations (2:4) with $\lambda_0=0$ are equivalent to the first n equations (3:1) with $(\eta) \equiv (0)$. Moreover the transversality condition (2:2) with $\lambda_0=0$ is equivalent to the conditions (3:3) with $(\eta) \equiv (0)$ and $(w) = (0)$.

Similarly we have

LEMMA 3:2. *The order q of anormality of g on the interval x^1x^2 is equal to the number of linearly independent secondary extremals η_i, μ_{β} having $(\eta) \equiv (0)$ on x^1x^2 .*

A further lemma is the following:

LEMMA 3:3. If u_i, v_i is a secondary extremal having $(u) \equiv (0)$ on $x^1 x^2$ then the relation $v_i \eta_i = \text{constant}$ holds for every differentially admissible arc $\eta_i(x)$ for the accessory minimum problem.

Let $\mu_\beta = \lambda_\beta(x)$ be the multipliers belonging to the secondary extremal u_i, v_i . The lemma now follows readily by multiplying the equations of variation

$$\phi_{\beta u_i} \eta_i + \phi_{\beta v_i} \eta_i' = 0$$

by the functions $\lambda_\beta(x)$, adding, and applying the usual integration by parts with the help of equations (2:4) with $\lambda_0 = 0$.

An important consequence of Lemma 3:3 is that the accessory minimum problem can be modified so that its admissible arcs are all normal. This can be done by replacing the secondary end conditions (3:2) by the conditions

$$(3:10) \quad \eta_i^1 = c_{ih}^1 w_h, \quad \eta_i^2 = c_{ih}^2 w_h + \zeta_{i\gamma}^2 w_{r+\gamma} \quad (\gamma = 1, \dots, p)$$

where p is the order of anormality of g and $\eta_{i\gamma}, \zeta_{i\gamma}$ are p linearly independent secondary extremals having the properties described in Lemma 3:1. We may suppose that these secondary extremals have been chosen so that the columns of the matrix $\|\zeta_{i\gamma}^2\|$ are normed and orthogonalized. By Lemma 3:1 we have

$$(3:11) \quad \zeta_{i\gamma}^2 c_{ih}^2 - \zeta_{i\gamma}^1 c_{ih}^1 = 0 \quad (\gamma = 1, \dots, p).$$

Multiplying the equations (3:10) by the values $-\zeta_{i\gamma}^1, \zeta_{i\gamma}^2$ and adding, it is found with the help of equations (3:11) and Lemma 3:3 that the equations

$$0 = -\zeta_{i\gamma}^1 \eta_i^1 + \zeta_{i\gamma}^2 \eta_i^2 = w_{r+\gamma} \quad (\gamma = 1, \dots, p)$$

hold for every admissible arc $\eta_i, w_h, w_{r+\gamma}$ for the new problem. The new problem is therefore equivalent to the original one. Moreover every admissible arc for the new problem is normal, by Lemma 3:1, since the secondary extremals $\eta_{i\gamma}, \zeta_{i\gamma}$, described above, do not satisfy the analogue of conditions (3:3) with the set $w_h = w_{r+\gamma} = 0$.

LEMMA 3:4. If g is non-singular then a minimizing arc g_2 for the accessory minimum problem must be an arc defined by a secondary extremal and hence can have no corners.

As was seen above we may assume that g_2 is normal. Since g_2 is a minimizing arc there exists for it a unique function $\Omega = \omega + \mu_\beta \Phi_\beta$ with which g_2 satisfies the conditions implied by Theorem 2:1. The functions $\zeta_i = \Omega_{q_i'}$ are therefore continuous along g_2 . The non-singularity of g and hence of g_2 implies that the functions η_i, μ_β belonging to g_2 define extremal segments between corners of g_2 (V, p. 684). The continuity of the functions ζ_i now implies that

the arc g_2 can have no corners since there is one and only one secondary extremal taking given values η_i^0, ζ_i^0 at a value $x = x^0$ on x^1x^2 . This proves the lemma.

4. Necessary conditions for the second variation to be positive. The second variation $J_2(\eta, w)$ is said to be *positive* along g if the inequality $J_2(\eta, w) \geq 0$ holds for every set of admissible variations η_i, w_λ belonging to g . The results of this section will remain valid if we further restrict these variations to have the continuity properties described in Theorem 3:1. The necessary conditions here given must therefore be satisfied if g is to be a *normal* minimizing arc for the original problem.

We have the following necessary condition in the separated end point case. The relations between this condition and those of Currier and Bliss have been explained in §1.

THEOREM 4:1. *If in the separated end point case the extremal g is non-singular, the secondary end conditions are regular, and the second variation J_2 is positive along g , then at each point x^3 on x^1x^2 the inequality*

$$(4:1) \quad (\zeta_{ij}u_{ik} - \eta_{ij}v_{ik})a_jb_k \geq 0 \quad (i, j, k = 1, \dots, n)$$

must hold for every set of constants (a_j, b_k) satisfying the equations

$$(4:2) \quad \eta_{ij}(x^3)a_j = u_{ik}(x^3)b_k$$

where η_{ij}, ζ_{ij} and u_{ik}, v_{ik} are the conjugate systems belonging to the sets (3:8) and (3:9) respectively. The coefficients in the bilinear form (4:1) are constants.

In order to prove the theorem we note that a set of constants (a_j, b_k) satisfying the equations (4:2) determines a broken secondary extremal η_i, ζ_i defined by the equations

$$(4:3) \quad \begin{aligned} \eta_i &= \eta_{ij}a_j, \quad \zeta_i = \zeta_{ij}a_j \quad \text{on } x^1 \leq x \leq x^3, \\ \eta_i &= u_{ik}b_k, \quad \zeta_i = v_{ik}b_k \quad \text{on } x^3 \leq x \leq x^2 \end{aligned}$$

and satisfying the conditions (3:6) and (3:7) with the set of constants $w_\mu = w_{\mu j}a_j, w_\sigma = w_{\sigma k}b_k$. Let $\mu_\beta(x)$ be the set of multipliers belonging to the broken extremal η_i, ζ_i . With the help of the formula

$$(4:4) \quad 2\Omega = \eta_i\Omega_{\eta_i} + \eta'_i\Omega_{\eta_i} + \mu_\beta\Omega_{\mu_\beta}$$

and the usual integration by parts it is found that along this broken extremal the second variation J_2 is expressible in the form

$$\begin{aligned} J_2 &= b_{\mu\sigma}^1 w_\mu w_\sigma - b_{\sigma\tau}^2 w_\sigma w_\tau + \int_{x^1}^{x^3} 2\Omega dx + \int_{x^3}^{x^2} 2\Omega dx \\ &= b_{\mu\sigma}^1 w_\mu w_\sigma - b_{\sigma\tau}^2 w_\sigma w_\tau + [\eta_i \zeta_i]_1^2 + [\eta_i \zeta_i]_{x^3+0}^{x^2-0}. \end{aligned}$$

By the use of equations (3:6), (3:7), (4:2), and (4:3) it follows readily that

$$\begin{aligned} J_2 &= \eta_i(x^3 + 0)\zeta_i(x^3 - 0) - \zeta_i(x^3 + 0)\eta_i(x^3 - 0) \\ &= (\zeta_{ij}u_{ik} - \eta_{ij}v_{ik})a_jb_k. \end{aligned}$$

This formula justifies the inequality (4:1). The last statement in the theorem follows from the remarks made in the paragraph containing the equations (3:4). The theorem is now proved.

Consider now the problem of Bolza in which the end conditions are not necessarily separated. Suppose for the moment that g is non-singular. Let $\eta_{ip}, \mu_{\beta p}, w_{hp}$ ($p=1, \dots, \nu$) be a maximum set of linearly independent secondary extremals and constants (w) satisfying the secondary end conditions (3:2). It is clear that the quadratic form

$$(4:5) \quad Q(z) = J_2(\eta_p z_p, w_p z_p)$$

in the constants (z_1, \dots, z_ν) must be positive on g if the second variation J_2 is to be positive along g . This proves the first part of the following theorem:

THEOREM 4:2. *If the extremal g is non-singular and the second variation J_2 is positive along g , then the quadratic form (4:5) must be positive on g . Moreover at each point x^3 on x^1x^2 the inequality (4:1) must hold for every set of constants (a_j, b_k) satisfying the equations (4:2), where η_{ij}, ζ_{ij} and u_{ik}, v_{ik} are conjugate systems of secondary extremals having $\eta_{ij}(x^1) = u_{ik}(x^2) = 0$.*

The last part of the theorem is obtained by applying Theorem 4:1 to the case in which the secondary end conditions are of the form $\eta_i^1 = 0, \eta_i^2 = 0$.

A value $x^3 \neq x^1$ is said to define a point 3 conjugate to 1 on g if there exists a secondary extremal $\eta_i = u_i(x), \mu_\beta = \mu_\beta(x)$ having $u_i(x^1) = u_i(x^3) = 0$ but not $(u) \equiv (0)$ on x^1x^3 .

The following necessary condition is a direct extension of a condition given by Bliss (IX, p. 266).

THEOREM 4:3. *If the extremal g is non-singular and the second variation J_2 is positive along g , then the quadratic form (4:5) must be positive on g . Moreover there can be no point 3 conjugate to 1 on g between its end points 1 and 2 defined by a secondary extremal $u_i(x), \rho_\beta(x)$ with $(u') \neq (0)$ at $x = x^3$. If the order q of anormality of g is the same on every sub-interval x^3x^2 of x^1x^2 , then there can be no point 3 conjugate to 1 on g between 1 and 2.*

For if there were a point 3 conjugate to 1 on g between 1 and 2 defined by a secondary extremal u_i, ρ_β , then along the arc

$$\eta_i = u_i(x) \quad (x^1 \leq x \leq x^3), \quad \eta_i = 0 \quad (x^3 \leq x \leq x^2), \quad w_h = 0$$

the second variation would take the value zero (V, p. 726). This arc would therefore be a minimizing arc for the accessory minimum problem and hence could have no corners, by Lemma 3:4. This proves the first statement concerning conjugate points.

In order to prove the last statement of the theorem we note that according to Lemma 3:4 the functions η_i just defined would belong to a secondary extremal η_i, μ_β . The functions η_i would then be identically zero on x^1x^2 since, as one easily sees, Lemma 3:2 and our assumption concerning anormality imply that a secondary extremal η_i, μ_β having $\eta_i \equiv 0$ on x^1x^2 has $\eta_i \equiv 0$ on the whole interval x^1x^2 . It follows that in this case there can be no point 3 on g conjugate to 1 between 1 and 2. This completes the proof of the theorem.

By the *accessory boundary value problem* is meant the equations

$$(4:6) \quad \begin{aligned} (d/dx)\Omega_{\eta_i} - \Omega_{\eta_i} + \sigma\eta_i &= 0, \quad \Phi_\beta = 0, \\ \eta_i &= c_{ih}w_h, \quad \zeta_i^2 c_{ih} - \zeta_i^1 c_{ih} + b_{hi}w_i = 0 \end{aligned} \quad (s = 1, 2),$$

where $\Omega = \omega + \mu_\beta \Phi_\beta$. A set of functions $\eta_i(x), \mu_\beta(x)$ having continuous derivatives $\eta_i', \eta_i'', \eta_\beta'$ and having $(\eta) \neq (0)$ on x^1x^2 is said to form a *characteristic solution* if it satisfies the equations (4:6) with a set of constants w_h, σ . The corresponding value σ is called a *characteristic root*.

We now have the further necessary condition:

THEOREM 4:4. *If the second variation $J_2(\eta, w)$ is positive along the extremal g then there can be no negative characteristic roots of the accessory boundary value problem.*

The proof of this theorem is well known (VIII, p. 524).

We also have the further necessary condition which is an analogue of the necessary condition of Clebsch:

THEOREM 4:5. *If the second variation $J_2(\eta, w)$ is positive along the extremal g , then at each element (x, y, y', λ) on g the inequality*

$$(4:7) \quad F_{y_i' y_k'} \pi_i \pi_k \geq 0$$

must hold for every set $(\pi) \neq (0)$ which is a solution of the equations $\Phi_{\beta y_i'} \pi_i = 0$. If g is non-singular then the condition (4:7) holds with the equality sign excluded.

According to the remarks preceding Lemma 3:4 we may suppose that g is normal. The first statement of the theorem can now be obtained by applying Theorem 2:2 to the accessory minimum problem and by the use of Taylor's expansion. The last statement follows readily from well known theorems on quadratic forms.

5. Criteria for conjugate points. A first criterion for conjugate points is the following one:

THEOREM 5:1. *If the extremal g is non-singular and if the functions u_{ij}, v_{ij} ($j=1, \dots, 2n$) form $2n$ linearly independent secondary extremals for g , then a value $x^3 \neq x^1$ defines a point 3 conjugate to 1 on g if and only if the matrix*

$$(5:1) \quad \begin{vmatrix} u_{ij}(x^3) \\ u_{ij}(x^1) \end{vmatrix} \quad (i = 1, \dots, n; j = 1, \dots, 2n)$$

has rank less than $2n-q$, where q is the order of anormality of g on the interval x^1x^3 .

The proof of this theorem can be made by the usual methods (V, p. 728) with the help of Corollary 3:2.

THEOREM 5:2. *If the extremal g is non-singular and the order q of anormality of g is the same on every sub-interval x^1x^3 of x^1x^2 , then there exists for g a conjugate system η_{ik}, ζ_{ik} of secondary extremals such that the points 3 conjugate to 1 on g are determined by the zeros $x^3 \neq x^1$ of the determinant $|\eta_{ik}|$.*

If $q=0$ then it suffices to choose the secondary extremals η_{ik}, ζ_{ik} which take the initial values $\eta_{ik}(x^1)=0, \zeta_{ik}(x^1)=\delta_{ik}$, where δ_{ik} is the Kronecker delta. This follows readily from Theorem 5:1 by choosing the first n secondary extremals of the set u_{ij}, v_{ij} to be the set η_{ik}, ζ_{ik} .

If $q>0$ we choose the first n secondary extremals of the set u_{ij}, v_{ij} of Theorem 5:1 such that $u_{ik}(x^1)=0$ ($k=1, \dots, n$), $u_{i\gamma}(x) \equiv 0$ ($\gamma=1, \dots, q$) on x^1x^2 , and such that the columns of the matrix $\|v_{ik}(x^1)\|$ are normed and orthogonalized. The second n secondary extremals of this set are chosen so as to take the initial values $u_{i,n+k}(x^1)=v_{ik}(x^1), v_{i,n+k}(x^1)=0$. The secondary extremals $\eta_{ik} \equiv u_{i,q+k}, \zeta_{ik} \equiv v_{i,q+k}$ can now be shown to have the properties described in the theorem. An examination of their values at $x=x^1$ will show that they are mutually conjugate. Moreover it is clear that the matrix (5:1) has rank $2n-q$ if and only if the matrix $\|\eta_{i\tau}\| = \|u_{i,q+\tau}\|$ ($\tau=1, \dots, n-q$) has rank $n-q$ at $x=x^3$. The theorem will now follow from Theorem 5:1 if we show that the determinant $|\eta_{ik}|$ is different from zero if and only if the matrix $\|\eta_{i\tau}\|$ has rank $n-q$. If the determinant $|\eta_{ik}|$ vanishes at a value $x=x^3$ then there exist constants a_k not all zero such that the equations $\eta_{ik}(x^3)a_k=0$ hold. By the use of Lemma 3:3 and by a consideration of the initial values of the secondary extremals under consideration it is found that

$$0 = v_{i\gamma}(x^3)\eta_{ik}(x^3)a_k = v_{i\gamma}(x^1)\eta_{ik}(x^1)a_k = a_{n-q+\gamma} \quad (\gamma = 1, \dots, q).$$

The matrix $\|\eta_{i\tau}\|$ ($\tau=1, \dots, n-q$) must therefore have rank less than $n-q$

whenever the determinant $|\eta_{ik}|$ vanishes. The converse is immediate, and the theorem is established.

6. A fundamental sufficiency theorem. The notion of a Mayer field \mathfrak{F} used here is that given by Bliss (V, p. 730). The slope functions and the multipliers belonging to \mathfrak{F} will be denoted by the symbols $p_i(x, y)$, $\lambda_\beta(x, y)$. The Hilbert integral

$$I^* = \int \{F(x, y, p, \lambda)dx + (dy_i - p_i dx)F_{y_i}(x, y, p, \lambda)\}$$

formed for these functions and $\lambda_0 = 1$ is independent of the path in \mathfrak{F} . The value of the integral I^* along an extremal of the field is equal to that of the integral

$$(6:1) \quad I = \int f(x, y, y')dx.$$

The Weierstrass E -function $E(x, y, p, \lambda, y')$ is the expression (2:3).

If g is an extremal of a Mayer field then the transversality condition (2:2) for g implies that the equation

$$(6:2) \quad [dI^*]_1^2 + d\theta = 0$$

is an identity in $d\alpha_h$ on g when the differentials $dx^1, dy^1, dx^2, dy^2, d\theta$ are expressed in terms of the differentials $d\alpha_h$. It follows readily that on g the second differential

$$(6:3) \quad [d^2I^*]_1^2 + d^2\theta$$

is a quadratic form in the variables $d\alpha_h$. With this in mind we can prove the following theorem:

THEOREM 6:1. *Let \mathfrak{F} be a Mayer field in which the inequality*

$$(6:4) \quad E[x, y, p(x, y), \lambda(x, y), y'] > 0$$

holds for every admissible set $(x, y, y') \neq (x, y, p)$. If g is an extremal of the field such that the equation (6:2) is an identity in $d\alpha_h$ on g and such that the quadratic form (6:3) is positive definite on g , then g affords a proper minimum to J relative to admissible arcs C in \mathfrak{F} with sets (α) near (0) .

Let A^1, A^2 be the arcs in \mathfrak{F} defined by the equations

$$(A^s) \quad x^s = x^s(t\alpha), \quad y_i^s = y_i^s(t\alpha) \quad (0 \leq t \leq 1; s = 1, 2)$$

for a set (α) near (0) , where the functions on the right are those appearing in equations (1:3). The condition (6:2) and the positive definiteness of the

quadratic form (6:3) tell us that the set $(\alpha) = (0)$ furnishes a proper minimum to the function

$$W(\alpha) = \theta(\alpha) - \theta(0) + I^*(A^2) - I^*(A^1)$$

relative to sets (α) near (0) .

Suppose now that the set (α) belongs to an admissible arc C in \mathfrak{F} . With the help of the formula (2:3) and the invariant property of the integral I^* it is found that

$$\begin{aligned} I(C) - I(g) &= \int_C E dx + I^*(C) - I^*(g) \\ &= \int_C E dx + I^*(A^2) - I^*(A^1), \end{aligned}$$

where I is the integral (6:1). When the expression $\theta(\alpha) - \theta(0)$ is added to both sides of the last equation, the formula

$$J(C) - J(g) = \int_C E dx + W(\alpha)$$

is obtained. Hence we have $J(C) \geq J(g)$ provided that the set (α) belonging to the arc C is near (0) . The equality holds only in case $(\alpha) = (0)$ and the integral of the E -function vanishes, that is, only in case the ends of C coincide with those of g and the equations $y'_i - p_i = 0$ hold along C . The arc C would then be an extremal of the field and would coincide with g since there is but one extremal of the field through each point of \mathfrak{F} (cf. V, pp. 731-2).

In the sequel we shall apply Theorem 6:1 only to the problem of Bolza with separated end points. If the end conditions are not of the form (1:4) then it is not always possible to construct a field such that the quadratic form (6:3) is positive definite on g . This can be seen by considering the special problem in (xy_1y_2) -space for which $\theta = \alpha$, $f \equiv 0$, $x^1(\alpha) = 0$, $y^1_1(\alpha) = 0$, $y^1_2(\alpha) = -\alpha$, $x^2(\alpha) = 1$, $y^2_1(\alpha) = 0$, $y^2_2(\alpha) = \alpha$, and $\phi_1 = (1 + y_1'^2)^{1/2} - y_2' = 0$. The sufficient conditions given in §9 below, however, are applicable to this problem.

7. Three lemmas. Consider first the problem of Bolza with separated end conditions. Suppose that g is non-singular and that the secondary end conditions are regular on g . The arc g will be said to satisfy the *condition IV'* if at each point 3 on g the inequality (4:1) holds subject to the conditions (4:2) and if furthermore the matrix

$$(7:1) \quad \|\xi_{ij}u_{ik} - \eta_{ij}v_{ik}\|$$

of the coefficients in the bilinear form (4:1) has rank $n - p$ on g , where p is the order of anormality of g . The matrix (7:1) has rank $n - p$ on g if and

only if the equations (3:4), (3:6), (3:7) have no solution (η, ζ, w) other than those described in Lemma 3:1. In the fixed end point case the matrix (7:1) has rank $n-p$ if and only if the end points of g are not conjugate to each other, as is readily verified.

LEMMA 7:1. *If in the separated end point case the extremal g is non-singular and satisfies the condition IV' and if the secondary end conditions are regular on g , then there exists for g a conjugate system U_{ik}, V_{ik} of secondary extremals whose determinant $|U_{ik}(x)|$ is different from zero on the interval $x^1 \leq x \leq x^2$ determined by the end points 1 and 2 of g . Moreover the inequalities*

$$(7:2) \quad \begin{aligned} & b_{\mu\nu}^1 z_\mu z_\nu - U_{ik}(x^1) V_{ij}(x^1) a_j a_k > 0 \quad (\mu, \nu = 1, \dots, \rho; i, j, k = 1, \dots, n), \\ & - b_{\sigma\tau}^2 z_\sigma z_\tau + U_{ik}(x^2) V_{ij}(x^2) b_j b_k > 0 \quad (\sigma, \tau = \rho + 1, \dots, r) \end{aligned}$$

hold for every set of constants $(a_k, z_\mu) \neq (0, 0)$ and $(b_k, z_\sigma) \neq (0, 0)$ satisfying the equations

$$(7:3) \quad U_{ik}(x^1) a_k = c_{i\mu}^1 z_\mu, \quad U_{ik}(x^2) b_k = c_{i\sigma}^2 z_\sigma.$$

For, if p is the order of anormality of g , then by virtue of Lemma 3:1 we can select the first p solutions of the sets (3:8) and (3:9) so that on $x^1 x^2$ we have

$$(7:4) \quad \eta_{i\gamma}(x) \equiv u_{i\gamma}(x) \equiv 0, \quad \zeta_{i\gamma}(x) \equiv v_{i\gamma}(x) \quad (\gamma = 1, \dots, p)$$

and so that the columns of the matrix $\|\zeta_{i\gamma}(x^1)\|$ are normed and orthogonalized. We then select the remaining solutions of these sets so that the relations

$$(7:5) \quad \zeta_{i\gamma}(x^1) \zeta_{i\alpha}(x^1) = 0, \quad \zeta_{i\gamma}(x^1) v_{i\alpha}(x^1) = v_{i\gamma}(x^1) v_{i\alpha}(x^1) = 0,$$

$$(7:6) \quad \zeta_{i\alpha} u_{i\beta} - \eta_{i\alpha} v_{i\beta} = \delta_{\alpha\beta} \quad (\alpha, \beta = p+1, \dots, n)$$

hold, where $\delta_{\alpha\beta}$ is the Kronecker delta. In order to obtain the relations (7:6) we note that, since the conjugate systems η_{ik}, ζ_{ik} and u_{ik}, v_{ik} have the secondary extremals (7:4) in common, it follows that

$$(7:7) \quad \zeta_{i\gamma} u_{ik} - \eta_{i\gamma} v_{ik} = 0, \quad \zeta_{ik} u_{i\gamma} - \eta_{ik} v_{i\gamma} = 0 \quad (\gamma = 1, \dots, p).$$

The determinant

$$(7:8) \quad |\zeta_{i\alpha} u_{i\beta} - \eta_{i\alpha} v_{i\beta}| \quad (\alpha, \beta = p+1, \dots, n)$$

must therefore be different from zero if the matrix (7:1) is to have rank $n-p$. The relations (7:6) are now obtained by replacing the solutions $\eta_{i\alpha}, \zeta_{i\alpha}, w_{\mu\alpha}$ by the solutions $\eta_{i\alpha} A_{\alpha\beta}, \zeta_{i\alpha} A_{\alpha\beta}, w_{\mu\alpha} A_{\alpha\beta}$, where the matrix $\|A_{\alpha\beta}\|$ is the reciprocal of the matrix (7:8).

The secondary extremals U_{ik}, V_{ik} taking the initial values

$$(7:9) \quad \begin{aligned} U_{i\gamma}(x^1) &= \zeta_{i\gamma}(x^1), & U_{i\alpha}(x^1) &= \eta_{i\alpha}(x^1) + u_{i\alpha}(x^1) \quad (\gamma = 1, \dots, p), \\ V_{i\gamma}(x^1) &= 0, & V_{i\alpha}(x^1) &= \zeta_{i\alpha}(x^1) + v_{i\alpha}(x^1) \quad (\alpha = p+1, \dots, n) \end{aligned}$$

can be shown to have the properties described in the theorem. In the first place these secondary extremals are mutually conjugate, as is easily seen, with the help of equations (7:5), (7:6), (7:9) and the conjugacy of the systems η_{ik} , ζ_{ik} and u_{ik} , v_{ik} . Moreover the determinant $|U_{ik}(x)|$ is different from zero on x^1x^2 . In order to prove this we use the relations

$$(7:10) \quad \zeta_{i\gamma}U_{i\epsilon} = \delta_{\gamma\epsilon}, \quad \zeta_{i\gamma}U_{i\alpha} = 0 \quad (\gamma, \epsilon = 1, \dots, p; \alpha = p+1, \dots, n)$$

which hold identically on x^1x^2 by virtue of Lemma 3:3 and the equations (7:4), (7:7), (7:9) together with the fact that the columns of the matrix $||\zeta_{i\gamma}(x^1)||$ are normed and orthogonalized. If now the determinant $|U_{ik}(x)|$ were zero at a point x^3 on x^1x^2 then there would exist constants c_k not all zero such that $U_{ik}(x^3)c_k = 0$. By multiplying these equations by $\zeta_{i\gamma}(x^3)$, adding, and using equations (7:10) it would follow that the constants c_1, \dots, c_p would all be zero and hence that

$$U_{i\alpha}(x^3)c_\alpha = \eta_{i\alpha}(x^3)c_\alpha + u_{i\alpha}(x^3)c_\alpha = 0 \quad (\alpha = p+1, \dots, n).$$

The equations (4:2) would then be satisfied by the set $a_k = c_k$, $b_k = -c_k$ and for these constants the bilinear form (4:1) would take the value $a_\alpha b_\alpha = -c_\alpha c_\alpha < 0$ by virtue of the equations (7:6) and (7:7). But this would contradict the condition IV'. The determinant $|U_{ik}|$ must therefore be different from zero on x^1x^2 .

We shall now establish the first of the inequalities (7:2). In order to do this we first note that the constants a_1, \dots, a_p in equations (7:3) are all zero, as can be easily seen, by multiplying the first n of these equations by $\zeta_{i\gamma}(x^1)$, adding, and applying the equations (7:10) and the analogue of equations (3:11) for the separated end point case. We use the abbreviations

$$(7:11) \quad \begin{aligned} \eta_i &= \eta_{i\alpha}a_\alpha, & \zeta_i &= \zeta_{i\alpha}a_\alpha, & w_\mu &= w_{\mu\alpha}a_\alpha, \\ u_i &= u_{i\alpha}a_\alpha, & v_i &= v_{i\alpha}a_\alpha, & z_\mu &= w_\mu + w'_\mu \quad (\alpha = p+1, \dots, n) \end{aligned}$$

and find that the set η_i, ζ_i, w_μ satisfies the equations (3:6). Moreover by the use of equations (3:6) and (7:3) it follows readily that at $x = x^1$

$$\begin{aligned} U_{ik}a_k - c_{i\mu}z_\mu &= u_i - c_{i\mu}w'_\mu = 0 \\ b_{\mu\nu}w_\nu w'_\mu &= \zeta_i c_{i\mu}w'_\mu = \zeta_i u_i, \\ b_{\mu\nu}w_\nu w_\mu &= \zeta_i c_{i\mu}w_\mu = \zeta_i \eta_i. \end{aligned}$$

With the help of the last two formulas and the equations (7:11) it is found that the first member of the relations (7:2) is expressible in the form

$$\begin{aligned} b_{\mu\nu}^1(w_\mu + w'_\mu)(w_\nu + w'_\nu) - (\eta_i + u_i)(\zeta_i + v_i) \\ = [b_{\mu\nu}^1 w'_\mu w'_\nu - u_i v_i] + [\zeta_i u_i - \eta_i v_i]. \end{aligned}$$

The second bracket is equal to the sum $a_\alpha a_\alpha$ by virtue of equations (7:6) and (7:11) and is positive unless the constants a_α are all zero, in which case the constants z_μ in equations (7:3) are also all zero since the secondary end conditions are regular. The first bracket in the last equation is positive or zero. For, as a consequence of the regularity of the secondary end conditions there exists for every set of constants w'_μ a secondary extremal η_{i0}, ζ_{i0} satisfying the conditions (3:6) with $w_\mu = w'_\mu$. The set $\eta_{i0}, \zeta_{i0}, w'_\mu$ is expressible linearly with constants c_k in terms of the set (3:8). Moreover it is clear that $\eta_{i0}(x^1) = u_i(x^1)$. Hence we have at $x = x^1$

$$\begin{aligned} b_{\mu\nu}^1 w'_\mu w'_\nu - u_i v_i &= \zeta_{i0} \eta_{i0} - u_i v_i \\ &= \zeta_{i0} u_i - \eta_{i0} v_i = (\zeta_{ij} u_{ik} - \eta_{ij} v_{ik}) c_{jk} \end{aligned}$$

and this expression must be positive or zero by IV'. This proves the first inequality (7:2). The second can be established by the same method. The proof of Lemma 7:1 is now complete

We also have the further useful lemma:

LEMMA 7:2. *If the extremal g is non-singular and its end points are not conjugate to each other, then the end points of every differentially admissible arc g_2 for the accessory minimum problem can be joined by a secondary extremal.*

To prove this let q be the order of anormality of g on $x^1 x^2$ and suppose that the first q secondary extremals $u_{i\gamma}, v_{i\gamma}$ ($\gamma = 1, \dots, q$) of the set u_{ij}, v_{ij} ($j = 1, \dots, 2n$) appearing in Theorem 5:1 have been chosen so that $u_{i\gamma} \equiv 0$ on $x^1 x^2$. Since the end points of g are not conjugate, the end values of the remaining $2n - q$ secondary extremals of this set form a set of $2n - q$ linearly independent solutions of the equations

$$v_{i\gamma}(x^2) \eta_i^2 = v_{i\gamma}(x^1) \eta_i^1 \quad (\gamma = 1, \dots, q),$$

by Lemma 3:3, and every solution η_i^1, η_i^2 of these equations is expressible linearly in terms of these $2n - q$ solutions. The end points of g_2 satisfy these equations, by Lemma 3:3. This proves Lemma 7:2.

By the *Clebsch condition III'* is meant the conditions of Theorem 4:5 with the equality sign excluded. The condition III' for g implies that g is non-singular (V, p. 735). We can now prove the following lemma:

LEMMA 7:3. *If the extremal g satisfies the condition III' and if there exists for g a conjugate system U_{ik}, V_{ik} of secondary extremals whose determinant $|U_{ik}(x)|$ is different from zero on x^1x^2 , then every secondary extremal u_i, v_i is an extremal of a Mayer field defined over the region \mathfrak{F}_Ω of points (x, η) whose x -projections lie on the interval x^1x^2 . Moreover the analogue of the condition (6:4) holds in \mathfrak{F}_Ω .*

For, the n -parameter family of secondary extremals

$$(7:12) \quad \eta_i = u_i + U_{ik}a_k, \quad \zeta_i = v_i + V_{ik}a_k$$

contains the extremal u_i, v_i for values $(a) = (0)$ and simply covers the region \mathfrak{F}_Ω . Moreover the Hilbert integral I^* formed for the function 2Ω is independent of the path on the hyperplane $x = x^2$. The family (7:12) therefore defines a field over \mathfrak{F}_Ω (V, p. 733; VII, p. 571). The last statement in the lemma follows at once from the condition III' by the use of Taylor's expansion.

8. Necessary and sufficient conditions for the second variation to be positive definite. The second variation J_2 is said to be *positive definite* along the extremal g if the inequality $J_2(\eta, w) > 0$ is true for every set of admissible variations $(\eta, w) \neq (0, 0)$ belonging to g .

THEOREM 8:1. *If in the separated end point case the extremal g is non-singular and the secondary end conditions are regular on g , then the second variation $J_2(\eta, w)$ is positive definite along g if and only if the conditions III' and IV' hold along g .*

The necessity of the conditions III' and IV' follows at once from Theorems 4:1, 4:5 and the remarks preceding Lemma 7:1. The sufficiency of these conditions follows readily from Lemmas 7:1, 7:3 and Theorem 6:1 applied to the secondary extremal $u_i = v_i = 0$, the conditions (7:2) and (7:3) implying the positive definiteness of the analogue of the quadratic form (6:3).

We now turn to the case in which the end conditions are of the form (1:3). By the condition V' is meant the necessary conditions of Theorem 4:2 with the added assumption that the equation $Q(z) = 0$ holds only in case $\eta_{ip}z_p = 0$, $w_{hp}z_p = 0$ on x^1x^2 . The condition V' for g prevents its end points from being conjugate to each other. For if the end points 1 and 2 of g were conjugate then there would exist a secondary extremal η_i, μ_β with $\eta_i(x^1) = \eta_i(x^2) = 0$ and $(\eta) \neq (0)$ on x^1x^2 . The set $\eta_i, \mu_\beta, w_h = 0$ would then be expressible linearly with constants z_p in terms of the set $\eta_{ip}, \mu_{\beta p}, w_{hp}$ appearing in the definition of the quadratic form (4:5). For these values of (z) we would have $Q(z) = 0$, as is easily seen, with the help of the formula (4:4) and the usual integration by

parts. It follows that the end points of g cannot be conjugate if the condition V' is to hold along g .

THEOREM 8:2. *If the extremal g is non-singular then the second variation $J_2(\eta, w)$ is positive definite along g if and only if the conditions III' and V' hold along g .*

It is clear that the conditions III', V' are necessary. In order to show that they are sufficient we note first that the condition V' for g implies the condition IV' for the fixed end point case. Lemma 7:1 now tells us that there exists a conjugate system U_{ik}, V_{ik} of secondary extremals whose determinant $|U_{ik}(x)|$ is different from zero on x^1x^2 . From Lemma 7:3 and Theorem 6:1 we conclude that every secondary extremal u_i, v_i affords a proper minimum to the integral

$$I_2 = \int_{x^1}^{x^2} 2\omega(x, \eta, \eta') dx$$

relative to differentially admissible arcs $\eta_i(x)$ joining its end points.

Suppose now that η_i, w_h is an admissible arc for the accessory minimum problem. By Lemma 7:2 there exists a secondary extremal u_i, v_i joining its end points. We have accordingly

$$(8:1) \quad J_2(\eta, w) - J_2(u, w) = I_2(\eta) - I_2(u) \geq 0,$$

the equality being valid only in case $(\eta) \equiv (u)$. From the definition of the quadratic form $Q(z)$ it is clear that there exist constants z_p such that $Q(z) = J_2(u, w)$. From the condition V' and the relation (8:1) we now conclude that $J_2(\eta, w) > 0$ unless $(\eta, w) \equiv (0, 0)$, as was to be proved.

The extremal g will be said to satisfy the condition VI' if the quadratic form (4:5) is positive on g and vanishes only in case $\eta_{ip}z_p \equiv 0, w_{hp}z_p = 0$ on x^1x^2 , and if furthermore there is no point 3 conjugate to the initial point 1 on g . We can now prove the following theorem:

THEOREM 8:3. *If the extremal g is non-singular and the order q of anormality of g is the same on every sub-interval x^1x^3 of x^1x^2 , then the second variation $J_2(\eta, w)$ is positive definite along g if and only if the conditions III', VI' hold along g .*

The proof of this theorem is like that of Theorem 8:2 provided that we can show that there exists for g a conjugate system U_{ik}, V_{ik} of secondary extremals having its determinant $|U_{ik}(x)|$ different from zero on x^1x^2 . This latter result will be obtained by a method first used by Morse (VII, pp. 574-6) for the problem of Lagrange and later adapted to the problem of

Mayer by Bliss and Hestenes (XVII, pp. 320-2). In the proof we suppose that the conjugate system η_{ik}, ζ_{ik} of Theorem 5:2 has been chosen to take the values δ_{ik}, B_{ik} at $x=x^2$, where δ_{ik} is the Kronecker delta and $B_{ik}=B_{ki}$. "Lemma 8:2" of Bliss and Hestenes now holds as before. Similarly "Lemma 8:3" is true, as is easily seen with the help of the following remarks. Although a secondary extremal η_i, ζ_i joining the points $(x, \eta) = (x^1, 0)$ and $(x, \eta) = (x^2, a)$ is not necessarily an extremal of the field it has associated with it a secondary extremal $\eta_i - c_\gamma u_{i\gamma}, \zeta_i - c_\gamma v_{i\gamma}$ ($\gamma=1, \dots, q$) belonging to the field, where q is the order of anormality of g on x^1x^2 and $u_{i\gamma}, v_{i\gamma}$ are q linearly independent secondary extremals having $u_{i\gamma} \equiv 0$ on x^1x^2 . Moreover the values of the integral " I_2 " along these two extremals are the same. The remainder of the proof is now like that of "Theorem 8:1" of Bliss and Hestenes.

We now turn to the accessory boundary value problem. Its characteristic roots are all real (XIV, p. 774; XIX, p. 394). We have the following theorem:

THEOREM 8:4. *If the extremal g is non-singular and the secondary end conditions are regular on g , then the second variation $J_2(\eta, w)$ is positive definite (positive) along g if and only if the condition III' holds along g and the characteristic roots of the accessory boundary value problem are all positive (non-negative).*

According to the remarks preceding Lemma 3:4 we may suppose that g is normal. The theorem then follows from a result given by Hu (XIX, p. 413).

The theorem can also be established with the help of the condition V'. A method will be outlined briefly as follows. We first replace the integrand 2ω in the functional $J_2(\eta, w)$ by $2\omega - \sigma\eta_i\eta_i$ and obtain a functional $J_2(\eta, w, \sigma)$. The preceding theorems concerning the functional $J_2(\eta, w)$ are valid also for the functional $J_2(\eta, w, \sigma)$ when the obvious changes due to the introduction of the parameter σ are made. By an argument like that given by Morse (VIII, pp. 533-4) it is found that for σ sufficiently large and negative the functional $J_2(\eta, w, \sigma)$ will be positive definite relative to sets of admissible variations $(\eta, w) \neq (0, 0)$. Let σ_0 be the least upper bound of the values of σ for which $J_2(\eta, w, \sigma)$ is positive definite. It will be shown below that σ_0 must be finite. We shall now show that σ_0 is a characteristic root. The functional $J_2(\eta, w, \sigma_0)$ must be positive since otherwise there would exist an admissible arc η_i, w_k such that $J_2(\eta, w, \sigma) < 0$ for $\sigma = \sigma_0$ and hence for $\sigma < \sigma_0$ and sufficiently near to σ_0 , which is not the case. If the functional $J_2(\eta, w, \sigma_0)$ were positive definite then by Theorem 8:2 the condition V' would hold for this functional. By the use of Lemma 3:2 applied to sub-intervals of the form x^1x^3 and x^3x^2 and with the help of well known theorems on quadratic forms it could then be shown that the condition V' would hold for the functional $J_2(\eta, w, \sigma)$ for values of σ slightly larger than σ_0 . The functional $J_2(\eta, w, \sigma)$ would then be positive definite for these values of σ , by Theorem 8:2, and σ_0

could not be the least upper bound for such values of σ . It follows that there exists at least one admissible arc η_i, w_h with $(\eta) \neq (0)$ on $x^1 x^2$ such that $J_2(\eta, w, \sigma_0) = 0$. As in the proof of Lemma 3:4 it is seen that this arc has associated with it a set of multipliers $\mu_0 = 1, \mu_\beta(x)$ such that the functions η_i, μ_β define a secondary extremal for the problem of minimizing the functional $J_2(\eta, w, \sigma_0)$ in the class of admissible variations η_i, w_h belonging to g . The functions η_i, μ_β therefore form a characteristic solution and σ_0 a characteristic root.

In order to show that σ_0 is finite we note that there exists at least one set of admissible variations η_i, w_h having $(\eta) \neq (0)$ on $x^1 x^2$ since the accessory minimum problem can be made normal. For this set the functional $J_2(\eta, w, \sigma)$ can be made negative by taking σ sufficiently large and positive. Consequently σ_0 must be finite. This proves the theorem.

9. Sufficient conditions for relative minima. The end conditions (1:3) are said to be *regular* on the admissible arc g under consideration if the matrix of the derivatives of the functions $x^s(\alpha), y_i^s(\alpha)$ has rank r for $(\alpha) = (0)$. The arc g is said to satisfy the *non-tangency condition* if the manifold $y_i^s = y_i(x^s)$ ($s = 1, 2$) and the terminal manifold $x^s = x^s(\alpha), y_i^s = y_i^s(\alpha)$ possess no common tangent line at the point $(\alpha) = (0)$ on the terminal manifold. The end conditions are regular and the non-tangency condition holds on g if and only if the secondary end conditions (3:2) are regular on g (VIII, pp. 525-6). No generality is lost in assuming that the end conditions are regular and the non-tangency condition holds on g , as can be seen from the proof of Theorem 9:2 below.

The symbol I will be used to denote the necessary condition of Theorem 2:1. An admissible arc g with a set of multipliers $\lambda_0, \lambda_\beta(x)$ is said to satisfy the *Weierstrass condition* II' \mathfrak{H} if at each element (x, y, y', λ) in a neighborhood \mathfrak{H} of those on g the inequality

$$E[x, y, y', \lambda, Y'] > 0$$

holds for every admissible set $(x, y, Y') \neq (x, y, y')$. The Clebsch condition III' and the conditions IV', V', VI' have been described in §§7 and 8. The last three conditions can readily be expressed in terms of the extremal family in a manner analogous to that given by Bliss (IX, pp. 265-6; cf. XVIII, p. 483).

THEOREM 9:1. *Let g be an admissible arc for the problem of Bolza with separated end conditions. Suppose that the end conditions (1:4) are regular and that the non-tangency condition holds on g . If g has no corners and satisfies the conditions I, II' \mathfrak{H} , III', IV' with a set of multipliers $\lambda_0 = 1, \lambda_\beta(x)$, then g affords a proper strong relative minimum to the functional J .*

From the conditions I and III' we conclude that g is a non-singular extremal since it has no corners (V, p. 735). The theorem will now be established by showing that the hypotheses of Theorem 6:1 are fulfilled.

As a first step we note that g is a member for values $x^1 \leq x \leq x^2$, $a_i = a_{i0}$ ($i = 1, \dots, n$) of an n -parameter family of extremals whose equations in the canonical variables $x, y_i, z_i = F_{y_i'}$, are of the form

$$(9:1) \quad y_i = y_i(x, a_1, \dots, a_n), \quad z_i = z_i(x, a_1, \dots, a_n).$$

The functions y_i, y_{ix}, z_i, z_{ix} have continuous first and second derivatives for all values (x, a) in a neighborhood of those belonging to g . The parameters (a) can be chosen so that along g

$$(9:2) \quad y_{ia_k}(x, a_0) \equiv U_{ik}(x), \quad z_{ia_k}(x, a_0) \equiv V_{ik}(x),$$

where the functions U_{ik}, V_{ik} are secondary extremals having the properties described in Lemma 7:1. Moreover the family of extremals (6:1) defines a Mayer field over a neighborhood \mathfrak{F} of g . The proof of the existence of such a family is like that given by Bliss and Hestenes (XVII, pp. 322-3) and by Morse (VII, p. 576) with help of Lemma 7:1. Let $p_i(x, y), \lambda_\beta(x, y)$ be the slope functions and the multipliers of the field. It is clear that the field \mathfrak{F} can be taken so small that the elements $[x, y, p(x, y), \lambda(x, y)]$ will lie in the neighborhood \mathfrak{R} specified by the condition II \mathfrak{R}' . The inequality (6:4) then holds at each point in \mathfrak{F} .

The identity (6:2) follows at once from the transversality condition (2:2). In order to show that the quadratic form (6:3) is positive definite on g it is convenient to express I^* in terms of the variables x, a_1, \dots, a_n instead of the variables x, y_1, \dots, y_n . In doing so we replace the functions $p_i(x, y), \lambda_\beta(x, y)$ by the functions $y_{ix}(x, a), \lambda_\beta(x, a)$, where $\lambda_\beta(x, a)$ are the multipliers belonging to the family (9:1). We use the following abbreviations:

$$\begin{aligned} \delta y_i &= y_{ia_k} da_k, & \delta z_i &= z_{ia_k} da_k, & dy_i &= y_{ix} dx + \delta y_i, \\ dy_i' &= y_{ixx} dx + \delta y_i', & d^2 y_i &= y_{ixx} d^2 x + dy_i' dx + \delta y_i' dx + \delta^2 y_i. \end{aligned}$$

With the help of the Euler-Lagrange equations (2:3) it is found that

$$d(F_{y_i'}) = F_{y_i} dx + \delta z_i.$$

Hence at the initial point 1 on g we have

$$\begin{aligned} -dI^* + d\theta^1 &= -(F - y_i' F_{y_i'}) dx - F_{y_i} dy_i + d\theta^1, \\ -d^2 I^* + d^2 \theta^1 &= -(F - y_i' F_{y_i'}) d^2 x - F_{y_i} d^2 y_i - (F_{xx} - y_i' F_{xy_i})(dx)^2 \\ &\quad - 2F_{xy_i} dy_i dx - \delta z_i \delta y_i + d^2 \theta^1. \end{aligned}$$

If now we replace the first and second differentials of x and y , by their values in terms of the differentials of α_μ it is found with the help of equations (3:5), (9:2), and Lemma 7:1 that the inequality

$$-d^2I^* + d^2\theta^1 = b_{\mu\nu}^1 d\alpha_\mu d\alpha_\nu - \delta y_i \delta z_i > 0 \quad (\mu, \nu = 1, \dots, \rho)$$

holds at the point 1 on g for every set $(da_k, d\alpha_\mu) \neq (0, 0)$ satisfying the conditions $\delta y_i(x^1) = c_{i\mu}^1 d\alpha_\mu$. By a similar argument it can be shown that at the final end point 2 on g the inequality

$$d^2I^* - d^2\theta^2 = -b_{\sigma\tau}^2 d\alpha_\sigma d\alpha_\tau + \delta y_i \delta z_i > 0 \quad (\sigma, \tau = \rho + 1, \dots, r)$$

holds for every set $(da_k, d\alpha_\mu) \neq (0, 0)$ satisfying the equations $\delta y_i(x^2) = c_{i\sigma}^2 d\alpha_\sigma$. The last two inequalities show that the quadratic form (6:3) is positive definite on g . Theorem 6:1 now justifies the theorem that was to be proved.

Theorem 9:1 will now be used in order to obtain sufficient conditions for the problem of Bolza with end conditions of the type (1:3). In the following theorem it should be noted that the assumptions of regularity of the end conditions and of non-tangency are not needed.

THEOREM 9:2. *If an admissible arc g without corners satisfies the conditions I, II', III' with a set of multipliers $\lambda_0=1, \lambda_\rho(x)$ and if the second variation $J_2(\eta, w)$ is positive definite along g , then g affords a proper strong relative minimum to the functional J .*

In order to prove the theorem we note first that a problem of Bolza with end conditions of the form (1:3) is equivalent to the problem of finding in the class of arcs

$$y_j = y_j(x) \quad (x^1 \leq x \leq x^2; j = 1, \dots, n+r)$$

and sets α_h, γ_h ($h=1, \dots, r$) satisfying the differential equations and end conditions

$$\begin{aligned} \phi_\theta(x, y_i, y_i') &= 0, & y_{n+h}' &= 0 & (i = 1, \dots, n; h = 1, \dots, r), \\ x^1 &= x^1(\alpha), & y_i^1 &= y_i^1(\alpha), & y_{n+h}^1 &= \alpha_h, \\ x^2 &= x^2(\gamma), & y_i^2 &= y_i^2(\gamma), & y_{n+h}^2 &= \gamma_h \end{aligned}$$

one which minimizes the functional J . This equivalence follows from the fact that the functions $y_{n+h}(x)$ are all constants and hence take the values $\alpha_h = \gamma_h$ at $x = x^1$ and $x = x^2$. The new problem is a problem of Bolza with separated end conditions and will be called the *transformed problem*. Let g_1 be the admissible arc for the transformed problem which corresponds to the arc g of the theorem. It is easily seen that the new end conditions are regular and that the non-tangency condition holds on g_1 . The arc g_1 also satisfies the con-

ditions I, II \mathfrak{N}' , III' for the transformed problem with the set of multipliers $\lambda_0=1, \lambda_\beta(x), \lambda_{m+\lambda}(x)$, where the multipliers $\lambda_{m+\lambda}(x)$ are constants determined by the transversality condition (2:2). Moreover there is one-to-one correspondence between the admissible variations for the two problems and along corresponding admissible variations the values of the second variation for the two problems are the same. Theorems 8:1 and 9:1 therefore tell us that g_1 furnishes a proper strong relative minimum for the transformed problem and hence that g furnishes a proper strong relative minimum for the original problem, as was to be proved.

From Theorems 8:2 and 9:2 we obtain the following result:

THEOREM 9:3. *If an admissible arc g without corners satisfies the conditions I', II \mathfrak{N}' , III', V' with a set of multipliers $\lambda_0=1, \lambda_\beta(x)$, then g affords a proper strong relative minimum to the functional J .*

Combining Theorems 8:3 and 9:2 we obtain the further

THEOREM 9:4. *The results of Theorem 9:3 remain valid when the condition V' is replaced by the condition VI' provided that the order q of anormality of g is the same on every sub-interval x^1x^3 of the interval x^1x^2 determined by the end points of g .*

The last two theorems are extensions of the sufficient conditions given by Bliss (IX, p. 271). The following theorem gives an extension of the sufficient conditions given by Morse (VIII, p. 528) and Hu (XIX, p. 417) and is obtained by combining Theorems 8:4 and 9:2.

THEOREM 9:5. *Suppose that the end conditions are regular and that the non-tangency condition holds on an admissible arc g having no corners. If g satisfies the conditions I, II \mathfrak{N}' , III' with a set of multipliers $\lambda_0=1, \lambda_\beta(x)$ and if the characteristic roots of the accessory boundary value problem are all positive, then g affords a proper strong relative minimum to the functional J .*

Sufficient conditions for a weak relative minimum can be obtained in the usual manner by omitting the condition II \mathfrak{N}' in the above theorems (V, pp. 736-7).

The following example shows clearly that the sufficient conditions here given actually do not imply the normality relations which we have proposed to exclude. Let h be a small positive constant and let $A(x), B(x)$ be functions satisfying the conditions

$$\begin{aligned} A(x) &> 0 \text{ on } x^1 \leq x < x^1 + h, & A(x) &\equiv 0 \text{ on } x^1 + h \leq x \leq x^2, \\ B(x) &\equiv 0 \text{ on } x^1 \leq x \leq x^2 - h, & B(x) &> 0 \text{ on } x^2 - h < x \leq x^2 \end{aligned}$$

and having continuous derivatives of the first three orders. The segment g of the x -axis between x^1 and x^2 then furnishes a proper strong minimum to the integral

$$J = \int_{x^1}^{x^2} (1 + y_1'^2)^{1/2} dx$$

in the class of arcs (1:1) with $n=4$ satisfying the differential equations

$$y_2' = y_1' + A(x)y_1, \quad y_3' = y_1' + B(x)y_1, \quad y_4' = 0$$

and joining the two fixed points $(x, y) = (x^1, 0)$ and $(x, y) = (x^2, 0)$. The order p of anormality of g is readily found to be unity. The order q of anormality of g is unity on every sub-interval $x'x''$ satisfying the conditions $x^1 \leq x' < x^1 + h$, $x^2 - h < x'' \leq x^2$. If one of these conditions holds, then $q=2$. If neither holds, then $q=3$. It follows that the sufficient conditions given heretofore are not applicable to g . However g satisfies the sufficient conditions here given with the set of multipliers $\lambda_0 = 1$, $\lambda_g(x) \equiv 0$, except for those in Theorem 9:4.

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GROUPS IN WHICH THE SQUARES OF THE ELEMENTS ARE A DIHEDRAL SUBGROUP*

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1. The dihedral subgroup is non-abelian. If a group G has the property that the squares of its operators constitute a given group then the direct product of G and any abelian group of order 2^n and of type $(1, 1, 1, \dots)$ has the same property. Such direct products will always be excluded in what follows. It has been noted that a necessary and sufficient condition that there is at least one group G which has the property that the squares of its operators constitute a given non-abelian dihedral group H is that the order h of H is not divisible by 8 and that every odd prime number which divides h is congruent to unity modulo 4.† To determine the number of these groups when h is given and satisfies these conditions we let n represent the number of the different prime numbers which divide h . Since the cyclic subgroup K of index 2 contained in H is the direct product of its Sylow subgroups and all the groups of isomorphisms of these Sylow subgroups are cyclic it results that when h is twice an odd number there is one and only one group of order $h \cdot 2^{n-2}$ which satisfies the condition that the squares of its operators constitute K .

The given group of order $h \cdot 2^{n-2}$ is the largest group which has the property that the squares of its operators constitute K when h is twice an odd number, as will be assumed until the contrary is explicitly stated. It is merely the direct product of $n-1$ dihedral groups, each of order twice the power of an odd prime. The commutator subgroup of every G is K and the corresponding quotient group is the abelian group of type $(2, 1, 1, \dots)$. This results directly from the fact that if two operators of G have squares which are contained in K then the square of their product is also contained therein since this square could not be an operator of order 2 in H in view of the fact that this product could not transform this cyclic subgroup according to an operator of order 4.

It should be emphasized that an operator of G which transforms some of the operators of the cyclic subgroup of index 2 in H according to an operator of order 2 does not necessarily transform all the operators of this subgroup, besides the identity, according to such an operator, but that every operator

* Presented to the Society, September 7, 1934; received by the editors April 13, 1934.

† G. A. Miller, *Proceedings of the National Academy of Sciences*, vol. 20 (1934), p. 129.

of G which transforms an operator of this cyclic subgroup according to an operator of order 4 transforms each of these operators besides the identity, according to such an operator. Hence G involves a subgroup of index 2 composed of all of its operators which transform the operators of the given cyclic subgroup either into themselves or into their inverses. Each of the remaining operators of G is of order 4 and transforms the operators of odd order in H according to one of the 2^{n-1} ways in which these operators can be transformed when they are transformed according to an operator of order 4. Since such an operator and its inverse transform the operators of odd order differently it results that we have to consider only 2^{n-2} of these different possible transformations.

The largest possible order of G is $h \cdot 2^{n-1}$ and there is one and only one G of this order. It involves as a subgroup of index 2 the given group of order $h \cdot 2^{n-2}$ composed of all the operators whose squares are the operators of odd order in H . To determine all the possible G 's it is desirable to note the different possible subgroups composed of the operators whose squares constitute the operators of odd order in H . If such a subgroup is of order $h \cdot 2^{n-2-a}$ the number of the possible G 's which involve it is 2^a since there are 2^a sets of operators of order 4 which can be added to it to obtain a G having the required properties and each of these sets transforms the operators of odd order in H in a different way. The number of the possible subgroups of order $h \cdot 2^{n-2-a}$ has been determined recently* and hence there results the following theorem:

The number of the groups which involve a given dihedral group whose order is twice an odd number as the group of the squares of their operators, when the number of the different prime factors of the order of this dihedral group is n and each of these odd factors is congruent to unity modulo 4 is equal to the sum of the indexes of all the subgroups, including the identity and the entire group, of the abelian group of order 2^{n-2} and of type $(1, 1, 1, \dots)$ under this group.

It was noted above that the operators of odd order in H constitute the commutator subgroup of G . This fact can also be established by noting that every such group can be represented as a transitive substitution group whose degree is equal to the order of K , since its Sylow subgroup whose order is a power of 2 does not involve any invariant subgroup of G besides the identity. This follows from the fact that direct products are excluded. Hence it results that each of the groups determined above is contained in the holomorph of K . The subgroup composed of all the substitutions which omit a fixed letter of this holomorph is therefore in the group of isomorphisms of K . Since

* G. A. Miller, Proceedings of the National Academy of Sciences, vol. 20 (1934), p. 203.

the group of isomorphisms of a cyclic group is abelian it results that the Sylow subgroups whose orders are a power of 2 in such a G are always abelian and hence all of these Sylow subgroups are of type $(2, 1, 1, \dots)$.

The number of these Sylow subgroups is $h/2$ and no two of them have an operator of order 4 in common. A necessary and sufficient condition that no two of them have an operator of order 2 in common is that the order of such a G is $2h$. The number of the groups of this order is 2^{n-2} and all of them are conformal. Every other G contains more than one of these conformal groups. In fact, if the order of such a G is $2^k \cdot h$ it contains 2^{k-1} of these groups. The number of the possible G 's increases very rapidly with the increase of n . In particular, when $n=5$ there are 51 such groups; viz., 8 of order $2h$, 28 of order $4h$, 14 of order $8h$, and one of order $16h$. This number depends only on the number of the distinct prime numbers which divide h and is independent of the values of these primes.

It remains to determine the possible groups when h is four times an odd number and each of the odd prime factors of h is again congruent to unity modulo 4. None of these groups is contained in the holomorph of K , since half of the operators of H are negative when it is thus represented. Each of the two dihedral subgroups of index 2 contained in H is invariant under G since all the operators of G which are either commutative with every operator of K or transform some of these operators into their inverses constitute a subgroup of index 2 under G . Each of the remaining operators of G is of order 4 and has for its square a non-invariant operator of order 2 in H . Since these operators cannot transform the two given dihedral subgroups of H into each other none of the operators of G can have this property and therefore each of these dihedral subgroups is invariant under G .

The commutator subgroup of G is again composed of the operators of odd order in K . Hence it results that all the operators of G whose squares appear in one of the two dihedral subgroups of index 2 in H constitute a subgroup of index 2 under G . That is, every such G contains as a subgroup of index 2 one of the groups enumerated above which has the property that the squares of its operators are a dihedral group whose order is twice an odd number. To extend one of the given groups so as to obtain the desired result we may represent it as a regular substitution group and make it simply isomorphic with itself represented on a different set of letters so as to obtain an intransitive substitution group. To this we may adjoin a substitution of order 4 which is commutative with every substitution of this intransitive group, interchanges its two systems of intransitivity and has for its square the invariant substitution of order 2 contained in H . Each such G contains two subgroups of index 2 such that the squares of their operators are the dihedral subgroups of index 2

in H , and one such subgroup such that the squares of its operators constitute the cyclic subgroup of index 2 in H . The cross-cut of these three subgroups is the subgroup of index 4 composed of the operators of odd order in H .

From the preceding paragraph it results that G involves an invariant cyclic subgroup of order h and is contained in the holomorph of this cyclic subgroup. Its operators whose squares are the non-invariant operators of order 2 in H transform the operators of this cyclic subgroup into powers which are congruent to unity modulo 4 as otherwise these squares of operators of order 4 would not give all the non-invariant operators of order 2 in H . It therefore results that every such G contains an operator of order 4 which is commutative with each of its operators and hence there results the following theorem:

Each group in which the squares of the operators constitute a dihedral group whose order is twice an odd number is a subgroup of index 2 under one and only one group in which the squares of the operators constitute a dihedral group whose order is four times an odd number and the number of distinct groups in both of these cases is the same.

2. The dihedral subgroup is the four group. The special case when the operators which are the squares of the operators of a given group G constitute the four group is much more difficult than the more general case when these squares constitute a non-abelian dihedral group. The only abelian group which comes under this special case is the group of order 16 and of type $(2, 2)$, and the order of every other group which comes thereunder is obviously also of the form 2^m . The commutator subgroup of every such non-abelian group is either of order 2 or of order 4. We shall first consider the former case and hence the operators of order 2 contained in G together with its operators of order 4 whose squares are equal to the commutator of order 2 constitute a subgroup of index 2 under G . This subgroup belongs to one of the three known infinite categories of groups involving separately two and only two operators which are squares. Its central involves at least three invariant operators of order 2 and at most seven such operators since G is supposed to have the property that it is not a direct product.

When the central of this subgroup involves only three operators of order 2 there are two such groups of order 2^m when m is odd and exceeds 3. These are the direct products of the cyclic group of order 4 and of the groups which involve only two operators which are squares but do not contain an invariant operator of order 4. There are also two such groups of order 2^m when m is even and exceeds 4. One of these two groups is also the direct product of the

cyclic group of order 4 and a non-abelian group which involves only two operators which are squares but involves an invariant operator of order 4, while the other is obtained by extending such a non-abelian group by an operator which does not transform into itself its invariant operator of order 4. When the central of the given subgroup of index 2 involves seven operators of order 2 there are two additional such groups when m is even and there is one such additional group when m is odd. Hence there results the following theorem:

There are four groups of order 2^m , m being even and larger than 4, which satisfy the condition that each of them has the four group for the group of its squares and involves a commutator subgroup of order 2. When m is odd and larger than 5 there are three such groups.

It remains to consider the case when the commutator subgroup of G is the same as the group of the squares of its operators and we shall first consider the special case when G involves an abelian subgroup of index 2. This subgroup is of one of the following three types: $(1, 1, 1, \dots)$, $(2, 1, 1, \dots)$, $(2, 2, \dots)$ and G involves only one abelian subgroup of this index. There is one and only one G which involves such a subgroup of the first of these three types. It is of order 32 and involves 12 operators of order 4. When this abelian subgroup is of the second type the order of G is either 32 or 64. In the former case there is one such G . This involves 20 operators of order 4. In the latter case there is also one and only one such G . This contains 40 operators of order 4. It remains to consider the case when the abelian subgroup of index 2 is of type $(2, 2, 1, 1, \dots)$ and hence the order of G is 32, 64, or 128.

In the first case there are two groups in which all of the remaining operators are of the same order. These are the generalized dihedral and the generalized dicyclic groups. When only four of the operators of the given abelian subgroup of index 2 are transformed into their inverses under G there are also two groups of order 32. In one of these each of the remaining operators is of order 4 while only half of these operators are of this order in the other. There is one additional such group of order 32 in which no one of the operators of order 4 in the given abelian subgroup of order 16 is transformed into its inverse under G . This group involves 24 operators of order 4 of which eight have a common square. When G is of order 64 it involves invariant operators of order 4 and there are two isomorphisms to be considered. One of these gives rise to two distinct groups while the other gives rise to only one group. There is obviously only one group of order 128 which involves this abelian subgroup of index 2 and hence the following theorem has been established:

There is one and only one group which satisfies the conditions that it involves the abelian group of type $(1, 1, 1, \dots)$ as a subgroup of index 2 and that the squares of its operators as well as its commutator subgroup constitute the four group. There are two such groups which involve the abelian group of type $(2, 1, 1, \dots)$ as such a subgroup, and there are nine such groups which involve the abelian group of type $(2, 2, 1, 1, \dots)$ as such a subgroup.

The most difficult case remains, viz., the one when G contains no abelian subgroup of index 2 and when the commutator subgroup of G coincides with the group of its squares. All these possible groups may be divided into three categories composed of those whose centrals are of order 4, 8, or 16 respectively. These centrals are of types $(1, 1)$, $(2, 1)$ and $(2, 2)$ respectively. For each of these categories it is possible to construct an infinite system of groups such that every operator which does not appear in the central has four conjugates under the group. To do this we may start with any abelian group whose order is four times the order of the central and whose squares appear in the four group contained therein. The group thus obtained is then extended twice successively by two operators which are relatively commutative and are commutative only with the operators of the given subgroup which appear in the central, and whose product has the same property. The order of the group thus obtained is sixteen times the order of its central, and each of its own invariant operators has four conjugates under it. This group can be extended successively by two operators which have their squares therein and are commutative with each other and with each operator of the given group. The resulting group can be extended as before. By continuing this process we obtain a group whose order is an arbitrary power of 16 times the order of the central and all of whose operators which do not appear in this central have four conjugates under the group.

The lowest order of a group G which belongs to the infinite system described in the preceding paragraph is 64. This is also the lowest order of G whenever it does not involve an abelian subgroup of index 2. If such a G is of order 64 and all of its operators except those which are squares have four conjugates under G , then every such operator appears in an abelian subgroup of order 16 and G contains exactly five such subgroups. These subgroups have the central of G in common but no two of them have any other operator in common. There is one such group which involves two abelian subgroups of order 16 and of type $(1, 1, 1, 1)$. The other three abelian subgroups of order 16 contained therein are of type $(2, 2)$. When there is one and only one such subgroup in G there is also a subgroup of type $(2, 2)$. Hence there is only one

such G . It also contains exactly 27 operators of order 2. There is one and only one G in which there is no abelian subgroup of type $(1, 1, 1, 1)$. It contains only eleven operators of order 2. This proves the following theorem:

There are three and only three groups of order 64 which separately satisfy the following conditions: the group of their squares and their commutator subgroup is the four group and each of the operators which is not in this four group has four conjugates under the group.

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A PROJECTIVE GENERALIZATION OF METRICALLY DEFINED ASSOCIATE SURFACES*

BY

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1. INTRODUCTION

In the metric differential geometry of surfaces in ordinary space, two surfaces are said by Bianchi to be *associate*† if the tangent planes at corresponding points are parallel and if the asymptotic curves on either surface correspond to a conjugate net on the other.

It is the purpose of this paper to develop a projective generalization of the relation of associateness of surfaces. Since associate surfaces are parallel in the metric sense, it will first be necessary to provide a projectively defined substitute for the property of metric parallelism. We shall employ as the basis of our study in this paper a projective generalization of euclidean parallelism of surfaces which the author has developed in his Chicago doctoral dissertation.

In §2, after stating a definition of projective parallelism of surfaces and briefly explaining this idea, we introduce a canonical form of our system of differential equations employed in the study of projectively parallel surfaces in ordinary space. In §3 we formulate a definition of projectively associate surfaces and investigate to some extent their properties and relations. A more general type of associateness which may be conveniently termed *modified projective associateness* is introduced in §4, and a somewhat different canonical form of our system of differential equations is employed in its study. Finally, in §5, we consider a rather general completely integrable system of partial differential equations, namely, the system for two surfaces in the general analytic one-to-one point correspondence in ordinary projective space S_3 , and a group of transformations that leaves this configuration invariant. We then reduce this system of equations to a new canonical form, and employ it to continue briefly the study of modified projective associateness introduced in the preceding section.

* Presented to the Society, September 7, 1934; received by the editors April 8, 1934.

† Eisenhart, *Differential Geometry*, Ginn and Company, 1909, p. 378. Hereinafter cited as Eisenhart. See also Bianchi, *Lezioni di Geometria Differenziale* (3d edition), vol. 2, p. 10.

2. PROJECTIVE PARALLELISM OF SURFACES

In formulating a projective generalization of metric parallelism of surfaces,* we begin by replacing the metric normal congruence by the *projective normal congruence*, and so consider two surfaces S_x, S_y , in ordinary projective space S_3 , with a common projective normal congruence. The developables of this congruence intersect both surfaces in the projective lines of curvature, which form conjugate nets. We then demand, in analogy to the metric parallelism of the tangent planes, that the tangent planes at corresponding points of the two surfaces intersect on a fixed plane. Two surfaces so related are said to be *parallel in the projective sense*.

For the basis of our study of projective parallelism we employ one of the well known transformations of surfaces, namely, the fundamental transformation. Two surfaces are said to be in the relation of a fundamental transformation, or transformation F^\dagger , in case their points are in a one-to-one correspondence such that the lines joining corresponding points form a congruence whose developables intersect both surfaces in conjugate nets, neither surface being a focal surface of the congruence. The congruence is called the conjugate congruence of the transformation because it is *conjugate* to both nets. The tangent planes at corresponding points of the two surfaces intersect in the lines of the harmonic congruence of the transformation, which is *harmonic* to both nets. By choosing the projective normal congruence as the common conjugate congruence of the transformation F , we provide, as previously stated, a projective substitute for the metric normal congruence. Furthermore, we assume that the developables of the harmonic congruence are indeterminate, that is, that corresponding tangent planes of the two surfaces intersect in the lines of a fixed plane. This assumption affords us a projective substitute for the metric parallelism of the tangent planes. Our definition of projective parallelism may now be stated in the following way:

Two surfaces S_x, S_y are said to be projectively parallel in case they are in the relation of a fundamental transformation with the projective normal congruence as the conjugate congruence and with the developables of the harmonic congruence indeterminate.

In order to represent analytically the definition which we have introduced, let us consider two projectively parallel surfaces S_x, S_y with the respective parametric vector equations

$$x = x(u, v), \quad y = y(u, v).$$

* M. L. MacQueen, *A Projective Generalisation of Euclidean Parallelism of Surfaces*, University of Chicago, December, 1933; unpublished doctoral dissertation. Hereinafter cited as Thesis.

† L. P. Eisenhart, *Transformations of Surfaces*, Princeton University Press, 1923, p. 34 et seq.

The four coordinates x and the four coordinates y form four pairs of solutions of a completely integrable system of differential equations of the form

$$\begin{aligned} x_{uu} &= px + \alpha x_u + \beta x_v + Ly, \\ x_{uv} &= cx + ax_u + bx_v, \\ (1) \quad x_{vv} &= qx + \gamma x_u + \delta x_v + Ny, \\ y_u &= fx + mx_u + Ay, \\ y_v &= gx + nx_v + By \end{aligned} \quad (mnLN \neq 0),$$

where the notation here employed is similar to that used by Lane in his recent book.*

Before stating the conditions which characterize this system we remark that in order to treat S_x, S_y in a symmetrical manner we see that x, y satisfy a system of equations of the form (1), but with the roles of x and y interchanged. The coefficients of such a system will be indicated by dashes and will be given later. In order that S_y may be non-developable we shall suppose that $\overline{LN} \neq 0$.

System (1) is characterized analytically by the following conditions:

$$\begin{aligned} (a) \quad & \alpha + b + A + (\log N)_u - 3(\log r)_u/2 - 2(\log R)_u = 0, \\ (b) \quad & \gamma/r + \alpha + (\log r)_u/2 = 0, \\ (c) \quad & \bar{r} = nr/m, \\ (2) \quad (d) \quad & f/m = -[\log(mn)^{1/2}R/L]_u, \\ (e) \quad & m(1-n)\mathfrak{B}'^2 + nr(1-m)\mathfrak{C}'^2 + m_v(\mathfrak{B}' + m_v/(4m)) \\ & \quad + n_u r(\mathfrak{C}' + n_u/(4n)) = 0 \end{aligned}$$

and by the counterpart of (a), (b), and (d) in the substitution

$$(3) \quad \begin{pmatrix} u & a & c & f & m & p & s & \alpha & \beta & A & L & M & r & R \\ v & b & c & g & n & q & t & \delta & \gamma & B & N & M & 1/r & rR \end{pmatrix}.$$

The invariants $\mathfrak{B}', \mathfrak{C}', R$ of Green, and the invariant r of Eisenhart, appearing in equations (2), are expressed for the projective lines of curvature on S_x in terms of the coefficients of system (1) by the following formulas:

$$\begin{aligned} 8\mathfrak{B}' &= 4a + 2N\beta/L - 2\delta + (\log N/L)_v, \\ 8\mathfrak{C}' &= 4b + 2L\gamma/N - 2\alpha + (\log L/N)_u, \\ (4) \quad R &= L\mathfrak{B}'^2/N + \mathfrak{C}'^2, \\ r &= N/L. \end{aligned}$$

* Lane, *Projective Differential Geometry of Curves and Surfaces*, University of Chicago Press, 1932, p. 183. Hereinafter cited as Lane.

Conditions (2) (a) imply that the line $x_u x_v$ is the reciprocal of the projective normal of S_x at P_x , and conditions (b) and (a) imply that the line xy is the projective normal of S_x at P_x ; condition (c) implies that the tangent planes at corresponding points of S_x, S_y intersect in the lines of a fixed plane; conditions (c) and (d) imply that the line $y_u y_v$ is the reciprocal of the projective normal of S_y at P_y , and conditions (c), (d), and (e) imply that the line xy is the projective normal of S_y at P_y .

It may be remarked that the choice of the proportionality factors which leads to our canonical form is precisely that which gives Fubini's normal coordinates.

The integrability conditions for system (1) are given by the following equations and those obtainable therefrom by the substitution (3):

$$\begin{aligned}
 (5) \quad & a_u + ab + c = \alpha_v + \beta\gamma, \\
 & b_u + b^2 + a\beta = \beta_v + b\alpha + \beta\delta + p + nL, \\
 & c_u + bc + ap = p_v + c\alpha + q\beta + gL, \\
 & g_u + cn + fB = f_v + cm + gA, \\
 & an + mB + g = m_v + am, \\
 & aL = L_v + BL + \beta N, \quad A_v = B_u.
 \end{aligned}$$

The coefficients of the equations corresponding to (1) when the roles of x and y are interchanged are given by

$$\begin{aligned}
 (6) \quad & \bar{p} = A_u + mL - A(m_u/m + f/m + \alpha) - m\beta B/n, \\
 & \bar{\alpha} = \alpha + f/m + m_u/m + A, \quad \bar{\beta} = m\beta/n, \\
 & \bar{L} = -m(\alpha f/m + \beta g/n + (f/m)^2 - p - (f/m)_u), \\
 & \bar{c} = A_v - A(m_v/m + a) - B(f/n + mb/n), \\
 & \quad = B_u - B(n_u/n + b) - A(g/m + na/m), \\
 & \bar{a} = a + m_v/m = B + g/m + na/m, \\
 & \bar{b} = b + n_u/n = A + f/n + mb/n, \\
 & \bar{q} = B_v + nN - B(n_v/n + g/n + \delta) - n\gamma A/m, \\
 & \bar{\gamma} = n\gamma/m, \quad \bar{\delta} = \delta + g/n + n_v/n + B, \\
 & \bar{N} = -n(\delta g/n + \gamma f/m + (g/n)^2 - q - (g/n)_v), \\
 & \bar{f} = -A/m, \quad \bar{m} = 1/m, \quad \bar{A} = -f/m, \\
 & \bar{g} = -B/n, \quad \bar{n} = 1/n, \quad \bar{B} = -g/n.
 \end{aligned}$$

The developables of the projective normal congruence intersect S_x and S_y in parametric conjugate nets which are the projective lines of curvature thereon, the foci P_v, P_t of a projective normal being given by

$$(7) \quad \eta = y - mx, \quad \zeta = y - nx.$$

The differential equation of the asymptotic curves on S_z is

$$(8) \quad Ldu^2 + Ndv^2 = 0,$$

and the asymptotic curves on S_y are given by the equation

$$(9) \quad \bar{L}du^2 + \bar{N}dv^2 = 0.$$

3. PROJECTIVELY ASSOCIATE SURFACES

The projective generalization of metric parallelism of surfaces summarized in the preceding section will now be employed in formulating a definition of projectively associate surfaces. In analogy to the metric definition of associate surfaces, *two surfaces S_z, S_y , in ordinary projective space, will be called projectively associate if they are projectively parallel and if the asymptotic curves on either surface correspond to a conjugate net on the other.*

A necessary and sufficient condition that the asymptotic curves on one of two projectively parallel surfaces S_z, S_y correspond to a conjugate net on the other is

$$(10) \quad L\bar{N} + \bar{L}N = 0,$$

i.e., the harmonic invariant of the asymptotic curves on the two surfaces vanishes. With the aid of (2) (c) this condition is seen to be equivalent to

$$(11) \quad m = -n.$$

By means of (7), condition (11) shows that P_y is the harmonic conjugate of P_z with respect to the two focal points of a projective normal. Thus we reach the following conclusion:

If two surfaces S_z, S_y are projectively parallel, a necessary and sufficient condition that they be projectively associate is that corresponding points on a projective normal separate harmonically the foci thereon.

The Laplace-Darboux point invariants, H, K , the Weingarten invariants $W^{(u)}, W^{(v)}$, and the tangential invariants $\mathfrak{F}, \mathfrak{R}$ are given for the projective lines of curvature on S_z in terms of the coefficients of system (1) by the formulas

$$\begin{aligned} H &= c + ab - a_u, & K &= c + ab - b_v, \\ W^{(u)} &= 2b_v + a_u - \delta_u - B_u - (\log L)_{uv}, \\ W^{(v)} &= 2a_u + b_v - \alpha_v - A_v - (\log N)_{uv}, \\ (12) \quad \mathfrak{F} &= K + W^{(u)} = a_u + \beta\gamma - B_u - (\log L)_{uv}, \\ &= N(\beta_u + \beta b - \beta A - \beta(\log L)_u)/L, \\ \mathfrak{R} &= H + W^{(v)} = b_v + \beta\gamma - A_v - (\log N)_{uv}, \\ &= L(\gamma_v + \gamma a - \gamma B - \gamma(\log N)_v)/N. \end{aligned}$$

The corresponding invariants, indicated by dashes, for the projective lines of curvature on S_v , projectively parallel to S_x , are found* to have the following expressions:

$$(13) \quad \begin{aligned} \bar{H} &= H - (\log m^2 n)_{uv}/2, & \bar{K} &= K - (\log mn^3)_{uv}/2, \\ \bar{W}^{(u)} &= W^{(u)} + (\log mn^3)_{uv}/2, & \bar{W}^{(v)} &= W^{(v)} + (\log m^2 n)_{uv}/2 \\ \bar{\mathfrak{S}} &= \mathfrak{S}, & \bar{\mathfrak{R}} &= \mathfrak{R}. \end{aligned}$$

It is evident from (2) (d) that

$$(14) \quad (f/m)_v = (g/n)_u.$$

By using (14) and the integrability conditions (5) a simple calculation is made which shows that in case $m = -n$ it follows that $a_u = b_v$, and the projective lines of curvature on S_x have equal point invariants. Moreover, in this case equations (13) show that the projective lines of curvature on S_v also have equal point invariants. We therefore reach the following conclusion:

If two surfaces S_x, S_v are projectively associate, the projective lines of curvature on each surface have equal point invariants.

We shall now investigate the conjugate nets on each of two projectively associate surfaces to which correspond the asymptotic curves on the other. When use is made of (10), equation (8), which defines the asymptotic curves on S_x , may be written

$$(15) \quad \bar{L}du^2 - \bar{N}dv^2 = 0.$$

This is the differential equation of the associate conjugate net of the projective lines of curvature on S_v , that is, the conjugate net whose tangents at each point of the surface S_v separate harmonically the tangents to the projective lines of curvature.

Similarly, by use of (10), we may write equation (9) in the form

$$Ldu^2 - Ndv^2 = 0,$$

which shows that the asymptotic curves on S_v correspond to the associate conjugate net of the projective lines of curvature on S_x . We may therefore state the following theorem:

If two surfaces S_x, S_v are projectively associate, and if the parametric net on each is the projective lines of curvature, then the asymptotic curves on either surface correspond to the associate conjugate net of the parametric conjugate net on the other.

* Thesis.

An interesting property of a conjugate net is isothermal conjugacy, the condition for which is $W^{(u)} = W^{(v)}$ or $(\log r)_{uv} = 0$. Let the projective lines of curvature on S_x be an isothermally conjugate net, and let S_y be projectively associate to S_x . From (10) or (12) it is then easy to obtain the following result:

If the projective lines of curvature are isothermally conjugate on one of two projectively associate surfaces, they are also isothermally conjugate on the other.

In this case the projective lines of curvature on S_x and S_y are called J nets, since they are isothermally conjugate and have equal point invariants.

4. MODIFIED PROJECTIVE ASSOCIATENESS OF SURFACES

In this section we shall drop the assumption that the common conjugate congruence of the transformation F is the projective normal congruence, and shall employ in its place a general conjugate congruence. The configuration composed of two surfaces in ordinary space in the relation of a fundamental transformation having a general conjugate congruence and with the developables of the harmonic congruence indeterminate leads us to a characterization of surfaces which are projectively parallel in a modified* sense. We shall use this type of parallelism in formulating our definition of modified projectively associate surfaces.

For the analytic basis of our work a somewhat different canonical form of the basic system of differential equations is employed. If S_x, S_y are a pair of surfaces projectively parallel in the modified sense, then the four coordinates x and the four coordinates y form four pairs of solutions of a completely integrable system of differential equations* of the form

$$\begin{aligned} x_{uu} &= L(x + y) + \alpha x_u + \beta x_v, \\ x_{uv} &= ax_u + bx_v, \\ x_{vv} &= N(x + y) + \gamma x_u + \delta x_v, \\ y_u &= mx_u, & y_v &= nx_v \end{aligned} \quad (mnLN \neq 0). \quad (16)$$

The integrability conditions for this system are found to be

$$\begin{aligned} a_u + ab &= \alpha_v + \beta\gamma, & b_v + ab &= \delta_u + \beta\gamma, \\ b_u + b^2 + a\beta &= \beta_v + b\alpha + nL + \beta\delta + L, \\ a_v + a^2 + b\gamma &= \gamma_u + a\delta + mN + \alpha\gamma + N, \\ L_v &= aL - \beta N, & N_u &= bN - \gamma L, \\ m_v &= a(n - m), & n_u &= b(m - n). \end{aligned} \quad (17)$$

* Thesis.

The coefficients of the equations corresponding to (16), when the roles of x and y are interchanged, are indicated by dashes and are given by the following expressions:

$$(18) \quad \begin{array}{lll} \bar{L} = mL, & \bar{\alpha} = \alpha + m_u/m, & \bar{\beta} = m\beta/n \\ \bar{a} = na/m, & \bar{b} = mb/n, & \bar{m} = 1/m, \quad \bar{n} = 1/n, \\ \bar{N} = nN, & \bar{\gamma} = n\gamma/m, & \bar{\delta} = \delta + n_v/n. \end{array}$$

We shall assume that $\bar{L}\bar{N} \neq 0$ in order that S_v may be non-developable.

The focal points P_η, P_ζ of a line xy joining corresponding points P_x, P_y of two surfaces S_x, S_y are defined by

$$(19) \quad \eta = y - mx, \quad \zeta = y - nx.$$

Several results similar to those in the preceding section will now be given. Inasmuch as the proofs of these results run parallel to those in §3 they will be omitted in this section. Precisely as in the preceding section, a necessary and sufficient condition that the asymptotic curves on either of two modified projectively parallel surfaces S_x, S_y correspond to a conjugate net on the other is found to be given by the condition $m = -n$. Hence the following result is readily obtained.

If two surfaces S_x, S_y are projectively parallel in the modified sense, a necessary and sufficient condition that they be projectively associate in the same sense is that corresponding points P_x, P_y of each line xy separate harmonically the foci thereon.

Some of the invariants of the parametric conjugate net N_x are found to have in our notation the following formulas:

$$(20) \quad \begin{aligned} H &= ab - a_u, & K &= ab - b_v, \\ W^{(u)} &= 2b_v + a_u - \delta_u - (\log L)_{uv}, \\ W^{(v)} &= 2a_u + b_v - \alpha_v - (\log N)_{uv}, \\ \mathfrak{H} &= K + W^{(u)} = a_u + \beta\gamma - (\log L)_{uv} \\ &= N(\beta_u + b\beta - \beta(\log L)_u)/L, \\ \mathfrak{K} &= H + W^{(v)} = b_v + \beta\gamma - (\log N)_{uv} \\ &= L(\gamma_v + a\gamma - \gamma(\log N)_v)/N, \\ 8\mathfrak{B}' &= 6a - 2\delta - 3(\log L)_v + (\log N)_v, \\ 8\mathfrak{C}' &= 6b - 2\alpha - 3(\log N)_u + (\log L)_u. \end{aligned}$$

By use of (17) and (18), the corresponding invariants for N_y , indicated by dashes, are given by the following expressions:

$$\begin{aligned}
 \overline{H} &= H - (\log m)_{uv}, & \overline{K} &= K - (\log n)_{uv}, \\
 \overline{W}^{(u)} &= W^{(u)} + (\log n)_{uv}, & \overline{W}^{(v)} &= W^{(v)} + (\log m)_{uv}, \\
 \overline{\mathfrak{S}} &= \mathfrak{S}, & \overline{\mathfrak{R}} &= \mathfrak{R}, \\
 8\overline{\mathfrak{B}}' &= 8\mathfrak{B}' + (\log m^2/n)_v, & 8\overline{\mathfrak{C}}' &= 8\mathfrak{C}' + (\log n^2/m)_u.
 \end{aligned}
 \tag{21}$$

The following result can be easily established.

If two surfaces S_x, S_y are projectively associate in the modified sense, the parametric conjugate net on each surface has equal point invariants.

The asymptotic curves on S_x and S_y are determined by the same equations as in the preceding section. Precisely as in §3 we arrive at the following result:

If two surfaces S_x, S_y are projectively associate in the modified sense, and if the parametric net on each surface is conjugate, then the asymptotic curves on either surface correspond to the associate conjugate net of the parametric net on the other.

The Laplace transformed points, or ray points, of the point P_x with respect to N_x are given by

$$x_1 = x_v - ax, \quad x_{-1} = x_u - bx,$$

and the ray points of P_y are defined by

$$y_1 = n(x_1 - a\eta/m), \quad y_{-1} = m(x_{-1} - b\zeta/n).$$

The ray points of the points P_η, P_ζ , defined by (19), are found to be

$$\begin{aligned}
 m_a\eta_1 &= (n-m)(H\eta + m_ax_1), & (n-m)\eta_{-1} &= m_a\zeta, \\
 (m-n)\zeta_1 &= n_v\eta, & n_v\zeta_{-1} &= (m-n)(K\zeta + n_vx_{-1}),
 \end{aligned}$$

where H, K are the point invariants of the net N_x . The points x_1, y_1, η, η_1 are collinear, as are also the points $x_{-1}, y_{-1}, \zeta, \zeta_{-1}$. The cross ratio* of the four points $x_{-1}, y_{-1}, \zeta, \zeta_{-1}$ is $-bn_v/(nK)$, and that of the points x_1, y_1, η, η_1 is $-am_u/(mH)$. Hence we have the following theorem:

If two surfaces S_x, S_y are projectively associate in the modified sense, the cross ratio of the four points x_1, y_1, η, η_1 is equal to that of the points $x_{-1}, y_{-1}, \zeta, \zeta_{-1}$, and the common value may be written $2ab/H$.

5. A CANONICAL FORM WITH PARTICULAR PARAMETRIC CURVES

In this section we place as fundamental the well known system of differential equations used in the study of the configuration composed of two surfaces S_x, S_y in ordinary projective space S_3 , with their points in a one-to-one

* Lane, op. cit., p. 214, exercise 11.

correspondence. We then reduce this system of equations to a canonical form so that every pair of integral surfaces is projectively associate in the modified sense, the surface S_x being referred to its asymptotic net as parametric, and the curves on S_y , corresponding to the asymptotic curves on S_x , forming the parametric conjugate net.

Let

$$x = x(u, v), \quad y = y(u, v)$$

be the parametric vector equations of two surfaces S_x, S_y in ordinary projective space. If these surfaces have their points in a one-to-one correspondence, such that corresponding points P_x, P_y have the same curvilinear coordinates u, v , and such that each point P_y does not lie in the tangent plane of S_x at the corresponding point P_x , then S_x, S_y are a pair of integral surfaces of a system of differential equations* of the form

$$\begin{aligned} x_{uu} &= px + \alpha x_u + \beta x_v + Ly, \\ x_{uv} &= cx + ax_u + bx_v + My, \\ x_{vv} &= qx + \gamma x_u + \delta x_v + Ny, \\ y_u &= fx + mx_u + sx_v + Ay, \\ y_v &= gx + tx_u + nx_v + By. \end{aligned} \quad (22)$$

The integrability conditions for this system are given by the following equations:

$$\begin{aligned} a_u + ab + c + mM &= \alpha_v + \beta\gamma + tL, \\ b_u + b^2 + a\beta + sM &= \beta_v + b\alpha + \beta\delta + p + nL, \\ c_u + bc + ap + fM &= p_v + c\alpha + q\beta + gL, \\ M_u + aL + (b + A)M &= L_v + BL + \alpha M + \beta N, \\ t_u + ta + an + mB + g &= m_v + am + s\gamma + tA, \\ g_u + pl + cn + fB &= f_v + qs + cm + gA, \\ B_u + tL + nM &= A_v + sN + mM \end{aligned} \quad (23)$$

and those obtainable therefrom by the substitution (3).

The lines xy joining pairs of corresponding points P_x, P_y of the surfaces S_x, S_y form a congruence, the developables† of which are given by

$$s du^2 - (m - n) du dv - t dv^2 = 0. \quad (24)$$

The focal points of a line xy are the points η, ζ given by

$$\eta = y + k_1 x, \quad \zeta = y + k_2 x,$$

* Lane, op. cit., p. 183 et seq.

† Ibid., p. 181.

where k_1, k_2 are the roots of the equation

$$(25) \quad k^2 + (m+n)k + mn - st = 0.$$

It is known that the asymptotic curves on S_x are parametric in case $L=N=0$. Let us suppose from now on that this condition is satisfied, and in order that the developables of the congruence of lines xy be determinate and intersect S_x in a conjugate net we shall suppose $m=n, st \neq 0$.

It is not difficult to calculate the system of equations corresponding to (22) when the roles of x and y are interchanged. We shall compute all of the coefficients of such a system later, but at the moment the only coefficients that are needed are those corresponding to L, M , and N which are indicated by dashes and given* by the following formulas:

$$(26) \quad \begin{aligned} \bar{L} &= f_u + np + cs + \bar{A}c_{11} + \bar{B}c_{12}, \\ \bar{M} &= f_v + cn + qs + \bar{A}c_{31} + \bar{B}c_{32} \\ &= g_u + cn + pt + \bar{B}c_{41} + \bar{A}c_{42}, \\ \bar{N} &= g_v + nq + ct + \bar{B}c_{21} + \bar{A}c_{22}, \end{aligned}$$

wherein the coefficients c_{ij} are defined by placing

$$\begin{aligned} c_{11} &= n_u + f + n\alpha + as, & c_{12} &= s_u + n\beta + bs, \\ c_{21} &= n_v + g + n\delta + bt, & c_{22} &= t_v + n\gamma + at, \\ c_{31} &= n_v + an + s\gamma, & c_{32} &= s_v + f + bn + s\delta, \\ c_{41} &= n_u + bn + t\beta, & c_{42} &= t_u + g + an + t\alpha, \end{aligned}$$

and where

$$\Delta\bar{A} = sg - nf, \quad \Delta\bar{B} = tf - ng, \quad \Delta = n^2 - st \neq 0.$$

The parametric curves on S_y form a conjugate net N_y in case $\bar{M}=0$. We shall suppose from now on that this condition is satisfied, and in order that S_y may be non-developable we shall suppose $\bar{L}\bar{N} \neq 0$. The developables of the congruence of lines xy intersect S_y in a conjugate net in case

$$(27) \quad t\bar{L} - s\bar{N} = 0,$$

a condition which we shall suppose from now on to be satisfied.

It is possible to simplify system (22) still more by a transformation of the form

$$(28) \quad x = \lambda\bar{x}, \quad y = \mu\bar{y}.$$

* Lane, op. cit., p. 185.

The effect of this transformation on the coefficients f, g, A, B is found to be given by the formulas

$$(29) \quad \begin{aligned} \mu \bar{f} &= \lambda(f + s\lambda_u/\lambda), & \mu \bar{g} &= \lambda(g + t\lambda_v/\lambda), \\ \bar{A} &= A - \mu_u/\mu, & \bar{B} &= B - \mu_v/\mu. \end{aligned}$$

The last of (23) shows that μ can be chosen so that $\bar{A} = \bar{B} = 0$. We shall suppose from now on that this choice has been made. A condition necessary and sufficient that λ can be chosen so that $\bar{f} = \bar{g} = 0$ is

$$(f/s)_u = (g/t)_v.$$

By means of (23) and (26) this condition can be shown to be equivalent to (27). We shall suppose from now on that this choice of λ has been made.

When $f = g = A = B = 0$, the line h of intersection of the tangent planes at two corresponding points P_x, P_y of the surfaces S_x, S_y joins the points P_ρ, P_σ defined by

$$(30) \quad \begin{aligned} \rho &= y_u = x_v + nx_u/s \\ \sigma &= y_v = x_u + nx_v/t, \end{aligned}$$

as is seen on inspecting the last two of equations (22).

When u, v vary, the line h generates a congruence $\rho\sigma$, whose developables will now be determined. If, as the point P_x describes a curve of the family $dv - \lambda du = 0$ on the surface S_x , the line h generates a developable of the congruence $\rho\sigma$, and if the point P_t defined by

$$\zeta = \rho + k\sigma \quad (k \text{ scalar})$$

is the corresponding focal point of the line h , then h is tangent to the locus of the point P_t ; consequently the derivative ζ' may be expressed as a linear combination of ρ, σ only. But by actual calculation it is found that ζ' appears as a linear combination of x, ρ, σ, y . Setting equal to zero the coefficients of x, y therein, we obtain conditions on the functions k, λ necessary and sufficient that the line h may generate a developable of the congruence $\rho\sigma$ and have P_t for focal point, namely,

$$(31) \quad \begin{aligned} (cst + npt) + (cns + pst)k + (qst + nct)\lambda + (cst + nqs)k\lambda &= 0, \\ st + nsk + nt\lambda + stk\lambda &= 0. \end{aligned}$$

Elimination of k and substitution of dv/du for λ give the differential equation of the developables of the congruence $\rho\sigma$, namely,

$$(32) \quad p \, du^2 - q \, dv^2 = 0.$$

A necessary and sufficient condition that the developables of the congruence $\rho\sigma$ be indeterminate is seen from (32) to be

$$(33) \quad p = q = 0.$$

We shall suppose from now on that this condition is satisfied. As a result of the conditions which we have thus far imposed we find from (26) that

$$(34) \quad \bar{L} = cs, \quad \bar{M} = cn = 0, \quad \bar{N} = ct.$$

In view of the previous assumptions it is therefore evident from (34) that $m = n = 0$, $c \neq 0$.

The most general transformation of the form (28) which leaves the form of system (22) invariant, has λ and μ constant. The only coefficients not absolutely invariant under such a transformation are s , t , M , for which we find

$$(35) \quad \bar{s} = \lambda s / \mu, \quad \bar{t} = \lambda t / \mu, \quad \bar{M} = \mu M / \lambda.$$

It is not difficult to show by means of the integrability conditions (23) that

$$M_u / M = c_u / c, \quad M_v / M = c_v / c,$$

and from these results it is evident that

$$(36) \quad M = kc \quad (k = \text{const.}).$$

It is therefore possible by a suitable choice of λ and μ to make the constant appearing in (36) equal to unity. We thus reach the following conclusion.

Any system (22) such that every pair of integral surfaces is projectively parallel in the modified sense can be reduced to the form

$$(37) \quad \begin{aligned} x_{uu} &= \alpha x_u + \beta x_v, \\ x_{uv} &= M(x + y) + ax_u + bx_v, \\ x_{vv} &= \gamma x_u + \delta x_v, \\ y_u &= sx_v, \quad y_v = tx_u \quad (stM \neq 0). \end{aligned}$$

The parametric net N_x is the asymptotic net and the parametric net N_y is conjugate.

The integrability conditions for system (37) are found to be

$$(38) \quad \begin{aligned} a_u + ab + c &= \alpha_v + \beta\gamma, \\ b_u + b^2 + a\beta + sM &= \beta_v + b\alpha + \beta\delta, \\ M_u + bM &= \alpha M, \\ t_u + t\alpha &= s\gamma, \end{aligned}$$

and the formulas obtainable from these by the substitution (3).

The system of equations corresponding to (37) when the roles of x and y are interchanged is found to be

$$(39) \quad \begin{aligned} y_{uu} &= \bar{L}(x+y) + \bar{\alpha}y_u + \bar{\beta}y_v, \\ y_{uv} &= \bar{a}y_u + \bar{b}y_v, \\ y_{vv} &= \bar{N}(x+y) + \bar{\gamma}y_u + \bar{\delta}y_v, \\ x_u &= \bar{s}y_v, \quad x_v = \bar{t}y_u, \end{aligned}$$

where

$$(40) \quad \begin{aligned} \bar{L} &= cs, & \bar{\alpha} &= b + s_u/s, & \bar{\beta} &= as/t, \\ \bar{a} &= \beta t/s, & \bar{b} &= \gamma s/t, \\ \bar{N} &= ct, & \bar{\gamma} &= bt/s, & \bar{\delta} &= a + t_v/t, \\ \bar{s} &= 1/t, & \bar{t} &= 1/s. \end{aligned}$$

It is evident that the asymptotic curves which are parametric on S_x correspond to the curves of the parametric conjugate net on S_y . The asymptotic curves on S_y are given by

$$(41) \quad \bar{L}du^2 + \bar{N}dv^2 = 0.$$

By use of (40) this equation becomes

$$sdu^2 + tdv^2 = 0$$

which defines the associate conjugate net of the net in which the developables of the congruence of lines xy intersect S_x . Thus since the asymptotic curves on each of the two surfaces S_x, S_y correspond to a conjugate net on the other, we therefore reach the following conclusion:

Any system (22) such that every pair of integral surfaces is projectively associate in the modified sense, can be reduced to the form (37).

The last two equations of (37) show that the tangent to an asymptotic u -curve (v -curve) through P_x on S_x intersects the tangent to the v -curve (u -curve) of the parametric conjugate net through P_y on S_y in a point which lies in a fixed plane. Since statements similar to the preceding can be made by choosing a conjugate net as parametric on S_x and the asymptotic net as parametric on S_y , we therefore have a projective generalization of the theorem* of Eisenhart:

* Eisenhart, op. cit., p. 381.

The tangents to the asymptotic curves on one of two projectively associate surfaces meet the tangents to the curves conjugate to the corresponding curves on the other surface in points of a fixed plane.

Inspection of (25) now shows that the two focal points P_η , P_ζ of a line xy are given by

$$(42) \quad \eta = y + (st)^{1/2}x, \quad \zeta = y - (st)^{1/2}x.$$

The cross ratio of the points P_x , P_y and the two focal points of the generator of the conjugate congruence is given by

$$(x, y, \eta, \zeta) = (\infty, 0, (st)^{1/2}, -(st)^{1/2}) = -1.$$

Corresponding points P_x , P_y of two projectively associate surfaces S_x , S_y are separated harmonically by the focal points of the line joining them.

If local coordinates x_1, \dots, x_4 based on the tetrahedron x, x_u, x_v, y with suitably chosen unit point are introduced, the first and second focal planes of a line xy are found to have the local equations

$$x_2 - (t/s)^{1/2}x_3 = 0, \quad x_2 + (t/s)^{1/2}x_3 = 0.$$

Therefore the planes $x_2=0$, $x_3=0$ containing a line xy and the asymptotic tangents through P_x on S_x separate the first and second focal planes of the line xy harmonically.

The developables of the congruence of lines xy intersect S_x in a conjugate net whose differential equation may be written

$$s du^2 - t dv^2 = 0.$$

Similarly, the developables intersect S_y in the conjugate net given by

$$\bar{s} du^2 - \bar{t} dv^2 = 0.$$

When reference is made to (40) it is evident that these curves on S_x and S_y correspond.

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ON THE POWER SERIES FOR ELLIPTIC FUNCTIONS*

BY

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1. Introduction. A more direct and more practicable method than those hitherto used for obtaining the coefficients in the power series expansions of doubly periodic and other elliptic theta quotients appears incidentally in some work relating to representations of rational integers as sums of integer squares. References to the literature will be found in the paper of Gruder†, and in the treatises of Enneper‡ and Krause.§ Most of the complications in some other methods, for example that of D. André||, enter with the use of the differential equation (or the equivalent difference equation, obtained by equating coefficients) satisfied by the function to be expanded. By avoiding the use of the differential equation entirely, the arithmetical nature of the coefficients in the power series becomes evident, and much tedious algebra is obviated. In the method used here the difference equations for the coefficients are *linear*; other methods introduce *non-linear* equations.

Hermite¶ proposed an extremely ingenious and elegant method, based on the transformation of the second order, for obtaining the coefficients when the functions are doubly periodic. Later** he remarked that this method is incapable, apparently, of leading to the desired end. However, Gruder (loc. cit., pp. 158-166) succeeded in obtaining explicit formulas for certain coefficients by this method, although the arithmetical character of the coefficients is perhaps not as evident as it might be.

Hermite** published only specimens of the results furnished by *another* method, without indicating what this method was. As noted by Picard††, some of Hermite's explicit formulas thus obtained are incorrect (owing to

* Presented to the Society, September 7, 1934; received by the editors April 30, 1934.

† O. Gruder, *Wiener Sitzungsberichte*, II a, vol. 126 (1917), pp. 125-183.

‡ A. Enneper, *Elliptische Functionen*, 1890, §47.

§ M. Krause, *Theorie der doppeltperiodischen Functionen*, 1895, §43.

|| D. André, *Annales de l'École Normale Supérieure*, (2), vol. 6 (1877), pp. 265-328. *Ibid.*, vol. 8 (1879), pp. 151-168; vol. 9 (1880), pp. 107-118.

¶ Ch. Hermite, *Comptes Rendus* (Paris), vol. 57 (1863), pp. 613-618; *Liouville's Journal*, (2), vol. 9 (1864), pp. 289-295.

** Ch. Hermite, *Lettre à M. Königsberger*, *Crelle's Journal*, vol. 81 (1876), pp. 220-228. *Oeuvres*, vol. 3, pp. 236-245. See also *Oeuvres*, vol. 3, pp. 222-231.

†† Hermite, *Oeuvres*, vol. 3, p. 237.

slips in calculation). The incorrect formulas have been reproduced in the treatises by Enneper and Krause cited above; they may be easily corrected by the present method. The expansion of $\text{cn } x$ being one of those containing errors, we shall consider it first in detail as an illustration of the *general method, which consists of comparing the MacLaurin and Fourier expansions of the function whose power series expansion is sought*. If the origin is a singularity of the function, the singularity is removed by any of the familiar devices used in obtaining the Fourier expansion; the procedure will be clear from the examples in §§4, 5. All of the series in the sequel are absolutely convergent for values of the variables different from zero.

2. **Expansion of $\text{cn } x$.** It is readily seen that the expansion is of the form

$$(1) \quad \text{cn } x = 1 + \sum_{s=1}^{\infty} (-1)^s Q_s(k^2) \frac{x^{2s}}{(2s)!},$$

$$(2) \quad Q_s(k^2) \equiv \sum_{r=0}^{s-1} q_r(s) k^{2r},$$

where the $q_r(s)$ are integers. The problem of expanding $\text{cn } x$ is thus reduced to that of calculating $q_r(s)$ as a function of r, s .

Replacing $\text{cn } x$ by its equivalent theta quotient, and expanding the latter in a cosine series,*

$$(3) \quad \partial_2^2 \text{cn}(x\partial_2^2) = \partial_0 \partial_2 \frac{\partial_2(x)}{\partial_0(x)} = 4 \sum q^{m/2} [\sum (-1|\tau) \cos tx],$$

$m=1, 3, 5, \dots$; $m=\tau t$, τ integers >0), $(-1|\tau) = (-1)^{(\tau-1)/2}$. In the last we now expand the cosines, rearrange the result (as is obviously permissible) as a power series in x , apply (1) to the left of (3), and finally equate coefficients of x^{2s} . Thus

$$(4) \quad \partial_2^2 \partial_3^{4s} Q_s(k^2) = 4 \sum q^{m/2} \xi_{2s}(m),$$

where $\xi_{2s}(m)$ denotes the sum of the $(2s)$ th powers of all those (positive) divisors of m whose conjugate divisors are of the form $4h+1$ minus the like sum in which the conjugates are of the form $4h+3$. In (4) we apply (2), replace q by q^4 in the result, and get

$$(5) \quad \sum_{r=0}^{s-1} q_r(s) \partial_2^{4r+2} (q^4) \partial_3^{4s-4r} (q^4) = 4 \sum q^{2m} \xi_{2s}(m).$$

* This series, with others of a similar kind in later sections, is given with many more in my paper, *Messenger of Mathematics*, vol. 54 (1924), pp. 116-176. The recurrence (6) occurs incidentally in my paper on sums of squares, *Bulletin of the American Mathematical Society*, vol. 26 (1919), pp. 19-25.

Let $N(n, f, g)$ denote the number of those representations of n as a sum of f squares, precisely g of which are odd with roots greater than zero, and occupy the first g places in the representations, and $f-g$ are even with roots greater than, equal to, or less than zero. Then, from (5) and the definition of N , we have the following recurrence for the $q_r(s)$:

$$(6) \quad \sum_{r=0}^{(m-1)/2} 2^{4r} N(2m, 4s+2, 4r+2) q_r(s) = \xi_{2s}(m).$$

To calculate $q_i(s)$ we take $m=1, 3, 5, \dots, 2j+1$ in (6) and solve the resulting linear equations for $q_i(s)$. An explicit determinant formula for the general coefficient $q_i(s)$ is thus obtained, but it is more practical to proceed step by step, evaluating the numbers N as they occur. To illustrate the process, we shall calculate $q_0(s)$, $q_1(s)$, $q_2(s)$, $q_3(s)$, the first three of which were given correctly, and the fourth incorrectly, by Hermite.

$$m = 1: \quad N(2, 4s+2, 2) q_0(s) = \xi_{2s}(1).$$

Referring to the definition of N , we see that $2=1^2+1^2+4s \cdot 0^2$ is the only representation enumerated by $N(2, 4s+2, 2)$. By the definition of ξ , $\xi_{2s}(1)=1$. Hence $q_0(s)=1$.

$$m = 3: \quad N(6, 4s+2, 2) q_0(s) + N(6, 4s+2, 6) 2^4 q_1(s) = \xi_{2s}(3);$$

$$6 = 1^2 + 1^2 + [2^2 + (4s-1)0^2],$$

$$N(6, 4s+2, 2) = \frac{2(4s)!}{1!(4s-1)!} = 8s; \quad \xi_{2s}(3) = 3^{2s} - 1;$$

$$2^4 q_1(s) = 3^{2s} - 8s - 1.$$

$$m = 5:$$

$$10 = 1^2 + 1^2 + [2^2 + 2^2 + (4s-2)0^2] = 1^2 + 3^2 + [4s(0^2)],$$

$$N(10, 4s+2, 2) = \frac{2^2(4s)!}{2!(4s-2)!} + 2 = 2(16s^2 - 4s + 1);$$

$$10 = 6 \cdot 1^2 + [2^2 + (4s-5)0^2],$$

$$N(10, 4s+2, 6) = \frac{2(4s-4)!}{1!(4s-5)!} = 8(s-1);$$

$$N(10, 4s+2, 10) = 1; \quad \xi_{2s}(5) = 5^{2s} + 1;$$

$$(32s^2 - 8s + 2) q_0(s) + 8(s-1) 2^4 q_1(s) + 2^8 q_2(s) = 5^{2s} + 1;$$

$$2^8 q_2(s) = 5^{2s} - 8(s-1) 2^{2s} + 32s^2 - 48s - 9.$$

$m = 7$:

$$14 = 2 \cdot 1^2 + [3 \cdot 2^2 + (4s - 3)0^2] = 1^2 + 3^2 + [2^2 + (4s - 1)0^2],$$

$$\begin{aligned} N(14, 4s + 2, 2) &= \frac{2^2(4s)!}{3!(4s - 3)!} + 2 \cdot 2 \cdot \frac{(4s)!}{1!(4s - 1)!}, \\ &= \frac{16}{3}s(16s^2 - 12s + 5); \end{aligned}$$

$$14 = 6 \cdot 1^2 + [2 \cdot 2^2 + (4s - 6)0^2] = 3^2 + 5 \cdot 1^2,$$

$$N(14, 4s + 2, 6) = \frac{2^2(4s - 4)!}{2!(4s - 6)!} + 6 = 2(16s^2 - 36s + 23);$$

$$14 = 10 \cdot 1^2 + [2^2 + (4s - 9)0^2],$$

$$N(14, 4s + 2, 10) = \frac{2(4s - 8)!}{1!(4s - 9)!} = 8(s - 2);$$

$$N(14, 4s + 2, 14) = 1; \quad \xi_{2s}(7) = 7^{2s} - 1;$$

$$\begin{aligned} 2^{12}q_3(s) &= 7^{2s} - 8(s - 2)5^{2s} + 2(16s^2 - 60s + 41)3^{2s} \\ &\quad - \frac{1}{3}(256s^3 - 1248s^2 + 1280s - 297). \end{aligned}$$

For $s = 1, 2, 3, \dots$ these values check with the numerical results given in the treatises. By the transformation of the first order,

$$\operatorname{sn}(ku, 1/k) = k \operatorname{sn}(x, k);$$

whence $q_j(s) = q_{s-j}(s)$.

From (6) and the definition of $\xi_{2s}(m)$ it is evident that

$$(7) \quad 2^{4j}q_j(s) = (2j + 1)^s + A_1(s)(2j - 1)^s + A_2(s)(2j - 3)^s + \dots + A_j(s)1^s,$$

where the A 's are polynomials in s with rational coefficients. It will be shown that the degree in s of $A_r(s)$ is r ($r = 1, \dots, j$). The last is an immediate consequence of (6) and the following lemma.

The degree in s of $N(2m, 4s + 2, 2m - 4h)$ is h ($h = 0, 1, \dots, (m - 1)/2$).

Before proving the lemma we shall examine it for $h = 0, 1, 2, 3$. Obviously $N(2m, 4s + 2, 2m) = 1$. For $h = 1$ we have, as the only possible decomposition of $2m$ of the kind enumerated by $N(2m, 4s + 2, 2m - 4)$,

$$2m = (2m - 4)1^2 + [2^2 + \{(4s + 2) - (2m - 3)\}0^2];$$

hence, enumerating the corresponding representations, we get

$$N(2m, 4s + 2, 2m - 4) = \frac{2 \cdot (4s - 2m + 6)!}{1!(4s - 2m + 5)!} = 4(2s - m + 3).$$

Here a negative result ($m > 2s+3$) is to be interpreted as zero (no representations), and likewise in all similar cases. When $h=2$, of the $2m-8$ odd squares in the representations enumerated by $N(2m, 4s+2, 2m-8)$ all may be 1's or precisely one may be 3^2 , and there are no other possibilities. Hence the only decompositions to be considered are

$$2m = (2m-8)1^2 + [2 \cdot 2^2 + (4s-2m+8)0^2] = (2m-9)1^2 + 3^2;$$

whence, counting the representations of the kind enumerated by N we have

$$N(2m, 4s+2, 2m-8) = \frac{2^2 \cdot (4s-2m+10)!}{2!(2s-2m+8)!} + \frac{(2m-8)!}{1!(2m-9)!},$$

the last fraction corresponding to the representations obtained by arranging the $(2m-9)$ 1's and the 3^2 in all possible ways. Thus

$$N(2m, 4s+2, 2m-8) = 2[16s^2 - 4(3m-14) + 2m^2 - 18m + 41].$$

The only decompositions of $2m$ to be considered when $h=3$ are

$$\begin{aligned} 2m &= (2m-12)1^2 + [3 \cdot 2^2 + (4s-2m+11)0^2], \\ &= (2m-13)1^2 + 3^2 + [2^2 + (4s-2m+13)0^2]; \end{aligned}$$

whence

$$\begin{aligned} N(2m, 4s+2, 2m-12) &= \frac{2^3(4s-2m+14)!}{3!(4s-2m+11)!} \\ &+ \frac{(2m-12)!}{1!(2m-13)!} \cdot \frac{2 \cdot (4s-2m+14)!}{1!(4s-2m+13)!}, \end{aligned}$$

which is of degree 3 in s .

To prove the lemma, consider the decomposition

$$2m = (2m-4h)1^2 + [h \cdot 2^2 + (4s+2-2m+3h)0^2],$$

which contributes to $N(2m, 4s+2, 2m-4h)$ precisely

$$A(t) \equiv \frac{2^h \cdot (t+4h)!}{h!(t+3h)!} \quad (t \equiv 4s+2-2m)$$

representations. If $h \cdot 2^2$ has a decomposition into a sum of $t+4h$ even squares other than that in [] above, it is of the form indicated in [] in the following decomposition of $2m$,

$$2m = (2m-4h)1^2 + [h_1 a_1^2 + \cdots + h_p a_p^2 + h_{p+1} 0^2],$$

where h_i, a_i ($i=1, \dots, p$) are >0 , $a_i \geq 2$, the a_1, \dots, a_p are distinct, and

$h_1 + \dots + h_p + h_{p+1} = t + 4h$. This decomposition contributes to $N(2m, 4s+2, 2m-4h)$ precisely

$$A'(t) \equiv \frac{2^{h_1+\dots+h_p}(t+4h)!}{h_1! \dots h_p!(t+4h-h_1-\dots-h_p)!}$$

representations. The degree in t , and hence also in s , of the polynomial $A(t)$ is h ; that of $A'(t)$ is $h_1 + \dots + h_p$. Hence, when it is shown that $h_1 + \dots + h_p < h$, the lemma will be proved. The required inequality is obviously implied by the following more general situation, which is of use in other questions of this kind. Both are practically obvious, but we give a formal proof.

If $p > 1$, and h, x, h_i, x_i ($i=1, \dots, p$) are any integers > 0 such that (A) $x_i \geq x$ ($i=1, \dots, p$); (B) $x_1 + \dots + x_p > px$; (C) $hx = h_1x_1 + \dots + h_px_p$; then $h > h_1 + \dots + h_p$.

To prove this, define the e_i by $x_i/x = 1 + e_i$. Then, from (A), $e_i \geq 0$; from (B) $e_1 + \dots + e_p > 0$; hence at least one of e_1, \dots, e_p is > 0 . From (C),

$$1 = \left(\frac{h_1}{h} + \dots + \frac{h_p}{h} \right) + \left(e_1 \frac{h_1}{h} + \dots + e_p \frac{h_p}{h} \right);$$

the second () is > 0 ; hence

$$1 > \frac{h_1}{h} + \dots + \frac{h_p}{h}, \quad h > h_1 + \dots + h_p.$$

The inequality is also easily seen from a simple contradiction. This completes the proof of the lemma.

3. Expansions of $\operatorname{sn} x, \operatorname{dn} x$. Proceeding as before from

$$\operatorname{sn} x = \sum_{s=0}^{\infty} \frac{(-1)^s P_s(k^2)}{(2s+1)!} x^{2s+1}, \quad P_s(k^2) \equiv \sum_{r=0}^s p_r(s) k^{2r},$$

$$\partial_2^2 \operatorname{sn}(x \partial_3^2) = \partial_2 \partial_3 \frac{\partial_1(x)}{\partial_0(x)} = 4 \sum q^{m/2} [\sum \sin tx]$$

$$(m = 1, 3, 5, \dots; m = t\tau, t > 0, \tau > 0),$$

we get the recurrence (8) for the $p_r(s)$:

$$(8) \quad \sum_{r=0}^s 2^{4r} p_r(s) N(2m, 4s+4, 4r+2) = \xi_{2s+1}(m),$$

where $\xi_{2s+1}(m)$ denotes the sum of the $(2s+1)$ th powers of the divisors of m .

To illustrate the calculations, let $m=1, 3, 5, 7$. Then

$$p_0(s) N(2, 4s+4, 2) = \xi_{2s+1}(1); \quad p_0(s) = 1.$$

For $m=3, 5, 7$ we need the following N 's:

$$N(2m, 4s+4, 2m) = 1;$$

$$N(6, 4s+4, 2) = \frac{2(4s+2)!}{1!(4s+1)!} = 8s+4;$$

$$N(10, 4s+4, 2) = 2 + \frac{2^2(4s+2)!}{2!(4s)!} = 2(16s^2+12s+3);$$

$$N(10, 4s+4, 6) = \frac{2(4s-2)!}{1!(4s-3)!} = 4(2s-1);$$

$$\begin{aligned} N(14, 4s+4, 2) &= \frac{1 \cdot 2^3(4s+2)!}{3!(4s-1)!} + 2 \cdot 2 \cdot \frac{(4s+2)!}{1!(4s+1)!}, \\ &= \frac{8}{3}(2s+1)(16s^2+4s+3); \end{aligned}$$

$$N(14, 4s+4, 6) = \frac{2^2 \cdot (4s-2)!}{2!(4s-4)!} + 6 = 32s^2 - 40s + 18;$$

$$N(14, 4s+4, 10) = \frac{2(4s-6)!}{1!(4s-7)!} = 4(2s-3).$$

It will be sufficient to indicate the origin of one of these, say $N(14, 4s+4, 2)$:

$$14 = 2 \cdot 1^2 + [3 \cdot 2^2 + (4s-1)0^2] = 1^2 + 3^2 + [2^2 + (4s+1)0^2].$$

Substituting these values in (8), and using $\zeta_{2s+1}(3) = 3^{2s+1} + 1$, etc., we find

$$\begin{aligned} p_0(s) &= 1; \quad 2^4 p_1(s) = 3^{2s+1} - 8s - 3; \\ 2^8 p_2(s) &= 5^{2s+1} - 4(2s-1)3^{2s+1} + 32s^2 - 32s - 17; \\ 2^{12} p_3(s) &= 7^{2s+1} - 4(2s-3)5^{2s+1} + (32s^2 - 88s + 30)3^{2s+1} \\ &\quad - \frac{1}{3}(256s^3 - 1056s^2 + 752s + 471), \end{aligned}$$

agreeing with the values stated by Hermite. As in §2, it can be shown that the general form is

$$(9) \quad 2^{4j} p_j(s) = (2j+1)^{2s+1} + B_1(s)(2j-1)^{2s+1} + \dots + B_j(s)1^{2s+1},$$

where $B_r(s)$ is a polynomial in s of degree r with rational coefficients. From the MacLaurin expansion the $p_j(s)$ are integers. An explicit (determinant) form follows from (8). The relation $\operatorname{sn}(kx, 1/k) = k \operatorname{sn}(x, k)$ gives $p_j(s) = p_{2-j}(s)$.

For $\operatorname{dn} x$ we have

$$\begin{aligned} \operatorname{dn} x &= 1 + \sum_{s=1}^{\infty} (-1)^s R_s(k^2) \frac{x^{2s}}{(2s)!}, \quad R_s(k^2) = \sum_{j=0}^{s-1} r_j(s) k^{2j+2}; \\ \vartheta_3^2 \operatorname{dn}(x\vartheta_3^2) &= \vartheta_0 \vartheta_3 \frac{\vartheta_3(x)}{\vartheta_0(x)} = 1 + 4 \sum q^n \left(\sum (-1 | \tau) \cos 2tx \right) \\ &\quad (n = 1, 2, 3, \dots; n = t\tau, \tau \text{ odd}, t > 0, \tau > 0); \\ (10) \quad \sum_{j=0}^{n-1} 2^{4j} N(4n, 4s+2, 4j+4) r_j(s) &= 2^{2s-2} \xi_{2s}(n), \end{aligned}$$

where ξ is as defined in §2. To calculate the successive $r_j(s)$, or to exhibit a determinant for $r_j(s)$, we take $n=1, 2, 3, \dots$, and proceed as before. Thus

$$\begin{aligned} r_0(s) &= 2^{2s-2}; \quad r_1(s) = 2^{2s-6}(2^{2s} - 8s + 4); \\ r_2(s) &= 2^{2s-10}[3^{2s} - 4(2s-3)2^{2s} + 32s^2 - 88s + 31]. \end{aligned}$$

The general form is

$$(11) \quad r_j(s) = 2^{2s-4j-2}[(j+1)^{2s} + C_1(s)j^{2s} + C_2(s)(j-1)^{2s} + \dots + C_j(s)1^s],$$

where $C_n(s)$ is a polynomial of degree n in s with rational coefficients. The relation $\operatorname{dn}(ku, 1/k) = \operatorname{cn}(u, k)$ gives $r_j(s) = q_{s-1-j}(s)$ (q as in §2); but this does not enable us to calculate the general $r_j(s)$ ($j=1, 2, \dots$) successively from the $q_j(s)$.

4. Reciprocal of $\operatorname{sn} x$. This will illustrate expansions in which the origin is a simple pole, and in which it is necessary to use the Bernoulli or Euler numbers to obtain the coefficients. From our paper already cited,* we have

$$\begin{aligned} \frac{x\vartheta_3^2}{\operatorname{sn}(x\vartheta_3^2)} &= x\vartheta_3\vartheta_3 \frac{\vartheta_0(x)}{\vartheta_1(x)} = x \csc x + 4x \sum q^n [\sum \sin \tau x] \\ &\quad (n = 1, 2, 3, \dots; n = t\tau, t > 0, \tau > 0, \tau \text{ odd}), \end{aligned}$$

the multiplier x being introduced to render the series regular at the origin. The form of the MacLaurin series is easily seen from the indicated division of x by the power series for $\operatorname{sn} x$ in §3, and we have

$$\frac{x}{\operatorname{sn} x} = 1 + \sum_{s=1}^{\infty} \frac{x^{2s}}{(2s-1)!} H_s(k^2); \quad H_s(k^2) = \sum_{r=0}^s (-1)^r h_r(s) k^{2r}.$$

To expand $x \csc x$ we shall use the numbers R of Lucas† defined by the symbolic identity

* Messenger of Mathematics, vol. 54 (1924), pp. 116-176, §14, p. 172.

† E. Lucas, *Théorie des Nombres*, chapter xiv.

$$x \csc x = 2 \cos Rx = 2 \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{(2s)!} R_{2s}.$$

In terms of the Bernoulli numbers B , in the even-suffix notation,

$$x \cot x = \cos 2Bx, \quad B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \dots,$$

we have

$$R_{2s} = (1 - 2^{2s-1})B_{2s}, \quad R_0 = \frac{1}{2}, \quad R_2 = -\frac{1}{6}, \quad R_4 = \frac{7}{30}, \dots$$

Proceeding as before we find

$$s\vartheta_3^{4s} H_s(k^2) = (-1)^s \left[R_{2s} - 4s \sum_{n=1}^{\infty} q^n \zeta'_{2s-1}(n) \right],$$

where $\zeta'_{2s-1}(n)$ denotes the sum of the $(2s-1)$ th powers of all the odd (positive) divisors of n . The left member is

$$s \left[h_0(s) \vartheta_3^{4s} + \sum_{r=1}^s h_r(s) \vartheta_3^{4r} \vartheta_3^{4s-4r} \right];$$

and

$$\vartheta_3^{4s} = 1 + \sum_{n=1}^{\infty} q^n N(n, 4s),$$

where $N(n, 4s)$ denotes the total number of representations of n as a sum of $4s$ squares. Hence

$$(12) \quad sh_0(s) = (-1)^s R_{2s},$$

$$h_0(s)N(n, 4s) + \sum_{r=1}^s (-1)^r h_r(s) 2^{4r} N(4n, 4s, 4r) = -4(-1)^s \zeta'_{2s-1}(n),$$

from which the successive $h_r(s)$ can be calculated as in previous examples, and the general form is easily determinable.

5. Expansion of $\wp(x)$. This is referred to the expansion of $x^2/\operatorname{sn}^2 x$ by means of

$$x^2 \wp(x; g_2, g_3) = \frac{x^2}{\operatorname{sn}^2(x, k)} - \frac{1+k^2}{3} x^2,$$

in the customary notation. As Gruder (loc. cit., §13) has shown the connection between the coefficients in the polynomials (in g_2, g_3 , or in the absolute invariant g_3^3/g_2^3) occurring as coefficients in the power series for $\wp(x; g_2, g_3)$

and the coefficients in the polynomials (in k^2) occurring as coefficients in the expansion of $x^2/\text{sn}^2(x, k)$, it will suffice here to give the recurrence for the latter.

From the expansion of $x/\text{sn } x$ it is easily seen that the MacLaurin series is of the form

$$\frac{x^2}{\text{sn}^2 x} = \sum_{s=0}^{\infty} \frac{x^{2s}}{(2s)!} T_{2s}(k^2), \quad T_{2s}(k^2) = \sum_{r=0}^s t_r(s) k^{2r}, \quad T_0(k^2) = 1;$$

and from the author's paper cited above,* we have

$$\begin{aligned} \frac{x^2 \vartheta_3^4}{\text{sn}^2(x \vartheta_3^2)} &= x^2 \vartheta_2^2 \vartheta_3^2 \frac{\vartheta_0^2(x)}{\vartheta_3^2(x)}, \\ &= x^2 [4 \sum q^n \sigma_1(n) + \csc^2 x - 8 \sum q^{2n} (\sum d \cos 2dx)] \\ &\quad (n = 1, 2, 3, \dots; n = d\delta, d > 0, \delta > 0), \end{aligned}$$

where $\sigma_1(n) \equiv \zeta_1(n) + \zeta'_1(n)$, ζ, ζ' being as defined in §§3, 4. To expand $x^2 \csc^2 x$ we may use the Bernoulli numbers of the second order,† or proceed as follows to obtain the coefficients at once in terms of ordinary Bernoulli numbers. The symbolic identity defining the Bernoulli numbers B is $x \text{ctn } x = \cos 2Bx$. Differentiating this with respect to x and multiplying the result throughout by x , we get

$$x^2 \csc^2 x = \cos 2Bx + 2Bx \sin 2Bx.$$

Hence, equating coefficients of like powers of x , we have

$$x^2 \csc^2 x \equiv \cos Dx, \quad D_{2n} = 2^{2n}(1 - 2n)B_{2n} \quad (n = 0, 1, \dots).$$

The rest of the work is like that in preceding sections, and we get (from the coefficients of $x^2, x^{2s}, s > 1$, in the identity between power series in x) the preliminary results

$$\begin{aligned} \vartheta_3^4 T_2(k^2) &= 8 [\sum q^n \sigma_1(n) - 2 \sum q^{2n} \zeta_1(n)] - D_2, \\ \vartheta_3^{4s} T_{2s}(k^2) &= (-1)^s [D_{2s} + 16s(2s-1) \sum q^{2n} 2^{2s-2} \zeta_{2s-1}(n)] \end{aligned}$$

for $s > 1$, the summations referring to $n = 1, 2, 3, \dots$. The first of these gives

$$t_0(1) \vartheta_3^4 + t_1(1) \vartheta_3^4 = 8 [\sum q^n \sigma_1(n) - 2 \sum q^{2n} \zeta_1(n)] - D_2;$$

and there are the known expansions

$$\begin{aligned} \vartheta_3^4 &= 1 + 8 \sum q^n (-1)^n \lambda_1(n), \quad \vartheta_2^4 = 16 \sum q^m \zeta_1(m) \\ &\quad (n = 1, 2, 3, \dots; m = 1, 3, 5, \dots), \end{aligned}$$

* Messenger of Mathematics, vol. 54 (1924), pp. 116-176, §16, p. 173.

† N. E. Nörlund, *Differenzenrechnung*, 1924, p. 129, et seq. The symbolic processes used here are justified (among other places) in my *Algebraic Arithmetic*, 1927.

where

$$\lambda_1(n) = [1 + 2(-1)^n] \zeta'_1(n).$$

From the definitions of the functions it is easily seen that

$$\sum q^n \sigma_1(n) - 2 \sum q^{2n} \zeta_1(n) = 2 \sum q^n \zeta'_1(n).$$

Hence, finally, we get

$$(13) \quad t_0(1) = t_1(1) = \frac{2}{3}.$$

All this detail for $t_0(1)$, $t_1(1)$ is of course unnecessary, as $T_2(k^2)$ is readily seen to be $\frac{2}{3}(1+k^2)$; but the reduction provides a check on the expansions.

Reducing the second of the above preliminary results as before we find

$$(14) \quad t_0(s) = (-1)^s D_{2s} = (-1)^s 2^{2s} (1-2s) B_{2s};$$

$$(15) \quad t_0(s) N(m, 4s) + \sum_{r=1}^m 2^{4r} t_r(s) N(4m, 4s, 4r) = 0$$

$$(m = 1, 3, 5, \dots);$$

$$(16) \quad t_0(s) N(2n, 4s) + \sum_{r=1}^{2n} 2^{4r} t_r(s) N(8n, 4s, 4r)$$

$$= (-1)^s (2s-1) 2^{2s+2} \zeta_{2s-1}(n) \quad (n = 1, 2, 3, \dots),$$

all of which hold only for $s > 1$. The functions N , ζ are as previously defined. From these the structure of $t_i(s)$ is seen as before (the few specimens given by Hermite, Oeuvres, vol. 3, p. 239, in another notation, do not indicate that the $(2s-1)$ th powers of integers > 1 enter the $t_r(s)$ for $r > 2$). Taking $m=1$ in (15) we get ($s > 1$)

$$t_1(s) = (-1)^s s(2s-1) 2^{2s-1} B_{2s},$$

and $n=1$ in (16),

$$t_2(s) = (-1)^s s(2s-1) 2^{2s-8} [1 - 2(4s-7) B_{2s}],$$

which check with tabulated results for $s=2, 3, 4, 5$.

6. Further developments. Hermite (Oeuvres, vol. 3, p. 245) was interested in these expansions partly on account of their possible applications to Gylden's methods (followed by Brendel) in the computation of perturbations, particularly for the so-called critical planets, whose mean motion is almost commensurable with Jupiter's. In this connection the expansions of power-products of $\operatorname{sn} x$, $\operatorname{cn} x$, $\operatorname{dn} x$ are required, the powers being positive or negative. From the series for $\operatorname{sn} x$, $\operatorname{cn} x$, $\operatorname{dn} x$ and their reciprocals, the general k^2 -polynomial form of the coefficient of x^n in the expansion of $\operatorname{sn}^a x \operatorname{cn}^b x \operatorname{dn}^c x$,

where a, b, c are integers, can be inferred. The general trigonometric series for use in the present method were investigated by Meyer,* from whose general results the types of arithmetical functions appearing in the coefficients can be determined. As this is quite an extensive subject we shall not go into it here, except to note a *necessary* change which occurs in the arithmetical character of the coefficients when any one of a, b, c passes the value 2: the functions are no longer expressible in terms of the divisors of a *single* integer (as they are for all the expansions in the present paper), but refer to representations in quadratic forms other than xy (which introduces the functions of divisors). For example, one function is $\sum (xyzw)^4$, the sum being taken over all representations of a fixed integer in the form $x^2 + y^2 + z^2 + w^2$. This is analogous to the similar situation concerning the number of representations of an integer as a sum of $2s$ squares when $s > 4$, where we have the classical theorems for $s = 5, 6$ which introduce quadratic forms other than xy .†

* C. O. Meyer, *Crelle's Journal*, vol. 37 (1848), pp. 273-304.

† For the following references to the astronomical applications, I am indebted to Professors A. O. Leuschner and R. H. Sciobereti.

(1) M. Brendel, *Abhandlungen der Königl. Gesellschaft der Wissenschaften zu Göttingen*, vol. 1, No. 2 (1898), part 1, pp. 45-51 and pp. 53-55; vol. 6, No. 4 (1909), part 2, chapter 2, p. 12.

(2) H. Gylden, *Studien auf dem Gebiete der Störungstheorie*, Academy of St. Petersburg, *Memoirs*, (7), vol. 16.

(3) F. Tisserand, *Mécanique Céleste*, vol. I, Chapter XVII: *Sur certaines fonctions des grands axes qui se présentent dans le développement de la fonction perturbatrice*.

(4) H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*. Poincaré's summary of Gylden's method is in vol. 2, p. 202, et seq., more particularly pp. 247-251-253.

(5) H. Gylden, *Traité des Orbites Absolues des 8 Planètes Principales*, 1893. vol. I, book II, chapter II; vol. I, book III, chapter II, p. 357, p. 394. Brendel's work on this particular subject is merely a reproduction of Gylden's treatment of the perturbative function.

SOME INEQUALITIES FOR NON-UNIFORMLY BOUNDED ORTHO-NORMAL POLYNOMIALS*

BY
M. F. ROSSKOPF

1. Introduction. Let the set $\{\phi_n(x)\}$ be an ortho-normal set of functions on the interval (a, b) and let M be a constant such that

$$|\phi_n(x)| \leq M \quad (n = 0, 1, 2, \dots; a \leq x \leq b);$$

then the Fourier expansion of any function $f(x)$ in terms of these functions is

$$f(x) \sim \sum_{n=0}^{\infty} c_n \phi_n(x), \text{ where } c_n = \int_a^b f(t) \phi_n(t) dt.$$

For sets of functions which satisfy the above assumptions the following two theorems of F. Riesz† are well known.

THEOREM A. *Let the set $\{\phi_n(x)\}$ of ortho-normal functions defined on the interval (a, b) satisfy the condition*

$$|\phi_n(x)| \leq M \quad (n = 0, 1, 2, \dots; a \leq x \leq b);$$

and let $f(x) \in L_p$ ($1 < p \leq 2$). Then

$$\left(\sum_{n=0}^{\infty} |c_n|^{p'} \right)^{1/p'} \leq M^{(2-p)/p} \left(\int_a^b |f(t)|^p dt \right)^{1/p},$$

where p' is determined by the relation $1/p + 1/p' = 1$.

THEOREM B. *If the series $\sum |c_n|^p$ is convergent, then the constants c_n are the Fourier coefficients of a function $f(x) \in L_{p'}$ ($p' \geq 2$), relative to a set of uniformly bounded ortho-normal functions; and moreover*

$$\left(\int_a^b |f(t)|^{p'} dt \right)^{1/p'} \leq M^{(2-p)/p} \left(\sum_{n=0}^{\infty} |c_n|^p \right)^{1/p},$$

where p and p' satisfy the relation $1/p + 1/p' = 1$.

* Presented to the Society, December 27, 1933; received by the editors May 8, 1934. The author is indebted to Professor J. D. Tamarkin for suggestions and criticisms.

† F. Riesz, *Über eine Verallgemeinerung der Parsevalschen Formel*, Mathematische Zeitschrift, vol. 18 (1923), pp. 117-124. For the case of trigonometric series see F. Hausdorff, *Eine Ausdehnung des Parsevalschen Satzes über Fourierreihen*, Mathematische Zeitschrift, vol. 16 (1923), pp. 163-169. In this paper is also given a list of W. H. Young's papers on the subject.

As was called to my attention by Professors Hille and Tamarkin, in the case of the expansion of the function

$$(1) \quad f(x) = \left(\frac{2}{1-x} \right)^\alpha$$

in normalized Legendre polynomials, F. Riesz's theorems do not hold. Stieltjes† considered this function and showed that for the convergence of its Legendre series, besides assuming $-1 < x < 1$, it is necessary to take $\alpha < \frac{3}{2}$; from the asymptotic value of the coefficients this is easily seen, since

$$c_n = \frac{2^{1/2}(2n+1)^{1/2}\Gamma(\alpha+n)\Gamma(1-\alpha)}{\Gamma(\alpha)\Gamma(n-\alpha+2)} \sim \frac{2\Gamma(1-\alpha)}{\Gamma(\alpha)} n^{2\alpha-3/2}.$$

The function (1) belongs to L_p for every $p < 1/\alpha$, whereas the series $\sum |c_n|^{p'}$ diverges whenever $\alpha \geq \frac{3}{2}$; thus it is seen F. Riesz's first theorem does not apply to Legendre series.

The problem now is to modify the inequalities which appear in Theorems A and B so that the Legendre coefficients of a certain class of functions would satisfy a new inequality. In particular it is desirable to obtain an inequality which would take care of this function of Stieltjes. In the first part of the present paper this problem is solved not only for the case of normalized Legendre, Jacobi and Hermite polynomials but also for a general class of ortho-normal polynomials possessing certain properties.

The end of the paper contains theorems for our general class of ortho-normal polynomials, which were suggested by a publication of R. E. A. C. Paley‡ in which he extended some results of Hardy and Littlewood§ from Fourier series to the case of a set of uniformly bounded ortho-normal functions. The following theorem is typical of Paley's results.

THEOREM C. Let c_0, c_1, c_2, \dots denote a bounded set of numbers such that $c_n \rightarrow 0$ as $n \rightarrow \infty$, and let

$$c_0^* \geq c_1^* \geq c_2^* \geq \dots$$

denote the set $|c_0|, |c_1|, |c_2|, \dots$ rearranged in descending order of magnitude. If the series $\sum c_n^* n^{p'-2}$ converges, where $p' \geq 2$, and if the ortho-normal set $\{\theta_n(x)\}$ satisfies the condition

† *Correspondence d'Hermite et Stieltjes*, Paris, Gauthier-Villars, 1905, vol. 2, letter 249, p. 46.

‡ R. E. A. C. Paley, *Some theorems on orthogonal functions* (1), *Studia Mathematica*, vol. 3 (1931), pp. 226-238.

§ G. H. Hardy and J. E. Littlewood, *Some new properties of Fourier constants*, *Mathematische Annalen*, vol. 97 (1926), pp. 159-209. *Notes on the theory of series* (XIII): *Some new properties of Fourier constants*, *Journal of the London Mathematical Society*, vol. 6 (1931), pp. 3-9.

$$|\theta_n(x)| \leq M \quad (n = 0, 1, 2, \dots; 0 \leq x \leq 1),$$

then the function $f(x) = \sum c_n \theta_n(x)$ is of class $L_{p'}$ and

$$(2) \quad \int_0^1 |f(t)|^{p'} dt \leq A_{p'} \sum_{n=0}^{\infty} c_n^{*p'} (n+1)^{p'-2}$$

where $A_{p'}$ depends only on p' and M .

As in the case of Theorems A and B modifications must be made in the inequality (2) in order to arrive at the theorems for our general class of ortho-normal polynomials.

2. Lemmas of M. Riesz. In this section we state two theorems of M. Riesz† which will be of fundamental importance in the proofs of our theorems. For convenience of reference we shall designate them as Lemmas 1 and 2. First we define a certain class of functions; the function $f(x)$ will be said to belong to the class L_{ϕ}^a if the following Lebesgue-Stieltjes integrals of $f(x)$ with respect to the non-decreasing function $\phi(x)$, defined on the interval (a, b) , exist and are finite:

$$\int_a^b f(t) d\phi(t), \quad \int_a^b |f(t)|^a d\phi(t) \quad (a \geq 1).$$

Similarly we can define the class of functions L_{ψ}^c ($c \geq 1$), corresponding to the non-decreasing function $\psi(x)$ defined on the interval (a', b') .

LEMMA 1. Let $T = T(f)$ be a linear limited functional transformation of certain classes L_{ϕ}^a into certain corresponding classes L_{ψ}^c ; i.e.,

(1) the transformation is distributive, so that for arbitrary constants λ_1, λ_2 ,

$$T(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 T(f_1) + \lambda_2 T(f_2);$$

(2) there exists a constant M^* such that

$$\left(\int |T(f)|^c d\psi \right)^{1/c} \leq M^* \left(\int |f|^a d\phi \right)^{1/a}.$$

Denote by $M^*(\alpha, \gamma)$ the least upper bound of the ratio

$$\left(\int |T(f)|^c d\psi \right)^{1/c} / \left(\int |f|^a d\phi \right)^{1/a}$$

for every couple of exponents a and c , where $a\alpha = c\gamma = 1$. If the relation between a and c is such that one always has $c \geq a$, and if the point (α, γ) describes a straight line segment in the triangle $0 \leq \gamma \leq \alpha \leq 1$, then $\log M^*(\alpha, \gamma)$ is a convex function of the points of the line segment.

† M. Riesz, *Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires*, Acta Mathematica, vol. 49 (1926), pp. 465-497. In particular see Theorems V and VI.

LEMMA 2. Every time that one has a linear limited functional transformation of L_{ψ^1} into L_{ψ^1} and of L_{ψ^2} into L_{ψ^2} , with $c_1 \geq a_1$, $c_2 \geq a_2$, the transformation can be extended to every couple of exponents corresponding to the points (α, γ) of the line segment joining the points (α_1, γ_1) and (α_2, γ_2) .

3. Notation and definitions. Let

$$A_0(x), A_1(x), \dots, A_n(x), \dots$$

be polynomials which are of exactly the n th degree for each value of n ; let $p(x)$ be a non-negative weight function, integrable and not identically equal to zero in the interval (a, b) . This set of polynomials will be said to be orthogonal if

$$\int_a^b p(t) A_n(t) A_m(t) dt = 0 \quad (n \neq m)$$

and normal if

$$\int_a^b p(t) A_n^2(t) dt = 1.$$

If the Fourier coefficients of a function $f(x)$ relative to these polynomials,

$$c_n = \int_a^b p(t) f(t) A_n(t) dt,$$

exist, the expansion of $f(x)$ in terms of these polynomials is

$$f(x) \sim \sum_{n=0}^{\infty} c_n A_n(x).$$

Let the function $\alpha(x)$ be absolutely continuous and such that

$$\alpha'(x) = \beta(x) \geq 0;$$

in addition let $\beta(x) > 0$ except for a set of measure zero. Set

$$(3) \quad W(x) = [p(x)/\beta(x)]^{1/2} \geq 0;$$

for convenience we shall write

$$J_p(f) = J_p = \left(\int_a^b |W(t)f(t)|^p d\alpha(t) \right)^{1/p}, \quad S_{p'}(f) = S_{p'} = \left(\sum_{n=0}^{\infty} |c_n|^{p'} \right)^{1/p'}.$$

If $J_p(f)$ exists, we shall write $W(x)f(x) \in L_{\alpha^p}$.

Throughout the paper we shall understand by p and p' two numbers which satisfy the relations $1 \leq p \leq 2$, $p' \geq 2$, $1/p + 1/p' = 1$; hence when $p=1$, the

corresponding value of p' is ∞ . Furthermore, A will be used in the generic sense to denote a constant independent of n and x .

We postulate the following properties of the set of polynomials $\{A_n(x)\}$:

- (1) the $A_n(x)$ are ortho-normal in the above sense;
- (2) $|W(x)A_n(x)| \leq A$, for all $(n=0, 1, 2, \dots)$ and $a \leq x \leq b$.

Property (2) is also a condition for the function $\beta(x)$ since it appears in $W(x)$.

It is interesting to see how the function $W(x)$ is introduced. Bessel's inequality for the polynomials $A_n(x)$ suggests putting

$$J_x^2 = \int_a^b p(t) |f(t)|^2 dt;$$

on the other hand, in order to use M. Riesz's lemmas we must have

$$J_x^2 = \int_a^b |W(t)f(t)|^2 d\alpha(t) = \int_a^b |W(t)f(t)|^2 \beta(t) dt.$$

Comparison of these two expressions for J_x^2 leads to setting

$$|W(t)|^2 = p(t)/\beta(t).$$

4. Generalizations of F. Riesz's theorems. Having agreed upon the above notation and properties of our ortho-normal polynomials, we can prove the following theorems.

THEOREM I. *If*

$$(4) \quad W(x)f(x) \in L_{\alpha^p},$$

then

$$S_{p'} \leq A^{(2-p)/p} J_p,$$

where A is a constant.

Set

$$(5) \quad F(x) = W(x)f(x),$$

and let $\phi(x) = \alpha(x)$, $\psi(x) = [x]$, where the symbol $[x]$ denotes the greatest integer in x . By definition the linear transformation T is

$$T(F) = c_n = \int_a^b p(t)f(t)A_n(t)dt$$

for all integral values n of x , and of arbitrary value for non-integral values of x .

In terms of this notation and the notation of Lemmas 1 and 2 what we wish to prove is that

$$(6) \quad M^*\left(\frac{1}{p}, \frac{1}{p'}\right) = \sup (S_{p'}/J_p) \leq A^{(2-p)/p}.$$

To prove the theorem it will suffice to show that $M^*(\frac{1}{2}, \frac{1}{2})$ and $M^*(1, 0)$ are bounded; then to interpolate for other values of $1/p$ and $1/p'$ on the line segment joining the two points $(\frac{1}{2}, \frac{1}{2})$ and $(1, 0)$ by Lemma 2. The desired inequality will result from Lemma 1.

Now $M^*(\frac{1}{2}, \frac{1}{2}) \leq 1$, by Bessel's inequality for our ortho-normal polynomials. For $M^*(1, 0)$ we write†

$$M^*(1, 0) = \sup \frac{\max_{0 \leq n < \infty} |c_n|}{J_1}.$$

By Property (2) we have‡

$$|c_n| \leq \int_a^b p(t) |f(t)| |A_n(t)| dt \leq A \int_a^b p^{1/2}(t) \beta^{1/2}(t) |f(t)| dt = AJ_1;$$

hence we have $M^*(1, 0) \leq A$.

The line segment with end points $(1, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ has the equation $\gamma = 1 - \alpha$, and lies in the triangle $0 \leq \gamma \leq \alpha \leq 1$. Consequently Lemma 2 applies and whenever $F(x) \in L_{\alpha^p}$, the series $\sum |c_n|^{p'}$ converges. The desired inequality (6) results from the convexity of $\log M^*(1/p, 1/p')$, assured by Lemma 1. Indeed, one has§

$$(7) \quad M^*\left(\frac{1}{p}, \frac{1}{p'}\right) \leq [M^*(1, 0)]^{(1/p-1/2)/(1-1/2)} \left[M^*\left(\frac{1}{2}, \frac{1}{2}\right)\right]^{(1-1/p)/(1-1/2)} \\ \leq A^{(2-p)/p}.$$

THEOREM II. *If the series $\sum |c_n|^p$ converges, then the numbers c_n are the Fourier coefficients of a function $f(x)$ such that*

$$W(x)f(x) \in L_{\alpha^{p'}},$$

and

$$J_{p'} \leq A^{(2-p)/p} S_p,$$

where A is a constant.

† The usual convention is made here. When $p' = \infty$, in order to compute $M^*(1, 0)$, the numerator is replaced by $\max |c_n|$ over all values ($n=0, 1, 2, \dots$). In the case of an integral appearing in the numerator, it is replaced by the upper measurable bound (in the sense of Lebesgue) of the integrand, which we shall designate simply as the maximum. Cf. M. Riesz, loc. cit., footnote 2, p. 477.

‡ The author is indebted to the referee for a simplification of the argument at this point.

§ For the origin of this inequality see M. Riesz, loc. cit., p. 484.

The notation of Lemmas 1 and 2 becomes $\psi(x) = \alpha(x)$, $\phi(x) = [x]$. In the case of Theorem I it was $f(x)$ which was varied but now the c_n are the quantities varied. By definition the transformation T will be such a transformation on the space of elements $c = (c_0, c_1, c_2, \dots, c_n, \dots)$, $\sum |c_n|^p < \infty$, which associates with c the Fourier expansion of the function $[W(x)]^{-1}F(x)$ which has the components of c for Fourier coefficients. We set

$$T(c) = F(x) \sim W(x) \sum_{n=0}^{\infty} c_n A_n(x);$$

then

$$M^*\left(\frac{1}{p}, \frac{1}{p'}\right) = \sup (J_{p'}/S_p).$$

Now $M^*(\frac{1}{2}, \frac{1}{2}) \leq 1$, by the Riesz-Fischer theorem. For $M^*(1, 0)$ we must write

$$M^*(1, 0) = \sup \frac{\max_{a \leq x \leq b} |F(x)|}{S_1},$$

but by Property (2) the numerator is bounded by AS_1 ; hence $M^*(1, 0) \leq A$. Using Lemmas 1 and 2 and the inequality (7), the statement of the theorem follows.

5. Jacobi polynomials. In order to show that Theorems I and II hold for normalized Jacobi polynomials we have only to prove that they possess Properties (1) and (2).

The Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$, $\alpha > -1$, $\beta > -1$, with the weight function $p(x) \equiv (1-x)^a(1+x)^b$ and $a = -1$ and $b = 1$, are orthogonal in the sense of Property (1). In fact†

$$\begin{aligned} \int_{-1}^1 (1-t)^a (1+t)^b P_n^{(\alpha, \beta)}(t) P_m^{(\alpha, \beta)}(t) dt \\ = \begin{cases} 0, & n \neq m; \\ \frac{2^{\alpha+\beta+1}}{2n + \alpha + \beta + 1} \frac{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}{\Gamma(n + 1)\Gamma(n + \alpha + \beta + 1)}, & n = m; \end{cases} \end{aligned}$$

then the set of polynomials

$$Q_n^{(\alpha, \beta)}(x) = \frac{P_n^{(\alpha, \beta)}(x)}{(k_n(\alpha, \beta))^{1/2}} \quad (n = 0, 1, 2, \dots),$$

† G. Pólya and G. Szegő, *Aufgaben und Lehrsätze aus der Analysis*, Berlin, Julius Springer, 1925, vol. II, pp. 93, 292.

where

$$k_n(\alpha, \beta) = \frac{2^{\alpha+\beta+1}}{2n + \alpha + \beta + 1} \frac{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}{\Gamma(n + 1)\Gamma(n + \alpha + \beta + 1)},$$

is an ortho-normal set of polynomials.

The function $\alpha(x) = \arcsin x$; this $\alpha(x)$ is obviously absolutely continuous in the interval $(-1, 1)$, and the function $\beta(x) = (1 - x^2)^{-1/2}$ is positive throughout this interval. The function $W(x)$ takes the form

$$W(x) = (1 - x)^{\alpha/2+1/4}(1 + x)^{\beta/2+1/4}.$$

It can be easily verified by applying Stirling's formula for $\Gamma(x)$ to the normalizing factor $[k_n(\alpha, \beta)]^{-1/2}$ that

$$(8) \quad Q_n^{(\alpha, \beta)}(x) = O(n^{1/2})P_n^{(\alpha, \beta)}(x).$$

Suppose $\alpha \geq -\frac{1}{2}$, $\beta \geq -\frac{1}{2}$; then the following result of S. Bernstein* is just a statement of the fact that the $Q_n^{(\alpha, \beta)}(x)$ possess Property (2).

LEMMA 3. Suppose that

$$\max_{-1 \leq x \leq 1} (1 - x)^{\alpha/2+1/4}(1 + x)^{\beta/2+1/4} |P_n^{(\alpha, \beta)}(x)| = M_n(\alpha, \beta).$$

Then

$$\lim_{n \rightarrow \infty} n^{1/2} M_n(\alpha, \beta) = 2^{(\alpha+\beta)/2} M(\alpha, \beta)$$

exists and

$$M(\alpha, \beta) = \begin{cases} \left(\frac{2}{\pi}\right)^{1/2}, & \text{if } -\frac{1}{2} \leq \alpha \leq \frac{1}{2}, \quad -\frac{1}{2} \leq \beta \leq \frac{1}{2}; \\ \text{finite and } > \left(\frac{2}{\pi}\right)^{1/2}, & \text{if } \alpha > \frac{1}{2}, \quad \beta \geq -\frac{1}{2} \\ & \text{or if } \alpha \geq -\frac{1}{2}, \quad \beta > \frac{1}{2}; \\ +\infty, & \text{if } \alpha \text{ or } \beta < -\frac{1}{2}. \end{cases}$$

THEOREM I'. The result of Theorem I is valid in the case of normalized Jacobi polynomials when $\alpha > -1$, $\beta > -1$, provided that $f(x)$ satisfies the further condition $(1-x)^\alpha(1+x)^\beta f(x) \in L$.

* S. Bernstein, *Sur les polynomes orthogonaux relatifs à un segment fini*, Journal de Mathématiques, (9), vol. 9 (1930), pp. 127-177; vol. 10 (1931), pp. 219-286; see pp. 225-232. These results are proved in a very simple way by G. Szegő, *Asymptotische Entwicklungen der Jacobischen Polynome*, Schriften der Königsberger Gelehrten Gesellschaft, vol. 10 (1933), pp. 35-110, p. 79.

The analysis is the same as that already given except that in showing that $|c_n|$ is bounded independently of n a discussion of the case $-1 < \alpha < -\frac{1}{2}$, $-1 < \beta < -\frac{1}{2}$ must be given. For this purpose we shall use the following bound for $P_n^{(\alpha, \beta)}(x)$ found by Szegő*,

$$(9) \quad |P_n^{(\alpha, \beta)}(x)| \leq A n^{\max(\alpha, -1/2)} \quad (-1 + \epsilon \leq x \leq 1),$$

the classical inequality for $P_n^{(\alpha, \beta)}(x)$ due to Darboux†,

$$(10) \quad |P_n^{(\alpha, \beta)}(\cos \theta)| \leq A n^{-1/2} (\sin \theta)^{-1/2} \left(\sin \frac{\theta}{2} \right)^{-\alpha} \left(\cos \frac{\theta}{2} \right)^{-\beta} \\ \left(\epsilon \leq \theta \leq \pi - \epsilon, 0 < \epsilon < \frac{\pi}{2}; n = 1, 2, \dots \right);$$

and the well known relation,

$$(11) \quad P_n^{(\alpha, \beta)}(-x) = (-1)^n P_n^{(\beta, \alpha)}(x).$$

Making use of (8), we write

$$|c_n| \leq O(n^{1/2}) \int_{-1}^1 (1-t)^{\alpha}(1+t)^{\beta} |f(t)| |P_n^{(\alpha, \beta)}(t)| dt \\ = O(n^{1/2}) \left\{ \int_{-1}^{-1+\delta} + \int_{-1+\delta}^{1-\delta} + \int_{1-\delta}^1 (1-t)^{\alpha}(1+t)^{\beta} |f(t)| |P_n^{(\alpha, \beta)}(t)| dt \right\} \\ = I_1 + I_2 + I_3,$$

where $0 < \delta < 1$. We have by (9) that

$$(12) \quad I_3 = O(1) \int_{1-\delta}^1 (1-t)^{\alpha}(1+t)^{\beta} |f(t)| dt;$$

using (11) and then (9), we can write

$$(13) \quad I_1 = O(1) \int_{-1}^{-1+\delta} (1-t)^{\alpha}(1+t)^{\beta} |f(t)| dt.$$

By the added condition on $f(x)$ the integrals (12) and (13) tend to zero as $\delta \rightarrow 0$. There remains to be considered I_2 ; in estimating it we make use of (10),

$$I_2 = O(1) \int_{-1+\delta}^{1-\delta} (1-t)^{\alpha/2-1/4} (1+t)^{\beta/2-1/4} |f(t)| dt \\ \leq A \int_{-1}^1 (1-t)^{\alpha/2-1/4} (1+t)^{\beta/2-1/4} |f(t)| dt;$$

* G. Szegő, loc. cit., p. 77.

† G. Darboux, *Sur l'approximation des fonctions de très grands nombres et sur une classe étendue de développements en série*, Journal de Mathématiques, (3), vol. 4 (1878), pp. 5-56, 377-416; p. 50.

hence

$$|c_n| \leq I_1 + I_2 + I_3 \leq A \int_{-1}^1 (1-t)^{\alpha/2-1/4} (1+t)^{\beta/2-1/4} |f(t)| dt = AJ_1.$$

This completes the proof that $|c_n|$ is bounded independently of n when $-1 < \alpha < -\frac{1}{2}$, $-1 < \beta < -\frac{1}{2}$; the proof in the case that $\alpha \geq -\frac{1}{2}$, $\beta \geq -\frac{1}{2}$ follows as before.

Our proof depends fundamentally on the applicability of Lemmas 1 and 2. The space of functions satisfying the conditions of Theorem I' is a sub-space of the space of functions satisfying the conditions of Theorem I. Hence what we have assumed is that M. Riesz's theorems hold in every sub-space of their space of definition. That this is true is apparent from the way in which M. Riesz derives his results.

That a similar extension of Theorem II is possible is not at all obvious. The lemma of S. Bernstein would seem to preclude that.

6. Legendre polynomials. The normalized Legendre polynomials defined on the interval $(-1, 1)$,

$$\left\{ \left(\frac{2n+1}{2} \right)^{1/2} P_n(x) \right\}, \left(\frac{2n+1}{2} \right)^{1/2} \left(\frac{2m+1}{2} \right)^{1/2} \int_{-1}^1 P_n(t) P_m(t) dt = \delta_{nm},$$

automatically possess Properties (1) and (2) since they correspond to the values $\alpha = \beta = 0$ of the parameters in Jacobi polynomials. The function $\alpha(x)$ has the same definition.

It can be shown that the Theorems I and II for Legendre polynomials sift out the correct intervals of convergence and divergence of the sum and integral involved in the inequalities for the Stieltjes function (1).

7. Hermite polynomials. The set of normalized Hermite polynomials,

$$\left\{ \frac{1}{(2^n n! \pi^{1/2})^{1/2}} H_n(x) \right\},$$

with $p(x) \equiv e^{-x^2}$, $a = -\infty$, $b = \infty$, possesses Property (1); indeed*

$$\int_{-\infty}^{\infty} e^{-t^2} H_n(t) H_m(t) dt = \begin{cases} 0, & n \neq m, \\ 2^n n! \pi^{1/2}, & n = m. \end{cases}$$

If $\alpha(x) = x$, it is easily seen that it is absolutely continuous and that $\beta(x) > 0$; by use of the inequality†

* E. Hille, *A class of reciprocal functions*, Annals of Mathematics, (2), vol. 27 (1925-26), pp. 427-464; pp. 431, 436.

† E. Hille, loc. cit., p. 435.

$$|H_n(x)| \leq A 2^{n/2} (n!)^{1/2} e^{x^2/2},$$

it is easy to see that normalized Hermite polynomials possess Property (2).

8. **Generalizations of Paley's theorems.** Let the set of polynomials $\{A_n(x)\}$ and the function $\alpha(x)$ have Properties (1) and (2).

Throughout we shall denote by c_0, c_1, c_2, \dots a bounded set of numbers such that $c_n \rightarrow 0$ as $n \rightarrow \infty$, and by

$$c_0^* \geq c_1^* \geq c_2^* \geq \dots$$

the set $|c_0|, |c_1|, |c_2|, \dots$ rearranged in descending order of magnitude.

THEOREM III. *If the series $\sum c_n^* n^{p'-2}$ converges where $p' \geq 2$, then*

$$(14) \quad W(x)f(x) \sim W(x) \sum_{n=0}^{\infty} c_n A_n(x)$$

is of class $L_{\alpha^{p'}}$, and

$$(J_{p'})^{p'} \leq A \sum_{n=0}^{\infty} c_n^* n^{p'-2},$$

where A is a constant.

It is observed that the series in the right member of (14) converges in the mean of order 2 and hence represents some function of the Lebesgue class L_2 ; for

$$\sum_{n=0}^{\infty} c_n^* \leq \left(\sum_{n=0}^{\infty} c_n^* n^{p'-2} \right)^{2/p'} \left(\sum_{n=0}^{\infty} (n+1)^{-2} \right)^{(p'-2)/p'} < \infty.$$

Consider the inequality

$$(15) \quad \int_a^b |W(t)f(t)|^{p'} d\alpha(t) \leq A \sum_{n=0}^{\infty} |c_n|^{p'} (n+1)^{p'-2};$$

for the case $p'=2$ it is well known. If Lemmas 1 and 2 applied, it would be sufficient to prove the theorem for positive even values of p' . Let us show first that these lemmas do apply.

Let the linear transformation T be by definition

$$T\{(n+1)c_n\} = W(x)f(x) \sim W(x) \sum_{n=0}^{\infty} (n+1)c_n \left\{ \frac{A_n(x)}{n+1} \right\};$$

let $\psi(x) = \alpha(x)$, and

$$\phi(x) = 1 + \frac{1}{2^2} + \dots + \frac{1}{([x]+1)^2} \quad (n \leq x < n+1; n = 0, 1, 2, \dots).$$

For the bound $M^*(1/p', 1/p')$ we have

$$M^*\left(\frac{1}{p'}, \frac{1}{p'}\right) = \sup \frac{\left(\int_a^b |W(t)f(t)|^{p'} d\alpha(t)\right)^{1/p'}}{\left(\sum_{n=0}^{\infty} |c_n(n+1)|^{p'(n+1)^{-2}}\right)^{1/p'}}.$$

Therefore Lemmas 1 and 2 will apply, and it suffices to prove the theorem when p' is an even integer.

To fix the ideas take $p'=4$; the proof for other even integral values of p' is similar. In this case (15) becomes

$$\int_a^b |W(t)f(t)|^4 d\alpha(t) \leq A \sum_{n=0}^{\infty} |c_n|^4 (n+1)^2.$$

Consider the sequence of functions,

$$f_0(x) = W(x)c_0A_0(x), \quad f_1(x) = W(x)c_1A_1(x),$$

$$f_m(x) = W(x) \sum_{n=2^{m-1}}^{2^m-1} c_n A_n(x), \quad m \geq 2;$$

and let

$$\epsilon_0 = c_0^4, \quad \epsilon_1 = c_1^4 2^2, \quad \epsilon_m = \sum_{n=2^{m-1}}^{2^m-1} c_n^4 (n+1)^2, \quad m \geq 2.$$

Then, if μ, ν are any two integers such that $0 < \mu \leq \nu$,

$$\begin{aligned} & \int_a^b f_\mu^2(t) f_\nu^2(t) d\alpha(t) \\ & \leq \int_a^b W^2(t) \left[\sum_{n=2^{\mu-1}}^{2^\mu-1} c_n A_n(t) \right]^2 d\alpha(t) \cdot \max_{a \leq t \leq b} \left[W(t) \sum_{n=2^{\nu-1}}^{2^\nu-1} c_n A_n(t) \right]^2 \\ & \leq \left[\sum_{n=2^{\mu-1}}^{2^\mu-1} c_n^2 \right] \left[A \sum_{n=2^{\mu-1}}^{2^\mu-1} |c_n| \right]^2 \leq A \left[\sum_{n=2^{\mu-1}}^{2^\mu-1} c_n^4 (n+1)^2 \right]^{1/2} \left[\sum_{n=2^{\nu-1}}^{2^\nu-1} (n+1)^{-2} \right]^{1/2} \\ & \quad \times \left[\sum_{n=2^{\mu-1}}^{2^\mu-1} c_n^4 (n+1)^2 \right]^{1/2} \left[\sum_{n=2^{\mu-1}}^{2^\mu-1} (n+1)^{-2/3} \right]^{3/2} \\ & \leq A \epsilon_\nu^{1/2} \epsilon_\mu^{1/2} 2^{(\nu-\mu)/2} \leq A(\epsilon_\nu + \epsilon_\mu) 2^{-|\mu-\nu|/2}, \end{aligned}$$

where use has been made of Hölder's inequality and Properties (1) and (2). Since this result is symmetric in μ and ν , it holds also if $\mu > \nu$.

It follows from the above equation that if m_1, m_2, m_3, m_4 are arbitrary positive integers, then

$$\begin{aligned}
& \int_a^b |f_{m_1}(t)f_{m_2}(t)f_{m_3}(t)f_{m_4}(t)| d\alpha(t) \\
& \leq \left(\int_a^b f_{m_1}^2 f_{m_2}^2 d\alpha(t) \right)^{1/6} \left(\int_a^b f_{m_1}^2 f_{m_4}^2 d\alpha(t) \right)^{1/6} \\
& \quad \times \left(\int_a^b f_{m_1}^2 f_{m_4}^2 d\alpha(t) \right)^{1/6} \left(\int_a^b f_{m_2}^2 f_{m_3}^2 d\alpha(t) \right)^{1/6} \\
& \quad \times \left(\int_a^b f_{m_2}^2 f_{m_4}^2 d\alpha(t) \right)^{1/6} \left(\int_a^b f_{m_3}^2 f_{m_4}^2 d\alpha(t) \right)^{1/6} \\
& \leq A(\epsilon_{m_1} + \epsilon_{m_2} + \epsilon_{m_3} + \epsilon_{m_4}) \\
& \quad \times 2^{-(1/12)(|m_1-m_2|+|m_1-m_3|+|m_1-m_4|+|m_2-m_3|+|m_2-m_4|+|m_3-m_4|)}.
\end{aligned}$$

Using this result we obtain,

$$\begin{aligned}
& \int_a^b (|f_1| + |f_2| + \cdots + |f_m| + \cdots)^4 d\alpha(t) \\
& \leq 6 \sum \int_a^b |f_{m_1}(t)f_{m_2}(t)f_{m_3}(t)f_{m_4}(t)| d\alpha(t) \\
& \leq A \sum (\epsilon_{m_1} + \epsilon_{m_2} + \epsilon_{m_3} + \epsilon_{m_4}) 2^{-(1/12)(|m_1-m_2|+|m_1-m_3|+|m_1-m_4|+|m_2-m_3|+|m_2-m_4|+|m_3-m_4|)}.
\end{aligned}$$

In the summation over m_4 the coefficient of an ϵ_m in the above sum is

$$\sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \sum_{m_3=1}^{\infty} 2^{-(1/12)(|m-m_1|+|m-m_2|+|m-m_3|+|m_1-m_2|+|m_1-m_3|+|m_2-m_3|)} \leq A;$$

it follows that

$$\int_a^b (|f_1| + |f_2| + \cdots + |f_m| + \cdots)^4 d\alpha(t) \leq A \sum_{m=1}^{\infty} \epsilon_m.$$

Now,

$$\int_a^b f_0^4(t) d\alpha(t) = c_0^4 \int_a^b [W(t)A_0(t)]^4 d\alpha(t) \leq A c_0^4 \int_a^b d\alpha(t) \leq A c_0^4 = A \epsilon_0,$$

by Property (2); consequently

$$(16) \quad \int_a^b \left[\sum_{n=0}^{\infty} |f_n(t)| \right]^4 d\alpha(t) \leq A \sum_{n=0}^{\infty} \epsilon_n = A \sum_{n=0}^{\infty} |c_n|^4 (n+1)^2.$$

From this we infer that the series

$$\sum_{n=0}^{\infty} f_n(x) = W(x) \sum_{n=0}^{\infty} c_n A_n(x)$$

converges almost everywhere, but the series $\sum c_n A_n(x)$ converges in the mean of order 2 to $f(x)$ as was remarked at the beginning of the proof. Since these two limits must be the same, we have finally

$$\int_a^b |W(t)f(t)|^2 d\alpha(t) \leq A \sum_{n=0}^{\infty} |c_n|^2 (n+1)^2.$$

It will be noticed that the inequality stated in the theorem was not proved but the inequality (15) was proved instead. The only point of the proof which depends on n is the use of Property (2); furthermore this estimate does not depend at all on the order in which it is made; hence we can assume the c_n are already rearranged in decreasing order of magnitude.

THEOREM IV. *If*

$$W(x)f(x) \sim W(x) \sum_{n=0}^{\infty} c_n A_n(x) \in L_a^p,$$

where $1 < p \leq 2$, then

$$\sum_{n=0}^{\infty} c_n^{*p} (n+1)^{p-2} \leq A J_p^p,$$

where A is a constant.

In view of our last remark above it suffices to prove

$$\sum_{n=0}^{\infty} |c_n|^p (n+1)^{p-2} \leq A J_p^p.$$

Let us write

$$d_n = |c_n|^{p-1} (n+1)^{p-2} \overline{\operatorname{sgn} c_n};$$

then

$$c_n = |d_n|^{p'-1} (n+1)^{p'-2} \overline{\operatorname{sgn} d_n};$$

where $p' \geq 2$, $1/p + 1/p' = 1$. Now

$$\sum_{n=0}^N |c_n|^p (n+1)^{p-2} = \sum_{n=0}^N c_n d_n = \sum_{n=0}^N |d_n|^{p'} (n+1)^{p'-2};$$

let

$$g_N(x) = \sum_{n=0}^N d_n A_n(x).$$

Using Hölder's inequality and Theorem III, we have

$$\begin{aligned}\sum_{n=0}^N |c_n|^p (n+1)^{p-2} &= \sum_{n=0}^N c_n d_n = \int_a^b p(t) f(t) g_N(t) dt \\ &\leq J_p(f) J_{p'}(g_N) \leq A J_p(f) \left(\sum_{n=0}^N |d_n|^{p'} (n+1)^{p'-2} \right)^{1/p'} \\ &= A J_p(f) \left(\sum_{n=0}^N |c_n|^p (n+1)^{p-2} \right)^{1/p'};\end{aligned}$$

therefore

$$\sum_{n=0}^N |c_n|^p (n+1)^{p-2} \leq A J_p(f).$$

Since A is independent of N , the theorem follows by making N tend to infinity.

The form of the expression (16) suggests the stronger inequality of the following theorem, the details of whose proof are analogous to those of the proof of Paley's† Theorem III. The only change is the introduction of the integrator function $\alpha(x)$ and the factor $W(x)$.

THEOREM V. Let $S(x)$ denote the upper bound

$$S(x) = \sup_{0 \leq m \leq \infty} \left| \sum_{n=0}^m c_n A_n(x) \right|.$$

Then, if $p' > 2$,

$$\int_a^b [W(t)S(t)]^{p'} d\alpha(t) \leq A \sum_{n=0}^{\infty} c_n^{*p'} (n+1)^{p'-2},$$

where A is a constant.

† R. E. A. C. Paley, loc. cit., pp. 232-238.

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ON BOUNDED LINEAR FUNCTIONAL OPERATIONS*

BY

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The set of all bounded linear functional operations on a given vector space† plays an important role in the consideration of linear functional operations. For the sake of greater definiteness it is desirable to know the form of such operations and a space determined thereby. This problem has been solved for a number of spaces, for instance, all continuous functions on a bounded closed interval, all Lebesgue p th power ($p \geq 1$) integrable functions, all sequences whose p th powers ($p \geq 1$) form absolutely convergent series, all sequences having a limit, and so on.‡ All of these spaces have the property of separability. For non-separable spaces, there is a recent determination of the operation for the space of all bounded functions on a finite interval, having at most discontinuities of the first kind, by H. S. Kaltenborn.§

In this paper we give a determination of the linear operation for the space of (a) all bounded sequences, (b) all bounded measurable functions, (c) all bounded functions on the infinite interval having at most discontinuities of the first kind, (d) all bounded continuous functions on the infinite interval, (e) all almost bounded functions. With the least upper bound as norm, all of these spaces are not separable.

1. Notations. The integral. We shall denote by

- (a) \mathfrak{P} a set of elements p .
- (b) ξ a real-valued function on \mathfrak{P} .
- (c) \mathfrak{X} a set of functions ξ .
- (d) \mathfrak{C} a set or class of subsets E of \mathfrak{P} , containing the null set and the set \mathfrak{P} .

* Presented to the Society, April 7, 1934; received by the editors April 17, 1934.

† By a linear vector space \mathfrak{X} , we shall mean a so-called Banach space (see Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 55) of elements ξ in which there is defined addition, and multiplication by constants, subject to the usual laws of algebra, a unique zero, and a distance function or norm $\|\xi\|$ subject to the condition $\|c_1\xi_1 + c_2\xi_2\| \leq |c_1| \cdot \|\xi_1\| + |c_2| \cdot \|\xi_2\|$ for all c_1 and c_2 . A linear operation L on \mathfrak{X} transforms \mathfrak{X} into real numbers and satisfies the condition $L(c_1\xi_1 + c_2\xi_2) = c_1L(\xi_1) + c_2L(\xi_2)$. L is bounded and therefore continuous if there exists an M such that, for all ξ , $|L(\xi)| \leq M\|\xi\|$. The smallest possible value for M is the modulus or norm M_L of L . We shall limit ourselves to real-valued linear operations since a complex-valued operation is expressible as the sum of two real-valued ones.

‡ See Banach, *Opérations Linéaires*, pp. 59–72; Hildebrandt, *Linear functional transformations in general spaces*, Bulletin of the American Mathematical Society, vol. 37 (1931), p. 189.

§ See Bulletin of the American Mathematical Society, vol. 40 (1934).

It will be assumed that \mathfrak{E} is additive and multiplicative, i.e., if E_1 and E_2 belong to \mathfrak{E} so do $E_1 + E_2$ and $E_1 E_2$.

(e) Π a finite partition or subdivision of \mathfrak{P} into mutually exclusive sets E_1, \dots, E_n belonging to \mathfrak{E} . $\Pi_1 \geq \Pi_2$ shall mean that every set $E^{(1)}$ of Π_1 is a subset of some set $E^{(2)}$ of Π_2 .

Because of the multiplicative property of \mathfrak{E} the partitions Π satisfy the conditions of a range on which the general limit of E. H. Moore-H. L. Smith* is definable, i.e., if β_Π is any function defined for all partitions Π of \mathfrak{P} , then $\lim \beta_\Pi = b$ has the following meaning: for every $\epsilon > 0$ there exists a Π_ϵ , such that if $\Pi \geq \Pi_\epsilon$, then $|\beta_\Pi - b| \leq \epsilon$.

(f) $\alpha(E)$ a function on \mathfrak{E} . $\alpha(E)$ is *additive* if $\alpha(E_1 + E_2) + \alpha(E_1 E_2) = \alpha(E_1) + \alpha(E_2)$, for every E_1 and E_2 of \mathfrak{E} . $\alpha(E)$ is of bounded variation on \mathfrak{P} if $\sum_\Pi |\alpha(E_i)|$ is bounded for all Π of \mathfrak{P} , and the least upper bound of this sum, which agrees with the limit in the Π sense if α is additive, is the total variation of α , $V(\alpha)$, on \mathfrak{P} . Obviously if α is additive, the boundedness of α on \mathfrak{E} is necessary and sufficient that α be of bounded variation.

For a bounded function ξ and a function α it is possible to define the Stieltjes integral $S \int \xi d\alpha$:

$$S \int \xi d\alpha = \lim_\Pi \sum \xi(p_i) \alpha(E_i),$$

where $\Pi = E_1, \dots, E_n$, and p_i is any point of E_i . We shall say that ξ is *S-integrable* relative to α if the limit on the left exists.†

For a bounded function ξ which is measurable relative to \mathfrak{E} , in the sense that for every c and d the set $E[c < \xi(p) \leq d]$ belongs to \mathfrak{E} , it is possible to define the Lebesgue integral $L \int \xi d\alpha$ by the Lebesgue process, viz., if (a, b) is an interval containing the range of values of ξ , and $a = y_0 < y_1 < \dots < y_n = b$ is any subdivision of (a, b) while $y_{i-1} < \eta_i \leq y_i$, then

$$L \int \xi d\alpha = \lim \sum \eta_i \alpha(E_i),$$

where $E_i = E[y_{i-1} < \xi(p) \leq y_i]$, and the limit is taken as the maximum of $y_i - y_{i-1}$ approaches zero.

If α is additive and bounded on \mathfrak{E} , and ξ is measurable relative to \mathfrak{E} , then obviously $L \int \xi d\alpha$ exists. The $S \int \xi d\alpha$ exists also in this case and agrees with the L -integral. The S -integral may exist even though ξ be not measurable

* A general theory of limits, American Journal of Mathematics, vol. 44 (1922), p. 103.

† This is a type of integral suggested by Moore-Smith (loc. cit., p. 114) and considered by Kolmogoroff, Untersuchungen ueber den Integralbegriff, Mathematische Annalen, vol. 103 (1930), pp. 682 ff.

relative to \mathfrak{E} . For example if $\mathfrak{P} = [0 < p \leq 1]$, \mathfrak{E} consists of all finite sets of non-overlapping subintervals of \mathfrak{P} open on the left, while $\alpha(E)$ is the length of E , then the set of all bounded Riemann integrable functions on \mathfrak{P} is obviously S -integrable with respect to α , but includes functions not measurable with respect to \mathfrak{E} . The same is true to a lesser degree if \mathfrak{E} is the set of all subsets of \mathfrak{P} having Jordan content and $\alpha(E) = \text{cont } E$.*

2. **Bounded sequences.** Let \mathfrak{P} be the set of all positive integers p . Let \mathfrak{E} be the set of all subsets of \mathfrak{P} , i.e., E is any set of positive integers. Π is then any division of \mathfrak{P} into a finite number of mutually exclusive sets of positive integers. At least one set in Π will contain an infinite number of elements, but they all may.

Let \mathfrak{X} be the vector space consisting of all bounded sequences, i.e., of all bounded real-valued functions ξ on \mathfrak{P} , with $\|\xi\|$ the least upper bound of the values $|\xi(p)|$. Then we have the following

THEOREM. Any bounded linear operation L on \mathfrak{X} is expressible in the form $L(\xi) = \int \xi d\alpha$, the integral being taken in either the L or S sense, and α being a bounded additive function on \mathfrak{E} whose total variation on \mathfrak{P} is the modulus of L . Conversely every such integral is a linear bounded operation on \mathfrak{X} .

Let $\chi(E, p)$ represent the characteristic function of the set E , i.e., zero for p not on E and unity for p on E . Then if $\alpha(E) = L(\chi(E, p))$ it is obviously additive and bounded on \mathfrak{E} .

Divide \mathfrak{P} by the partition $\Pi = E_1, \dots, E_n$. Define

$$\xi(\Pi) = \sum_{i=1}^n \xi(p_i) \chi(E_i, p).$$

Then $\lim \|\xi(\Pi) - \xi\| = 0$. For suppose that the range of values of $\xi(p)$ is contained in the interval (a, b) , and divide (a, b) into n equal parts by the points $a = y_0 < y_1 < y_2 < \dots < y_n = b$, so that $(b-a)/n$ is less than a given ϵ . If E_i is the set $E[y_{i-1} < \xi(p) \leq y_i]$ and Π_ϵ consists of E_1, \dots, E_n , then obviously $\|\xi(\Pi_\epsilon) - \xi\| \leq \epsilon$. The same inequality will also hold for any repartition Π of Π_ϵ , which demonstrates the assertion.

Now by the linearity of L ,

$$L(\xi(\Pi)) = \sum_i \xi(p_i) L(\chi(E_i, p)) = \sum_i \xi(p_i) \alpha(E_i).$$

By the boundedness of L and the convergence of the right-hand side, it follows that

$$L(\xi) = \int \xi d\alpha,$$

* See J. Ridder, *Nieuw Archiv der Wiskunde*, (2), vol. 15 (1928), pp. 321-9; O. Frink, *Annals of Mathematics*, (2), vol. 34 (1933), pp. 518-527.

where it is obvious that since ξ is measurable relative to \mathfrak{E} , the integral on the left may be defined in either the S or L sense. The fact that the total variation of α is the modulus of L follows from the fact that if

$$\xi(p) = \sum \chi(E_i, p) \operatorname{sgn} \alpha(E_i),$$

then $\|\xi\| = 1$ and $L(\xi) = \sum |\alpha(E_i)|$.

The converse theorem follows from obvious properties of the integral.

It is possible to give the result another form. Suppose p_i is the first integer in the set E_i . Define the sequence or function $\beta_\Pi(p) = 0$ if $p \neq p_i$, while $\beta_\Pi(p) = \alpha(E_i)$ if $p = p_i$. Then the approximating sum $\sum \xi(p_i) \alpha(E_i)$ can be written $\sum_p \beta_\Pi(p) \xi(p)$, where only a finite number of the $\beta_\Pi(p)$ are not zero for a given Π . We can consequently state the following:

To every linear functional operation L there corresponds a set of sequences $\beta_\Pi(p)$ whose elements are different from zero at most for a finite number of p , such that

$$(1) \quad \lim_\Pi \sum_p \beta_\Pi(p) \xi(p) = L(\xi);$$

$$(2) \quad \sum_p |\beta_\Pi(p)| \leq M_L$$

and

$$(3) \quad \lim_\Pi \sum_p |\beta_\Pi(p)| = M_L.$$

This result parallels a result due to Banach* for separable subspaces of the space \mathfrak{X} . While the limit involved in this result can be reduced to a sequential limit for each particular ξ , a non-sequential limit is needed for the whole space. The import of the Banach theorem is that for the case of a separable subspace \mathfrak{X}_0 of \mathfrak{X} , there exists a sequence of partitions Π_n , such that for every ξ of \mathfrak{X}_0 ,

$$\lim_n \|\xi(\Pi_n) - \xi\| = 0 \text{ and } \lim_n \sum_p \beta_{\Pi_n}(p) \xi(p) = L(\xi).$$

It is possible to deduce this result from our general considerations. For this purpose we note that if ξ_n is any sequence of functions of the space \mathfrak{X} , it is possible to select a sequence of partitions Π_n by the diagonal process, such that $\lim_n \xi(\Pi_n) = \xi$ for every ξ_n of the given sequence. If \mathfrak{X}_0 is a separable subspace of \mathfrak{X} and ξ_n is dense in \mathfrak{X}_0 , then if ξ belongs to \mathfrak{X}_0 , there exists a sequence ξ_{n_m} approaching ξ . Let $\Pi_1, \dots, \Pi_{k_1}, \dots$ be the partitions such that for every n

$$\lim_k \xi_n(\Pi_k) = \xi_n.$$

* *Opérations Linéaires*, p. 72.

Now

$$\lim_m \|\xi_{n_m}(\Pi_k) - \xi(\Pi_k)\| = 0$$

uniformly in k , since

$$\|\xi_{n_m}(\Pi_k) - \xi(\Pi_k)\| \leq \|\xi_{n_m} - \xi\|.$$

Hence by the iterated limits theorem on double sequences it follows that $\lim_k \|\xi(\Pi_k) - \xi\| = 0$. Consequently, for every ξ of \mathfrak{X}_0 ,

$$\lim_k \sum_p \beta_{\Pi_k}(p) \xi(p) = \lim_k \sum_{n_k} \xi(p_i) \alpha(E_i) = \int \xi d\alpha = L(\xi).$$

We note that if ξ is a special sequence, it may not be necessary to use all of the values of α . For instance if ξ is a sequence converging to zero, it is sufficient to know the values of $\alpha(E_p)$ where E_p consists of the integer p only. Obviously in this case $L(\xi)$ reduces to $\sum_1^\infty \xi(p) \alpha(E_p)$, with $\sum |\alpha(E_p)| < \infty$, which is a well known result. Similarly for any sequence converging to a limit, the values $\alpha(E_p)$ and $\alpha(\mathfrak{P})$ suffice.

The effect of the fundamental theorem established is that a conjugate space to the space of all bounded sequences is the space of all additive bounded functions on subsets of integers, with norm the total variation of the function.

The question naturally arises whether additive functions on the set \mathfrak{E} exist, which are not absolutely additive, i.e., whether this form of operation is effective. The instance of sequences approaching a limit must come from such a function. Banach's measure function* on subsets of positive integers gives a complete example.

3. **Bounded measurable functions.** The procedure in this case is entirely analogous to the preceding case.

We let $\mathfrak{P} = -\infty < p < \infty$, \mathfrak{E} the set of all measurable subsets of \mathfrak{P} , \mathfrak{X} the set of all bounded measurable functions ξ on \mathfrak{P} , with $\|\xi\|$ the least upper bound of $|\xi(p)|$. Then we have

THEOREM. Any linear functional operation on the space of all bounded measurable functions is expressible in the form $\int \xi d\alpha$, where the integral is to be taken in either the S or L sense, the function α is additive and bounded on \mathfrak{E} , and the total variation of α is the modulus M_L of L . $\alpha(E)$ is the value of $L(\chi(E))$, where $\chi(E)$ is the characteristic function of E .

Obviously it is possible to give a theorem corresponding to the Banach result, viz., that the operation L is the Π -limit of a set of finite sums, each involving the function ξ at only a finite number of points.

* *Opérations Linéaires*, p. 231.

4. **Bounded functions on the infinite interval having at most discontinuities of the first kind.** This class of functions is obviously a subclass of the set of bounded measurable functions. As a consequence it is to be expected that a smaller set \mathfrak{E} will suffice. Let again $\mathfrak{P} = -\infty < p < \infty$. Then the set \mathfrak{E} consists of all subsets E of \mathfrak{P} , which consist of a finite or infinite number of open intervals and single points, there being at most a finite number of intervals and individual points in the finite part of the fundamental interval, i.e., a set E consists of disjoint open intervals $a_n < p < b_n$, together with points p_n either end points of (a_n, b_n) or not belonging to any (a_n, b_n) , where a_n, b_n, p_n have at most $+\infty$ and $-\infty$ as limiting points. The intervals $-\infty < p < a$ and $a < p < \infty$ will be considered open intervals.

With this definition we have the following

THEOREM. *Every bounded linear operation on the set of all bounded functions on $-\infty < p < \infty$ having at most discontinuities of the first kind is expressible in the form $\int f d\alpha$ where the integral is of the S type, α is additive and bounded on \mathfrak{E} and has total variation M_L .*

In order to prove this theorem it is sufficient to show that the functions

$$\xi(\Pi) = \sum_i \xi(p_i) \chi(E_i, p)$$

approach ξ uniformly in the Π -sense. For this purpose we utilize the theorem of Lebesgue* that if ξ is bounded and has only discontinuities of the first kind and is limited to a finite interval (a, b) then there exists for any given $\epsilon > 0$ a subdivision of (a, b) into a finite number of intervals such that on each open subinterval the oscillation of ξ is less than ϵ . It follows that for any given ξ and any $\epsilon > 0$, there exists a sequence of points $\dots p_{-n} < \dots < p_{-1} < p_0 < p_1 < \dots < p_n < \dots$ approaching $-\infty$ on the left and $+\infty$ on the right, such that interior to each interval (p_{i-1}, p_i) the oscillation of ξ is less than ϵ . Suppose now (c, d) contains the region of variation of $\xi(p)$, i.e., $c < \xi(p) < d$ for all p . Divide (c, d) into a finite number of equal parts of length ϵ_0 , by the points $c = y_0 < y_1 < \dots < y_n = d$. Let the set E_1 consist of all the intervals (p_i, p_{i+1}) containing in their interior a point p such that $y_0 < \xi(p) \leq y_1$, together with all points p_i satisfying the same condition. Let E_2 consist of all the intervals not belonging to E_1 which contain a point p for which $y_1 < \xi(p) \leq y_2$, and the points p_i satisfying the same condition, and so on. Then since the oscillation of ξ on any of the intervals (p_i, p_{i+1}) is at most ϵ , it follows that the oscillation of $\xi(p)$ on any E_k is at most $\epsilon + \epsilon_0$. Consequently

$$\|\xi(p) - \sum_1^n \xi(p_i) \chi(E_i, p)\| \leq \epsilon + \epsilon_0,$$

* Annales de la Faculté des Sciences de Toulouse, (3), vol. 1 (1909), p. 60.

\bar{p}_i being any point of E_i . Since the same type of inequality will be valid for any partition $\Pi \geq \Pi_0$, where Π_0 consists of E_1, \dots, E_n , we have the result desired.

The case in which the infinite interval is replaced by a finite interval has been considered by Kaltenborn.* In that case the infinite parts of our partitions drop away, and it can be shown that the integral depends only on a point function of bounded variation and a function zero except at a denumerable set of points, but it is simpler to proceed directly in this case.

5. **Bounded continuous functions on the infinite interval.** Obviously the class of functions considered in §4 contains the set of bounded continuous functions as a subset. As a consequence we can effect a further reduction in the set \mathfrak{E} . We shall assume that \mathfrak{E} contains all sets E which consist of a finite or denumerable set of non-overlapping half open intervals $(a_n < p \leq b_n)$, whose end points have at most $-\infty$ and $+\infty$ as limiting points. The intervals $-\infty < p \leq a$ and $a < p < \infty$ will be considered to be half open intervals.

With this definition of \mathfrak{E} we can state the same theorem as in the preceding paragraph. It is to be noted, however, that in this case the function $\alpha(E)$ is defined in terms of an extension of the linear operation L on continuous functions to functions having discontinuities of the first kind.

If we limit ourselves to a finite interval, the ordinary Stieltjes integral applies, since because of the continuity of ξ , the successive partition limit agrees with the limit as the maximum length of subdivisions approaches zero.

It is possible to give a form to the general theorem which is comparable to the Banach result for sequences. Let Π be any partition of \mathfrak{P} into sets E_1, \dots, E_n . Let p_i be any point in the interval of E_i nearest to $p=0$. Let $\beta_\Pi(p)$ be a point function such that $\beta_\Pi(0)=0$, and constant except at the points $p=p_i$, where it has a break or saltus of magnitude $\alpha(E_i)$. Then obviously

$$L(\xi(\Pi)) = L\left(\sum_1^n \xi(p_i)\chi(E_i, p)\right) = \sum \xi(p_i)\alpha(E_i) = \int_{-\infty}^{\infty} \xi(p)d\beta_\Pi(p),$$

where the infinite limits could be replaced by any finite interval containing the points p_1, \dots, p_n in its interior. It follows that we have the following alternative theorem:

If L is any bounded linear operation on the class of bounded continuous functions on $-\infty < p < \infty$, then there exists a set of point functions $\beta_\Pi(p)$ constant except at a finite number of points such that

$$L(\xi) = \lim_\Pi \int_{-\infty}^{\infty} \xi(p)d\beta_\Pi(p),$$

* Loc. cit.

the integral being an ordinary Stieltjes integral. The functions β_{Π} are uniformly of bounded variation and $\lim_{\Pi} V(\beta_{\Pi}) = M_L$.

For any separable subset we can obviously proceed as in §2 and replace the limit in the Π -sense by a sequential limit, i.e., we can find a sequence Π_n of partitions which is effective in the limit for all functions of the set.

6. Almost bounded measurable functions. In agreement with common usage the measurable function is almost bounded if it is bounded except for a set of zero measure. The $\|\xi\|$ is defined as the greatest lower bound of positive numbers a such that the set $E[|\xi(p)| > a]$ is of zero measure.

The only difference between this case and that of §3 is that if E is a set of zero measure then $\alpha(E) = 0$, since then $L(\chi(E)) = 0$. It cannot however be concluded that if $\alpha(E) = 0$ for any set of zero measure then $\alpha(E)$ is absolutely continuous and consequently the indefinite integral of a Lebesgue integrable function.

An example on the finite interval $0 \leq p \leq 1$ of an additive bounded function $\alpha(E)$ on measurable sets which satisfies the condition that $\alpha(E) = 0$ for $\text{meas } E = 0$, but is not absolutely continuous nor absolutely additive, can be constructed. Let (a_n, b_n) be a sequence of disjoint intervals whose end points have 1 as their only limiting point. If E_0 is any measurable subset of (a_n, b_n) , then define $\beta(E_0) = mE_0/(a_n - b_n)$. If now E is any subset of $0 \leq p \leq 1$, and E_n the part of E lying on (a_n, b_n) , then $\beta(E_n)$ defines a bounded sequence of numbers. The function $\alpha(E) = \int \beta(E_n) d\mu$ (in the sense of §2), where μ is a measure function of Banach on subsets of positive integers,* will be additive on measurable subsets of $(0, 1)$, will satisfy the condition

$$\alpha(E) = 0 \text{ if } \text{meas } E = 0,$$

but will not be absolutely additive, nor absolutely continuous. For if E is the set $(1 - e \leq p \leq 1)$ then for all $e > 0$, $\alpha(E) = 1$. Incidentally it appears that if $\xi(p)$ is continuous on $(0, 1)$ then $\int_0^1 \xi d\alpha = \xi(1)$, i.e., as far as the integration of continuous functions is concerned $\alpha(E)$ is equivalent to the function $\gamma(p) = 0$ for $0 \leq p < 1$, $\gamma(1) = 1$.

It is obvious that the results of §§3, 5, and 6 can be extended to corresponding situations in n -dimensional space. Also that it would be possible to set up a general theorem reducing to the special cases considered by a proper choice of the set \mathfrak{P} and \mathfrak{E} .

* *Opérations Linéaires*, p. 231.

A SET OF FOUR POSTULATES FOR BOOLEAN ALGEBRA IN TERMS OF THE "IMPLICATIVE" OPERATION*

BY

B. A. BERNSTEIN

1. Introduction. Whitehead and Russell's *Principia Mathematica* makes fundamental the notion " \supset " of "implication," defined by

$$p \supset q = \sim p \vee q, \text{ Df.}$$

The main object of my paper is to present in terms of this "implicative" operation† \supset a set of four postulates for Boolean algebra. This will secure for Boolean algebra, for the first time, a set of postulates expressed in terms of an operation other than "rejection" having as few postulates as the present minimum sets.‡ Of course, by the principle of duality in Boolean algebra, my postulates will also be a set in terms of the dual of $p \supset q$, namely $\sim pq$.

I prove for my postulates (a) their *consistency*, (b) their *mutual independence*, (c) their *sufficiency* for Boolean algebra, (d) their *necessariness* for Boolean algebra.§ The consistency and independence systems are all Boolean

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† The *Principia* calls \supset a *relation*.

‡ For the present minimum sets, see B. A. Bernstein, (I) *A set of four independent postulates for Boolean algebras*, these Transactions, vol. 17 (1916), pp. 50–51; (II) *Simplification of the set of four postulates for Boolean algebras in terms of rejection*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 783–787. For another set of postulates in terms of \supset , the first set in terms of \supset , see E. V. Huntington, (I) *A new set of independent postulates for the algebra of logic, with special reference to Whitehead and Russell's Principia Mathematica*, Proceedings of the National Academy of Sciences, vol. 18 (1932), pp. 179–180. Huntington's postulates are eight in number (including an inadvertently omitted existence postulate).

§ I offer (a)–(d) as a set of *defining* properties of a set of postulates: a system of propositions S is a set of *postulates* for a (consistent) system Σ if and only if the propositions of S are (a) consistent, (b) mutually independent, (c) sufficient for Σ , (d) necessary for Σ . More simply and less formally stated, a set of postulates for a system is a set of propositions of the system which cannot be derived from one another but from which all the other propositions of the system *can* be derived. This view of postulates is opposed to the view, seemingly held widely, that demands of postulates only sufficiency and necessariness (hence also consistency). The latter view would have to accept as a set of postulates for euclidean geometry *all* of Euclid's *Elements*, and would violate the generally accepted distinction between *postulate* and *theorem*. My view of postulates is of course opposed to the view, seemingly held by some, that demands of postulates only sufficiency. This view would have to accept S as a set of postulates for Σ , not only when S is the *whole* of Σ , but also when S is *inconsistent* (since "a false proposition implies any proposition") and when S is only a *special case* of Σ (when S , for example, is the theory of Abelian groups and Σ the theory of groups in general). If I am correct in my view, the term "independent postulates," found in the literature, must be understood to mean "postulates whose independence has been proved," and the term "postulates" applied to S when only (c) and (d) have

and very simple. The proof of sufficiency consists in deriving from the postulates my second set of postulates in terms of rejection (see my Paper II, loc. cit.); the proof of necessariness consists in the converse derivation.

I shall derive from my postulates the "theory of deduction" of the *Principia*. This will verify the fact, obtained elsewhere* less directly and from another point of view, that the theory of deduction is derivable from the general logic of classes.

There is a close relation between \supset and the operation " $-$ " of "exception" used by me†, and later by Taylor‡, in postulates for Boolean algebra. I shall bring out this relation.

If in a set of independent postulates for the logic of classes there is a proposition demanding that the number of elements be *at least* two, and if this proposition be replaced by a proposition demanding that the number of elements be *just* two, then the propositions resulting from the change will be sufficient for the logic of propositions as a two-element Boolean algebra. But these propositions will, in general, not be independent. I have so chosen my postulates that the change in question will render them a set of *independent* postulates for the logic of propositions.§

2. The postulates. The postulates with which we are mainly concerned have as primitive ideas a class K and a binary operation \supset , and are the propositions A_1 - A_4 below.|| In Postulates A_2 and A_3 , there is to be understood the supposition that *the elements involved and their indicated combinations belong to K* . This must especially be borne in mind when the independence of the postulates is considered. The postulates follow.

POSTULATE A_1 . $a \supset b$ is an element of K whenever a and b are elements of K .

been proved for S , must be understood to mean "provisional postulates" for Σ . If desired, "provisional postulates" might have a distinctive name, say *basic propositions* of Σ , or *defining conditions* for Σ .

* B. A. Bernstein, (III) *On section A of Principia Mathematica*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 788-792.

† B. A. Bernstein, (IV) *A complete set of postulates for the logic of classes expressed in terms of the operation "exception," and a proof of the independence of a set of postulates due to Dele Re*, University of California Publications in Mathematics, vol. 1, pp. 87-96.

‡ J. S. Taylor, *A set of five postulates for Boolean algebras in terms of the operation "exception,"* University of California Publications in Mathematics, vol. 1, pp. 242-248.

§ For a discussion of the nature of the logic of propositions, see B. A. Bernstein, (V) *Sets of postulates for the logic of propositions*, these Transactions, vol. 28 (1926), pp. 472-478.

|| The symbol " $=$ " used in the postulates is taken as an idea *outside* the system. Compare my (VI) *Whitehead and Russell's theory of deduction as a mathematical science*, Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 480-488. For sets of postulates for Boolean algebra in which " $=$ " is taken as an idea *within* the system, see E. V. Huntington, (II) *New sets of independent postulates for the algebra of logic, with special reference to Whitehead and Russell's Principia Mathematica*, these Transactions, vol. 35 (1933), pp. 274-304.

POSTULATE A₂. $(a \supset b) \supset a = a$.

POSTULATE A₃. There is an element z in K such that

$$(d \supset d) \supset [(a \supset b) \supset c] = \{[(c \supset z) \supset a] \supset [(b \supset c) \supset z]\} \supset z.$$

POSTULATE A₄. K consists of at least two distinct elements.

3. Consistency and independence of the postulates. The consistency and the independence of Postulates A₁-A₄ are given by the following systems S₀-S₄, in which S₀ is the consistency system, and S₁, S₂, S₃, S₄ are the independence systems for A₁, A₂, A₃, A₄ respectively. The systems are all Boolean.

| System | K | $a \supset b$ |
|----------------|-----|---------------|
| S ₀ | 0,1 | $a' + b$ |
| S ₁ | 0,1 | 0/0* |
| S ₂ | 0,1 | 0 |
| S ₃ | 0,1 | a |
| S ₄ | 0 | 0 |

4. Theorems. The proof of the sufficiency of Postulates A₁-A₄ for Boolean algebra, and for the theory of deduction of the *Principia*, will be effected with the help of the following theorems 1a-26a.

1a. $(a \supset z) \supset z = a.$

2a. $(a \supset a) \supset b = b.$

3a. $(a \supset b) \supset c = \{[(c \supset z) \supset a] \supset [(b \supset c) \supset z]\} \supset z.$

4a. $(a \supset z) \supset b = (b \supset z) \supset a.$

5a. $a \supset b = (b \supset z) \supset (a \supset z).$

6a. $a \supset (b \supset z) = b \supset (a \supset z).$

7a. The element z of Postulate A₃ is unique.

DEFINITION 1a. $a_1 = a \supset z.$

8a. $a_{11} = a,$ where $a_{11} = (a_1)_1.$

9a. $(a \supset b) \supset c = [(c_1 \supset a) \supset (b \supset c)_1]_1.$

10a. $a_1 \supset b = b_1 \supset a.$

11a. $a \supset b = b_1 \supset a_1.$

12a. $a \supset b_1 = b \supset a_1.$

13a. $a_1 = a \supset a_1.$

14a. $a \supset a = b \supset b.$

* In a two-element Boolean algebra, we may define the quotient precisely as in the case of the algebra of number.

DEFINITION 2a. $u = a \supset a$.

- 15a. $z \supset a = u$.
 16a. $u \supset a = a$.
 17a. $z_1 = u; u_1 = z$.
 18a. $a \supset b = (b \supset a_1) \supset a_1$.
 19a. $(a \supset b) \supset b = b_1 \supset a$.
 20a. $a \supset b = b_1 \supset (a \supset b)$.
 21a. $a \supset b = a \supset (a \supset b)$.
 22a. $a \supset b = (b \supset a) \supset (a \supset b)$.
 23a. $a \supset (b \supset a) = u$.
 24a. $a \supset (b \supset c) = (b \supset a) \supset (b \supset c)$.

DEFINITION 3a. $a|b = b \supset a_1$.

DEFINITION 4a. $a' = a|a$.

- 25a. $a' = a_1$.

DEFINITION 5a. $\sim a = a_1$.

DEFINITION 6a. $a \vee b = a_1 \supset b$.

- 26a. $\sim a \vee b = a \supset b$.

DEFINITION 7a. $\vdash a = (a = u)$.

5. Proofs of the theorems. The proofs of the theorems 1a-26a follow.

Proof of 1a. $a = (a \supset a) \supset a = (a \supset a) \supset [(a \supset z) \supset a] = \{ [(a \supset z) \supset a] \supset [(z \supset a) \supset z] \} \supset z = (a \supset z) \supset z$, by A_2, A_3, A_3, A_2 .

Proof of 2a. $(a \supset a) \supset b = (a \supset a) \supset [(b \supset z) \supset b] = \{ [(b \supset z) \supset b] \supset [(z \supset b) \supset z] \} \supset z = (b \supset z) \supset z = b$, by $A_2, A_3, A_2, 1a$.

Proof of 3a. $(a \supset b) \supset c = (d \supset d) \supset [(a \supset b) \supset c] = \{ [(c \supset z) \supset a] \supset [(b \supset c) \supset z] \} \supset z$, by 2a, A_3 .

Proof of 4a. $(a \supset z) \supset b = \{ [(b \supset z) \supset a] \supset [(z \supset b) \supset z] \} \supset z = \{ [(b \supset z) \supset a] \supset z \} \supset z = (b \supset z) \supset a$, by 3a, $A_3, 1a$.

Proof of 5a. $a \supset b = [(a \supset z) \supset z] \supset b = (b \supset z) \supset (a \supset z)$, by 1a, 4a.

Proof of 6a. $a \supset (b \supset z) = [(b \supset z) \supset z] \supset (a \supset z) = b \supset (a \supset z)$, by 5a, 1a.

Proof of 7a. Suppose that two elements, y and z , have the property of z . Then $y = (y \supset z) \supset z = \{ [(z \supset y) \supset y] \supset [(z \supset z) \supset y] \} \supset y = \{ z \supset [(z \supset z) \supset y] \} \supset y = (z \supset y) \supset y = z$, by 1a, 3a, 1a, 2a, 1a.

Proof of 8a. By def. 1a, 1a.

Proof of 9a. By def. 1a, 3a.

Proof of 10a. By def. 1a, 4a.

Proof of 11a. By def. 1a, 5a.

Proof of 12a. By def. 1a, 6a.

Proof of 13a. $a_1 = (a_1 \supset z) \supset a_1 = [(a \supset z) \supset z] \supset a_1 = a \supset a_1$, by A_2 , def. 1a, 1a.

Proof of 14a. $a \supset a = [(a \supset a) \supset z] \supset z = z \supset z = [(b \supset b) \supset z] \supset z = b \supset b$, by 1a, 2a, 2a, 1a.

Proof of 15a. $z \supset a = [(z \supset a) \supset z] \supset z = z \supset z = u$, by 1a, A_2 , def. 2a.

Proof of 16a. $u \supset a = (a \supset a) \supset a = a$, by def. 2a, A_2 .

Proof of 17a. $z_1 = z \supset z = u$, by def. 1a, def. 2a; $u_1 = (z \supset z)_1 = (z \supset z) \supset z = z$, by def. 2a, def. 1a, A_2 .

Proof of 18a. $(b \supset a_1) \supset a_1 = [(a_{11} \supset b) \supset (a_1 \supset a_1)_1]_1 = [(a_1 \supset a_1) \supset (a_{11} \supset b)_1]_1 = (a_{11} \supset b)_{11} = a \supset b$, by 9a, 12a, 2a, 8a.

Proof of 19a. $b_1 \supset a = (a \supset b_{11}) \supset b_{11} = (a \supset b) \supset b$, by 18a, 8a.

Proof of 20a. $b_1 \supset (a \supset b) = [(a \supset b) \supset b] \supset b = (b_1 \supset a) \supset b = (a_1 \supset b) \supset b = b_1 \supset a_1 = a \supset b$, by 19a, 19a, 10a, 19a, 11a.

Proof of 21a. $a \supset (a \supset b) = [(a \supset b) \supset a_1] \supset a_1 = [(a_{11} \supset a) \supset (b \supset a_1)_1]_1 \supset a_1 = [(a \supset a) \supset (b \supset a_1)_1]_1 \supset a_1 = (b \supset a_1)_{11} \supset a_1 = (b \supset a_1) \supset a_1 = a \supset b$, by 18a, 9a, 8a, 2a, 8a, 18a.

Proof of 22a. $(b \supset a) \supset (a \supset b) = \{ [(a \supset b)_1 \supset b] \supset [a \supset (a \supset b)]_1 \}_1 = \{ [(a \supset b)_1 \supset b] \supset (a \supset b)_1 \}_1 = (a \supset b)_{11} = a \supset b$, by 9a, 21a, A_2 , 8a.

Proof of 23a. $a \supset (b \supset a) = [(b \supset a) \supset a_1] \supset a_1 = [(a_{11} \supset b) \supset (a \supset a_1)_1]_1 \supset a_1 = [(a_{11} \supset b) \supset a_{11}]_1 \supset a_1 = [(a \supset b) \supset a]_1 \supset a_1 = a_1 \supset a_1 = u$, by 18a, 9a, 13a, 8a, A_2 , def. 2a.

Proof of 24a. $(b \supset a) \supset (b \supset c) = \{ [(b \supset c)_1 \supset b] \supset [a \supset (b \supset c)]_1 \}_1 = \{ [b_1 \supset (b \supset c)] \supset [a \supset (b \supset c)]_1 \}_1 = \{ [b_1 \supset (c_1 \supset b_1)] \supset [a \supset (b \supset c)]_1 \}_1 = \{ u \supset [a \supset (b \supset c)]_1 \}_1 = [a \supset (b \supset c)]_{11} = a \supset (b \supset c)$, by 9a, 10a, 11a, 23a, 16a, 8a.

Proof of 25a. $a' = a \mid a = a \supset a_1 = a_1$, by def. 4a, def. 3a, 13a.

Proof of 26a. $\sim a \vee b = a_1 \vee b = a_{11} \supset b = a \supset b$, by def. 5a, def. 6a, 8a.

6. Sufficiency of the postulates. I shall now prove the sufficiency of postulates A_1 - A_4 by deriving from them my second set of postulates for Boolean algebra in terms of rejection.* This set has as primitive ideas K and " \mid ," and as postulates the propositions B_1 - B_4 following (in postulates B_3 and B_4 there is to be understood the supposition that *the elements involved and their indicated combinations belong to K*).

B_1 . K contains at least two distinct elements.

B_2 . If a and b are elements of K , $a \mid b$ is an element of K .

DEFINITION 1b. $a' = a \mid a$.

B_3 . $a = (b \mid a) \mid (b' \mid a)$.

B_4 . $a \mid (b \mid c) = [(c' \mid a) \mid (b' \mid a)]'$.

* See my Paper II, loc. cit.

The derivations of B_1 - B_4 from A_1 - A_4 follow.

Proof of B_1 . By A_4 .

Proof of B_2 . By def. 3a, def. 1a, A_1 .

Proof of B_3 . $a = (a_1)_1 = [(b \supset b) \supset a_1]_1 = [(a_{11} \supset b) \supset (b \supset a_1)]_{11} = (a \supset b) \supset (b \supset a_1)_1 = (b \supset a_1) \supset (a \supset b)_1 = (b \supset a_1) \supset (b_1 \supset a_1)_1 = (b|a)|(b_1|a) = (b|a)|(b'|a)$, by 8a, 2a, 9a, 8a, 12a, 11a, def. 3a, 25a.

Proof of B_4 . $a|(b|c) = a \supset (b \supset c)_1 = (b \supset c_1) \supset a_1 = [(a_{11} \supset b) \supset (c_1 \supset a_1)]_1 = [(b_1 \supset a_1) \supset (c_1 \supset a_1)]_1 = [(c_1 \supset a_1) \supset (b_1 \supset a_1)]_1 = [(c' \supset a_1) \supset (b' \supset a_1)]'_1 = [(c'|a)|(b'|a)]'$, by def. 3a, 12a, 9a, 10a, 12a, 25a, def. 3a.

7. Necessariness of the postulates. I shall prove that A_1 - A_4 are necessary for Boolean algebra by deriving A_1 - A_4 from the rejection postulates B_1 - B_4 above. For this derivation I shall use as auxiliary theorems propositions 1b-8b following, derivable from B_1 - B_4 .

$$1b. \quad a'' = a, \text{ where } a'' = (a')'.$$

$$2b. \quad a|b = b|a.$$

$$3b. \quad a|(b|b') = a'.$$

$$4b. \quad (a|c)|(b|c) = [c|(a'|b')]'.$$

$$5b. \quad a|a' = b|b'.$$

DEFINITION 2b. $u = a|a'$.

$$6b. \quad a|u = a'.$$

$$7b. \quad a|u' = u.$$

DEFINITION 3b. $a \supset b = a|b'$.

$$8b. \quad a \supset u' = a'.$$

Propositions 1b, 2b, 3b, 5b are respectively Sheffer's Postulate 3, Theorem A, Postulate 4, Theorem B.* The proofs of 4b, 6b, 7b, and 8b follow.

Proof of 4b. $[c|(a'|b')]' = [(b''|c)|(a''|c)]'' = (b|c)|(a|c)$, by B_4 , 1b.

Proof of 6b. $a|u = a|(a|a') = a'$, by def. 2b, 3b.

Proof of 7b. $a|u' = [(a|u')]' = [(a|u')|(a|a')]' = [(u'|a)|(a'|a)]' = [a|(u''|a'')]' = a|(u|a) = a|(a|u) = a|a' = u$, by 1b, 3b, 2b, 4b, 1b, 2b, 6b, def. 2b.

Proof of 8b. $a \supset u' = a|u'' = a|u = a'$, by def. 3b, 1b, 6b.

The derivations of A_1 - A_4 from B_1 - B_4 now follow.

Proof of A_1 . By def. 3b, def. 1b, B_2 .

Proof of A_2 . $(a \supset b) \supset a = (a|b')|a' = (a|b')|[a|(b|b')]' = (b'|a)|[(b'|b)|a]$

* See H. M. Sheffer, *A set of five independent postulates for Boolean algebras, with application to logical constants*, these Transactions, vol. 14 (1913), pp. 481-488.

$= \{a \mid [b'' \mid (b' \mid b)']\}' = \{a \mid [b \mid (b' \mid b)']\}' = \{a \mid [b \mid (b \mid b')']\}' = [a \mid (b \mid u')]' = (a \mid u')' = a'' = a$, by def. 3b, 3b, 2b, 4b, 1b, 2b, def. 2b, 7b, 6b, 1b.

Proof of A₃. The element u' will serve as the required element z . For, $\{[(c \supset u') \supset a] \supset [(b \supset c) \supset u']\} \supset u' = [(c' \supset a) \supset (b \supset c)']' = [(c' \mid a') \mid (b \mid c')']' = [(c' \mid a') \mid (b \mid c')]' = [(a' \mid c') \mid (b \mid c')]' = \{[c' \mid (a'' \mid b')]' \}' = [c' \mid (a'' \mid b')]' \mid (d \mid d') = [c' \mid (a \mid b')]' \mid (d \mid d') = (d \mid d') \mid [(a \mid b') \mid c']' = (d \supset d) \supset [(a \supset b) \supset c]$, by 8b, def. 3b, 1b, 2b, 4b, 3b, 1b, 2b, def. 3b.

Proof of A₄. By B₁.

8. **Derivation of the theory of deduction.** I now come to the derivation from A₁-A₄ of the theory of deduction of *Principia Mathematica*. The primitive ideas of this theory are a class K , a unary operation " \sim ," a binary operation " \vee ," and a notion " \vdash ," which may perhaps be termed a *predicative relation*. The postulates of the theory are the propositions C₁-C₇ below. These postulates are expressed in terms of K , \sim , \vee , \vdash , and an operation " \supset " defined by

DEFINITION 1c. $a \supset b = \sim a \vee b$.

By 26a, the " \supset " of Definition 1c is seen to be the same as the " \supset " of postulates A₁-A₄. This fact will be used hereafter without further mention. The postulates C₁-C₇ follow.†

C₁[*1·1]. If $\vdash a$ and $\vdash (a \supset b)$ then $\vdash b$.

C₂[*1·2]. $\vdash [(a \vee a) \supset a]$.

C₃[*1·3]. $\vdash [a \supset (b \vee a)]$.

C₄[*1·4]. $\vdash [(a \vee b) \supset (b \vee a)]$.

C₅[*1·6]. $\vdash \{ (a \supset b) \supset [(c \vee a) \supset (c \vee b)] \}$.

C₆[*1·7]. If a is in K , then $\sim a$ is in K .

C₇[*1·71]. If a and b are in K , then $a \vee b$ is in K .

The derivations of C₁-C₇ from A₁-A₄ follow.

Proof of C₁. Let $\vdash a$ and $\vdash (a \supset b)$. Then $a = u$ and $a \supset b = u$, by def. 7a. Hence $u \supset b = u$. Hence $b = u$, by 16a; hence $\vdash b$, by def. 7a.

Proof of C₂. $(a \vee a) \supset a = (a_1 \supset a) \supset a = a_1 \supset a_1 = u$, by def. 6a, 19a, def. 2a. Hence the theorem, by def. 7a.

Proof of C₃. $a \supset (b \vee a) = a \supset (b_1 \supset a) = u$, by def. 6a, 23a. Hence the theorem, by def. 7a.

Proof of C₄. $(a \vee b) \supset (b \vee a) = (a_1 \supset b) \supset (b_1 \supset a) = (a_1 \supset b) \supset (a_1 \supset b) = u$, by def. 6a, 10a, def. 2a. Hence the theorem, by def. 7a.

Proof of C₅. $(a \supset b) \supset [(c \vee a) \supset (c \vee b)] = (a \supset b) \supset [(c_1 \supset a) \supset (c_1 \supset b)] =$

† The numbers associated with C₁-C₇ are those of the *Principia*. For the form of *1·1, see my (VII) *Remarks on propositions *1·1 and *3·35 of Principia Mathematica*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 111-114. The *Principia* proposition *1·5 has been omitted, since *1·5 is redundant (see P. Bernays, *Mathematische Zeitschrift*, vol. 25 (1926), pp. 305-320).

$(a \supset b) \supset [a \supset (c_1 \supset b)] = b \supset [a \supset (c_1 \supset b)] = \{[a \supset (c_1 \supset b)] \supset b_1\} \supset b_1 = \{(b_{11} \supset a) \supset [(c_1 \supset b) \supset b_1]_1\}_1 \supset b_1 = \{(b \supset a) \supset [(c_1 \supset b) \supset b_1]_1\}_1 \supset b_1 = \{(b \supset a) \supset [(b_1 \supset c) \supset b_1]_1\}_1 \supset b_1 = [(b \supset a) \supset b_{11}]_1 \supset b_1 = [(b \supset a) \supset b]_1 \supset b_1 = b_1 \supset b_1 = u$, by def. 6a, 24a, 24a, 18a, 9a, 8a, 10a, A₂, 8a, A₂, def. 2a. Hence the theorem, by def. 7a.

Proof of C₆. By def. 5a, def. 1a, A₁.

Proof of C₇. By def. 6a, def. 1a, A₁.

9. Relation between the implicative operation and the operation exception. I shall now bring out the relation existing between the implicative operation \supset and the operation " $-$ " of "exception."

The considerations are simple. The element $a - b$ is, in the usual Boolean notation, the element ab' . Since $a \supset b$ is the element $a' + b$, the elements $a \supset b$ and $b - a$ are the duals of each other. Hence, a postulate-set in terms of \supset is essentially also a set in terms of " $-$," and vice versa.

Let me actually transform Postulates A₁-A₄ into a set in terms of " $-$." To do this, it will be convenient to re-letter the formulas in A₁-A₄. If we write $b - a$ for $a \supset b$, z for u (the dual of z), and re-letter, Postulates A₁-A₄ become the following postulates D₁-D₄ in terms of "exception."

D₁. $a - b$ is an element of K whenever a and b are elements of K .

D₂. $a - (b - a) = a$.

D₃. There is an element u in K such that

$$[a - (b - c)] - (d - d) = u - \{[u - (a - b)] - [c - (u - a)]\}.$$

D₄. K consists of at least two distinct elements.

To actually transform a set of "exception" postulates into a set of "implication" postulates, let me take a set due to Taylor.* This set consists of the postulates E₁-E₅ following.†

E₁. K contains at least two distinct elements.

E₂. If a and b are elements of K , $a - b$ is an element of K .

E₃. $a - (b - b) = a$.

E₄. There exists a unique element u in K such that $a - (u - b) = b - (u - a)$.

DEFINITION 1e. $a_1 = u - a$.

E₅. $a - (b - c) = [(a - b)_1 - (a - c_1)]_1$.

If we replace $a - b$ by $b \supset a$ and u by z , and re-letter, E₁-E₅ become the following postulates F₁-F₅ in terms of implication.

F₁. K contains at least two distinct elements.

F₂. If a and b are elements of K , $a \supset b$ is an element of K .

F₃. $(a \supset a) \supset b = b$.

* J. S. Taylor, loc. cit.

† In E₃, E₄, E₅ there is to be understood the supposition that the elements involved and their indicated combinations belong to K . In E₅ there is to be understood the further supposition that E₄ holds.

F₄. There exists a unique element z such that $(a \supset z) \supset b = (b \supset z) \supset a$.

DEFINITION 1f. $a_1 = a \supset z$.

F₅. $(a \supset b) \supset c = [(a_1 \supset c) \supset (b \supset c)_1]_1$.

10. **Postulates for the logic of propositions.** I take up finally the last item of my paper: to show that a simple change in one of the postulates A₁-A₄ will make these postulates a set of *independent* postulates for the logic of propositions as a two-element Boolean algebra. The change consists in replacing Postulate A₄ by Postulate A'₄ following:

POSTULATE A'₄. K consists of two distinct elements.

That A₁, A₂, A₃, A'₄ are *necessary* and *sufficient* for a two-element Boolean algebra, is obvious. That A₁, A₂, A₃, A'₄ are mutually *independent*, is seen from the table of §3: in that table *each of the independence systems for A₁, A₂, A₃ consists of only two elements*.*

* In the same way, and for the same reasons, one can transform into independent postulate-sets for the logic of propositions my two sets for Boolean algebra in terms of rejection (see my Papers I, II, loc. cit.). A like remark applies to Huntington's first set of postulates for Boolean algebra (E. V. Huntington, (III) *Sets of independent postulates for the algebra of logic*, these Transactions, vol. 5 (1904), pp. 288-309).

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ON NORMAL KUMMER FIELDS OVER A NON-MODULAR FIELD*

BY

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1. Let F be any non-modular field, p an odd prime, $\zeta \neq 1$ a p th root of unity. Suppose that μ in $F(\zeta)$ is not the p th power of any quantity of $F(\zeta)$ so that the equation $y^p = \mu$ is irreducible in $F(\zeta)$. Then the field $F(y, \zeta)$ is called a *Kummer† field over F* .

In the present paper we shall give a formal construction of all *normal* Kummer fields over F . This is equivalent to a construction of all fields $F(x)$ of degree p over F such that $F(x, \zeta)$ is cyclic of degree p over $F(\zeta)$. In particular we provide a construction of *all cyclic fields of degree p over F* .

We shall also apply the cyclic case to prove that a normal division algebra D of degree p over F is cyclic if and only if D contains a quantity y not in F such that $y^p = \gamma$ in F .

2. The equation

$$g(\xi) \equiv \xi^{p-1} + \xi^{p-2} + \cdots + \xi + 1 = 0$$

is irreducible in the field R of all rational numbers and has all the primitive p th roots of unity as roots. If F is any non-modular field, then $g(\xi)$ has an irreducible factor $h(\xi) = 0$ in F and with ζ as a root. The roots of $h(\xi) = 0$ are all powers of ζ and hence are in a sub-field L of $R(\zeta)$. But then the coefficients of $h(\xi) = 0$ are in L so that the group of $h(\xi)$ with respect to F is its group with respect to L . This latter group is the group of all the automorphisms of the cyclic field $R(\zeta)$ leaving the quantities of L invariant and is a sub-group of the group of $R(\zeta)$. Every sub-group of a cyclic group is cyclic, so that $h(\xi) = 0$ has a cyclic group generated by

$$T: \quad \zeta \longleftrightarrow \zeta^t,$$

where t is an integer *belonging* to the degree n of $h(\xi) = 0$, $t^n \equiv 1 \pmod{p}$. We may write

$$(1) \quad \zeta_k = \zeta^{t^{k-1}}, \quad \zeta_{n+1} = \zeta_1 = \zeta^n \quad (k = 1, \dots, n),$$

so that we have

* This paper is a revision and amplification of the paper *On cyclic equations of prime degree*, which I presented to the Society on December 27, 1933; it was received by the editors March 17, 1934.

† If F is the field of all rational numbers, then $F(y, \zeta)$ is the ordinary Kummer field of modern arithmetic. Our work is a generalization to any non-modular field of that special case.

$$(2) \quad \zeta_k = \zeta^{t_k}, \quad t_k \equiv t^{k-1} \pmod{p}, \quad 1 \leq t_k < p.$$

Then T is equivalent to the cyclic substitution $(\zeta_1, \zeta_2, \dots, \zeta_n)$ on the roots of $h(\xi) = 0$.

If λ and μ are any two quantities of $K = F(\zeta)$ we say that λ is p -equal to μ and write

$$(3) \quad \lambda \underset{(p)}{=} \mu.$$

H. Hasse* has then given a purely algebraic proof of

LEMMA 1. *If*

$$y^p \underset{(p)}{=} \mu \neq 1,$$

then $Z = K(y)$ is cyclic of prime degree p over K and with generating automorphism

$$S: \quad y \longleftrightarrow \zeta y.$$

Conversely every cyclic field Z of degree p over K is equal to a field $K(y)$,

$$y^p \underset{(p)}{=} \mu \neq 1.$$

Moreover if also $Z = K(z)$, $z^p \underset{(p)}{=} \mu'$ in K , then

$$\mu' \underset{(p)}{=} \mu^a,$$

so that $z = \lambda y^a$ where λ is in K .

3. We now assume that Z is any normal field of degree pn over F containing $K = F(\zeta)$ of degree n over F . Then K is the set of all quantities of Z unaltered by a cyclic sub-group H of Z of order p and Z is cyclic of degree p over K . By Lemma 1, $Z = F(y, \zeta)$, $y^p \underset{(p)}{=} \mu$ in K and $H = (I, S, \dots, S^{p-1})$ where S is given above. We can then decompose the group G of Z relative to H and write $G = H + H\sigma_1 + \dots + H\sigma_{n-1}$. Then $I, \sigma_1, \dots, \sigma_{n-1}$ carry ζ to the other roots of the irreducible equation $h(\xi) = 0$. In particular one $\sigma_i = \tau$ carries ζ to ζ^t .

We let $T = \tau^p$ so that T also carries ζ to ζ^t since $t^p \equiv t \pmod{p}$. Then τ^n leaves ζ unaltered and is in H . Hence $\tau^n = S^r$, $T^n = S^{pr} = I$.

The group G now has the decomposition $G = H + HT + \dots + HT^{n-1}$. For otherwise $T^r = S^i T^j$ where $n > r > j$ so that $T^{r-j} = S^i$ leaves ζ unaltered, which is impossible. We have proved that

* Bericht über Klassenkörper, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 36 (1927), pp. 232-311, p. 262.

$$G = \langle S^i T^j \rangle \quad (i = 0, 1, \dots, p-1; j = 0, 1, \dots, n-1).$$

The group G has a cyclic sub-group $\langle T^j \rangle$ of order n and hence Z has a sub-field $F(x)$ of degree p over F . Moreover

$$y^{(T)} = \lambda y^r \quad (\lambda \text{ in } K).$$

For $y^{(T)}$ in Z evidently generates $K(y)$ and we may apply Lemma 1. But

$$(4) \quad y^{(TS)} = \lambda \zeta^r y^r = y^{(S^e T)} = \zeta^{er} \lambda y^r,$$

where $er \equiv r \pmod{p}$ so that $e \equiv r t^{n-1} \pmod{p}$. Hence $TS = S^e T$. Conversely if $TS = S^e T$ then $r \equiv et \pmod{p}$ is determined and we have proved*

THEOREM 1. Let $F(x)$ have degree p over F and $F(x, \zeta) \equiv Z$ be normal over F . Then Z has the group

$$(5) \quad S^i T^j \quad (i = 0, 1, \dots, p-1; j = 0, 1, \dots, n-1),$$

such that $S^p = T^n = I$, the identity automorphism, and

$$(6) \quad TS = S^e T \quad (0 < e < p).$$

Moreover $Z = F(y, \zeta)$ where $y^p = \mu$ in $F(\zeta)$,

$$(7) \quad \zeta^{(T)} = \zeta^t, \quad y^{(T)} = \lambda y^r, \quad \zeta^{(S)} = \zeta, \quad y^{(S)} = \zeta y, \quad \mu^{(T)} = \mu^r, \quad (\mu \text{ in } F(\zeta))$$

and $r \equiv et \pmod{p}$.

Conversely every normal field $Z > F(\zeta)$ of degree p^n over $K = F(\zeta)$ is generated as a field $Z = F(y, \zeta)$, $y^p = \mu = \mu(\zeta)$ in $F(\zeta)$ such that

$$(8) \quad \mu \not\equiv 1, \quad \mu(\zeta^t) = \mu^r \quad (1 \leq r < p).$$

The group of Z is then given by (5), (6), (7) where e is determined by $r \equiv et \pmod{p}$ and Z contains a sub-field $F(x)$ of degree p over F , the field of all quantities of Z unaltered by the automorphism T .

It is evident that $F(x)$ is uniquely determined in the sense of equivalence and is generated by any quantity

$$(9) \quad x = \sum_{i=0}^{p-1} \alpha_i(\zeta) y^i = \sum_{i=0}^{p-1} \alpha_i(\zeta^t) \lambda^i y^{ri}$$

for which at least one $\alpha_i \neq 0$ for $i > 0$. Moreover the equation

$$(10) \quad \phi(\eta) \equiv (\eta - x)(\eta - x^{(S)}) \cdots (\eta - x^{(S^{p-1})})$$

has coefficients in F , is irreducible in F , and has x as a root. Hence Theorem 1

* A similar result was obtained by Hilbert for the case $F = R$.

gives a formal construction of all fields $F(x)$ of degree p over F with the property that $F(x, \zeta)$ is normal over F in terms of the construction of all quantities μ satisfying (8).

If in particular $F(y, \zeta)$ has an abelian group, then $F(y, \zeta) = F(x) \times F(\zeta)$, where $F(x)$ is cyclic over F . Conversely if $F(x)$ is cyclic over F , then $F(x) \times F(\zeta) = F(y, \zeta)$ has an abelian group, $e=1$, $r=t$ and we have

THEOREM 2. *Let μ range over all quantities of $F(\zeta)$ such that*

$$(11) \quad \mu \not\equiv 1 \pmod{p}, \quad \mu(\zeta^t) = \mu^t.$$

Then $Z = F(x) \times F(\zeta)$ where $F(x)$ is cyclic of degree p over F . Conversely every cyclic field $F(x)$ of degree p over F is the uniquely defined sub-field of such an $F(\mu^{1/p}, \zeta)$.

4. We proceed now to the construction of the quantities μ . The condition

$$\mu \not\equiv 1 \pmod{p}$$

is evidently an irreducibility condition depending intrinsically on F itself and so must remain in our final conditions. We first prove

LEMMA 2. *The integer r satisfies the congruence*

$$(12) \quad r^n \equiv 1 \pmod{p}.$$

For

$$\text{if } \mu^{(T)} = \mu^r \text{ then } \mu = \mu^{r^n}$$

and hence

$$\mu^{r^n-1} = 1.$$

But then if $y^p = \mu$ the quantity $y^{r^n-1} = \lambda y^n$ where $r^n - 1 \equiv s \pmod{p}$, $0 \leq s < p$ and λ is in $F(\zeta)$. But y^{sp} is then in $F(\zeta)$ so that $s=0$.

We have observed that $0 < r < p$ so that there exists an integer ρ such that

$$(13) \quad \rho r \equiv 1 \pmod{p}.$$

We define

$$(14) \quad \rho_k \equiv \rho^{k-1} \pmod{p}, \quad 1 \leq \rho_k < p,$$

for all integer values of k , where $\rho_{n+1} = \rho_1 = 1$, and $\rho^{-\alpha}$, $\alpha > 0$, is to be defined as a corresponding positive power of ρ . Then

$$(15) \quad r \rho_k \equiv \rho_{k-1} \pmod{p}.$$

We may then prove

LEMMA 3. Let λ be any quantity of $F(\zeta)$ and define

$$(16) \quad \mu = \prod_{k=1}^n \lambda(\zeta_k)^{\rho_k}.$$

Then

$$(17) \quad \mu^{(T)} = \mu(\zeta^t) = \mu^{(p)}.$$

For the automorphism T carrying ζ to ζ^t carries each ζ_k to ζ_{k+1} . Hence

$$(18) \quad \mu^{(T)} = \prod_{k=1}^n \lambda(\zeta_{k+1})^{\rho_k} \equiv \prod_{k=1}^n \lambda(\zeta_k)^{\rho_{k-1}},$$

while, by (15),

$$\mu^r = \prod_{k=1}^n \lambda(\zeta_k)^{r\rho_k} = \mu(\zeta^t)_{(p)}$$

as desired.

Let now

$$\mu(\zeta^t)_{(p)} = \mu^r \text{ and } \mu \not\equiv 1_{(p)}.$$

Then define

$$(19) \quad M = \prod_{k=1}^n \Lambda(\zeta_k)^{\rho_k}$$

where $\Lambda = \mu$. Then $\Lambda(\zeta_k) = \mu^{k-1}$ so that

$$(20) \quad \Lambda(\zeta_k)^{\rho_k} = \mu^{(r\rho)^{k-1}}_{(p)} = \mu_{(p)}$$

and hence

$$(21) \quad M = \mu^n_{(p)}.$$

But n is not divisible by p so that $z = y^n$ generates $K(y)$,

$$z^p = M_{(p)}.$$

Hence $F(y, \zeta) = F(w, \zeta)$ where $w^p = M$ is a quantity of the form (16). Conversely if μ has the form (16) and

$$\mu \not\equiv 1_{(p)}$$

then $F(y, \zeta)$, $y^p = \mu$, is normal of degree np over F . We have proved

THEOREM 3. *Let λ range over all quantities of $F(\zeta)$ such that*

$$(22) \quad y^p = \mu \equiv \prod_{k=1}^n \lambda(\zeta_k)^{p_k} \not\equiv 1. \quad (p)$$

Then $F(y, \zeta)$ is a normal field of Theorem 1. Conversely every normal field of Theorem 1 is generated by a μ defined by (22).

We have now succeeded in giving a formal construction of all the fields of Theorem 1. In particular we have constructed all cyclic fields of prime degree over F . For this case we have $pt \equiv 1 \pmod{p}$, and may state

THEOREM 4. *Let $p_k \equiv t^{p-k} \pmod{p}$ so that $tp_k \equiv t^{p-(k-1)} \equiv p_{k-1} \pmod{p}$ and let λ range over all quantities of $F(\zeta)$ such that*

$$(23) \quad a = \prod_{k=1}^n \lambda(\zeta_k)^{p_k}$$

is not the p th power of any quantity b of $F(\zeta)$. Then if

$$(24) \quad z^p = a,$$

the field $F(z, \zeta)$ is cyclic of degree np over F and

$$F(z) = F(x) \times F(\zeta),$$

where $F(x)$ is cyclic of degree p over F . Conversely every cyclic field $F(x)$ of degree p over F is generated as the uniquely defined sub-field of such an $F(z, \zeta)$.

We have thus given a construction of all cyclic fields of prime degree over any non-modular field F where the condition $a \not\equiv b^p$ is the irreducibility condition.

5. On normal division algebras of degree p . Let Z be a cyclic field of degree p over F so that every automorphism of Z is a power of an automorphism S given by $z \mapsto z^S$ for every z and corresponding z^S of Z . Define an algebra D whose quantities have the form

$$(25) \quad \sum_{i=0}^{p-1} z_i y^i \quad (z_i \text{ in } Z),$$

such that

$$(26) \quad y^i z = z^S y^i, \quad y^p = \gamma \neq 0 \text{ in } F.$$

Then D is a cyclic algebra over F and is a normal division algebra if and only

if $\gamma \neq N(z)$ for any z in Z . Evidently D is uniquely defined by Z, S, γ and we write

$$(27) \quad D = (Z, S, \gamma) = (Z, S, \delta), \quad \delta = N(c)\gamma$$

for any c of Z . For γ is replaced by δ when we replace y by cy . Also*

$$(28) \quad (Z, S, \gamma) \times (Z, S, \delta) \sim (Z, S, \gamma\delta).$$

If D is a cyclic normal division algebra of degree p over F , then D has the above form and hence contains a sub-field $F(y)$, $y^p = (\gamma)$ in F .

Conversely, let D be any normal division algebra of degree p over F with $F(x)$, $x^p = \beta$ in F as sub-field. Let $K = F(\zeta)$ of degree n over F . The algebra

$$(29) \quad M = (K, T, 1),$$

a cyclic algebra of degree n over F , is a total matric algebra. We form the direct product $M \times D$ which evidently contains $K \times D = D_0$ as sub-algebra. Algebra D_0 is a normal division algebra of degree p over K and has the cyclic sub-field $Z = K(x)$. Moreover

$$(30) \quad D_0 = (Z, S, \gamma),$$

where γ is in K and the automorphism S is given by the transformation

$$(31) \quad yx = \zeta xy, \quad x^S \equiv \zeta x.$$

Let M have a basis $(\epsilon^i j^k)$ ($i, k = 0, 1, \dots, n$) such that $j^n = 1$. Then in $D \times M$ we have

$$(32) \quad j(yx)j^{-1} = y_T x = j(\zeta xy)j^{-1} = \zeta^i x y_T,$$

where $y_T = j y j^{-1}$ is in $D \times M$. But y is commutative with ζ since y is in D_0 . Also $y\zeta = \zeta y$ implies that $y_T \zeta^i = \zeta^i y_T$ and hence y_T is also commutative with ζ . For $F(\zeta^i) = F(\zeta)$. The algebra of all quantities of $D \times M$ commutative with ζ is evidently D_0 so that y_T is in D_0 .

Since $y_T x = \zeta^i x y_T$ while $y^i x = \zeta^i x y^i$, we then have $y_T = d y^i$ where d is in Z . Then

$$(33) \quad (y_T)^p = j \gamma j^{-1} = \gamma(\zeta^i) = N(d) \gamma^i,$$

where $N(d)$ is the norm of the quantity d of the cyclic field Z . But

$$(34) \quad D_0^i \sim (Z, S, \gamma^i) = (Z, S, \gamma(\zeta^i)),$$

by (33), (27).

* If A is any normal simple algebra, then $A = M \times D$, where the total matric algebra M and the normal division algebra D are uniquely determined in the sense of equivalence. If A and B are two normal simple algebras with the same D , we say that A and B are similar, and write $A \sim B$.

By applying (34) we have $D_0^{t^2} \sim (Z, S, \gamma(\xi^{t^2}))$, and hence

$$D_0^{t^k} \sim (Z, S, \gamma(\xi^k)),$$

from which, if $u = \sum \rho_k t_k = n + \lambda p$ by (25),

$$D_0^u \sim D_0^n \sim (Z, S, \alpha),$$

where

$$\alpha = \prod_{k=1}^n \gamma(\xi^k)^{\rho_k}.$$

If D is any normal simple algebra of prime degree p over F , and K is a field of degree n not divisible by p , then D is a total matrix algebra if and only if $D \times K$ over K is a total matrix algebra. Moreover, if r is prime to p , then D^r is total matrix if and only if D is total matrix. Hence, if $D_0 = D \times K$ and D_0^r is a total matrix algebra, then so is D .

Algebra D_0^n is a normal division algebra since D is a normal division algebra. Hence $\alpha \neq N(c)$ for any c of Z . In particular $\alpha \neq b^p$ for any b of K . Thus D_0 contains a cyclic field* W of prime degree p over F . But then $D_0^n \times W'$ over $W' \cong W_K$, the composite of W and K , is a total matrix algebra. Hence $D_0 \times W'$ is a total matrix algebra and so must be $D \times \overline{W}$ over \overline{W} , $\overline{W} \cong W$. But then D has a sub-field equivalent to W and is cyclic.

THEOREM 5. *A normal division algebra D of prime degree p over F is cyclic if and only if D has a sub-field $F(x)$, $x^p = \gamma$ in F .*

* The cyclic sub-field of $F(\alpha^{1/p})$ defined by Theorem 4.

CORRECTION TO A PAPER ON THE WHITEHEAD- HUNTINGTON POSTULATES

BY

A. H. DIAMOND

Professor E. W. Chittenden has called to my attention an error which occurs in the last footnote on page 940 of my paper in volume 35 of these Transactions. It should read as follows:

The largest number hitherto published is $2^8 = 256$. See Dorothy McCoy, *The complete existential theory of eight fundamental properties of topological spaces*, Tôhoku Mathematical Journal, vol. 33 (1931), pp. 88-116. $2^6 = 64$ propositions occur in a paper of B. A. Bernstein, *The complete existential theory of Hurwitz's postulates for abelian groups and fields*, etc.

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